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On anti–invariant submanifolds in Sasakian manifolds with vanishing contact Bochner curvature tensor

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In a Sasakian manifold, MATSUMOTO and CHŪMAN [4] defined a contact Bochner curvature tensor, which is constructed from the Bochner curvature tensor by the fibering of BOOTHBY and WANG [1] (see also YANO [6]). HASEGAWA and NAKANE [2] and IKAWA and KON [3] have studied a Sasakian manifold with vanishing contact Bochner curvature tensor. YANO [6], [7] studied it in the theory of submanifolds. In this paper we shall give a sufficient condition for an anti–invariant submanifold in a Sasakian manifold with vanishing contact Bochner curvature tensor to be totally geodesic.

1. Preliminaries

Let $\overline{M}^{2r+1}$ be a $(2r+1)$–dimensional contact metric manifold with the structure $(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$, where $\overline{\phi}$ is a linear mapping $T\overline{M} \to T\overline{M}$ ($T\overline{M}$ is the tangent bundle over $\overline{M}^{2r+1}$), $\overline{\xi}$ is a vector field, $\overline{\eta}$ a 1–form, and $\overline{g}$ a Riemannian metric on $\overline{M}^{2r+1}$, such that

$$\overline{\phi}\,\overline{\xi} = 0, \quad \overline{\eta}(\overline{\xi}) = 1, \quad \overline{\phi}^2 = -I + \overline{\eta} \otimes \overline{\xi}, \quad \overline{\eta}(\overline{X}) = \overline{g}(\overline{\xi}, \overline{X})$$

$$\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \overline{\eta}(\overline{X})\overline{\eta}(\overline{Y}), \quad \overline{g}(\overline{X}, \overline{\phi}\,\overline{Y}) = d\,\overline{\eta}(\overline{X}, \overline{Y})$$

for any vector fields $\overline{X}$ and $\overline{Y}$ on $\overline{M}^{2r+1}$. If a contact metric manifold $\overline{M}^{2r+1}$ is normal (i.e., $\overline{N} + d\,\overline{\eta} \otimes \overline{\xi} = 0$, where $\overline{N}$ denotes the Nijenhuis tensor formed with $\overline{\phi}$), $\overline{M}^{2r+1}$ is called a Sasakian manifold. It is well–known that in a Sasakian manifold $\overline{\xi}$ is a Killing vector field.

The contact Bochner curvature tensor $\overline{B}$ of a Sasakian manifold $\overline{M}^{2r+1}$ is given by

$$
\begin{aligned}
\overline{B}(\overline{X},\overline{Y}) &= \overline{R}(\overline{X},\overline{Y}) + \frac{1}{m+4}(\overline{Q}\,\overline{Y}\wedge\overline{X} - \overline{Q}\,\overline{X}\wedge\overline{Y} \\
&+ \overline{Q}\,\overline{\phi}\,\overline{Y}\wedge\overline{\phi}\,\overline{X} - \overline{Q}\,\overline{\phi}\,\overline{X}\wedge\overline{\phi}\,\overline{Y} + 2\overline{g}(\overline{Q}\,\overline{\phi}\,\overline{X},\overline{Y})\overline{\phi} \\
&+ 2\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\overline{Q}\,\overline{\phi} + \overline{\eta}(\overline{Y})\overline{Q}\,\overline{X}\wedge\overline{\xi} + \overline{\eta}(\overline{X})\overline{\xi}\wedge\overline{Q}\,\overline{Y}) \\
&- \frac{k+m}{m+4}(\overline{\phi}\,\overline{Y}\wedge\overline{\phi}\,\overline{X} + 2\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\overline{\phi}) - \frac{k-4}{m+4}\overline{Y}\wedge\overline{X} \\
&+ \frac{k}{m+4}(\overline{\eta}(\overline{Y})\overline{\xi}\wedge\overline{X} + \overline{\eta}(\overline{X})\overline{Y}\wedge\overline{\xi}),
\end{aligned}
\tag{1.1}
$$

where $\overline{R}$ is the Riemannian curvature tensor of $\overline{M}^{2r+1}$, $\overline{Q}$ is the Ricci operator of $\overline{M}^{2r+1}$, $k = (\overline{S}+m)/(m+2)$ ($m = 2r$, and $\overline{S}$ is the scalar curvature tensor of $\overline{M}^{2r+1}$), and $(\overline{X}\wedge\overline{Y})\overline{Z} = \overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y}$ (see [4]).

Let M^n be an n–dimensional submanifold of $\overline{M}^{2r+1}$. By N_A ($A = 1,2,\ldots,2r+1-n$) we denote local mutually orthogonal unit vector fields normal to M^n. Let $\overline{\nabla}$ (resp. ∇) be the Riemannian connection on $\overline{M}^{2r+1}$ (resp. M^n) determined by the metric $\overline{g}$ (resp. the induced metric g). Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_A h_A(X,Y)N_A,\ \overline{\nabla}_X N_A = -H_A X + \sum_B L_{BA}(X)N_B,$$

where h_A and H_A are the second fundamental forms and L_{BA} the third fundamental forms of M^n. h_A and H_A satisfy $h_A(X,Y) = g(H_A X,Y) = g(X,H_A Y) = h_A(Y,X)$. For any vector field X,Y,Z and W on M^n the Gauss equation is given by

$$
\begin{aligned}
&\overline{g}(\overline{R}(X,Y)Z,W) = g(R(X,Y)Z,W) \\
&- \sum_B g(H_B Y,Z)g(H_B X,W) + \sum_B g(H_B X,Z)g(H_B Y,W),
\end{aligned}
\tag{1.2}
$$

where R is the Riemannian curvature tensor of M^n.

M^n immersed in $\overline{M}^{2r+1}$ is said to be anti–invariant in $\overline{M}^{2r+1}$ if $\overline{\phi}T_x(M^n) \subset T_x(M^n)^\perp$ for each $x \in M^n$, where $T_x(M^n)$ denotes the tangent space of M^n at x and $T_x(M^n)^\perp$ denotes the normal space of M^n at x. Then we see that $\overline{\phi}$ is of rank $2r$, $n \le r+1$.

Define $\overline{s}^\star = \sum_{i,j=0}^{2r} \overline{g}(\overline{R}(\overline{E}_i,\overline{E}_j)\overline{\phi}\,\overline{E}_j,\overline{\phi}\,\overline{E}_i)$ on $\overline{M}^{2r+1}$, where $\{\overline{E}_i\}$ is an orthonormal basis. We need, in the sequel, the following two lemmas

Lemma 1.1. *(e.g., [8])* *If $\overline{M}^{2r+1}$ is a Sasakian manifold, we have*

$$\bar{g}(\overline{Q}\,\overline{X},\bar{\xi}) = 2r\bar{\eta}(\overline{X}) \tag{1}$$

$$\bar{g}(\overline{Q}\,\bar{\phi}\,\overline{X},\bar{\phi}\,\overline{Y}) = \bar{g}(\overline{Q}\,\overline{X},\overline{Y}) - 2r\bar{\eta}(\overline{X})\bar{\eta}(\overline{Y}) \tag{2}$$

Lemma 1.2. *([5])* *For any $(2r+1)$–dimensional Sasakian manifold $\overline{M}^{2r+1}$, we have*

$$\bar{s}^{\star} - \overline{S} + 4r^2 = 0.$$

2. Some Lemmas

Lemma 2.1. *Let $\overline{M}^{2r+1}$ be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M^n (resp. M^{n+1}) is an anti–invariant submanifold of $\overline{M}^{2r+1}$ normal to the structure vector field $\bar{\xi}$ (resp. tangent to the structure vector field $\bar{\xi}$), then we have*

$$(r+2)\bar{s}^{\star} = (r+2)\overline{S} - n\sum_{i=1}^{n}\bar{g}(\overline{Q}\tilde{e}_i,\tilde{e}_i) - 4r^2(r+2),$$

where $\{\tilde{e}_i\}$ is an orthonormal basis of $T_x(M^n)$ (resp. $\{\tilde{e}_i\}$ is an orthonormal basis of $T_x(M^{n+1})$, that is orthogonal to $\bar{\xi}$).

PROOF. Since the contact Bochner curvature tensor of $\overline{M}^{2r+1}$ vanishes, we have

$$\begin{aligned}
\bar{g}(\overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W}) = &-\frac{1}{m+4}(\bar{g}(\overline{X},\overline{Z})\bar{g}(\overline{Q}\,\overline{Y},\overline{W})\\
&- \bar{g}(\overline{Q}\,\overline{Y},\overline{Z})\bar{g}(\overline{X},\overline{W}) - \bar{g}(\overline{Y},\overline{Z})\bar{g}(\overline{Q}\,\overline{X},\overline{W})\\
&+ \bar{g}(\overline{Q}\,\overline{X},\overline{Z})\bar{g}(\overline{Y},\overline{W}) + \bar{g}(\bar{\phi}\,\overline{X},\overline{Z})\bar{g}(\overline{Q}\,\bar{\phi}\,\overline{Y},\overline{W})\\
&- \bar{g}(\overline{Q}\,\bar{\phi}\,\overline{Y},\overline{Z})\bar{g}(\bar{\phi}\,\overline{X},\overline{W}) - \bar{g}(\bar{\phi}\,\overline{Y},\overline{Z})\bar{g}(\overline{Q}\,\bar{\phi}\,\overline{X},\overline{W})\\
&+ g(\overline{Q}\,\bar{\phi}\,\overline{X},\overline{Z})\bar{g}(\bar{\phi}\,\overline{Y},\overline{W}) + 2\bar{g}(\overline{Q}\,\bar{\phi}\,\overline{X},\overline{Y})\bar{g}(\bar{\phi}\,\overline{Z},\overline{W})\\
&+ 2\bar{g}(\bar{\phi}\,\overline{X},\overline{Y})\bar{g}(\overline{Q}\,\bar{\phi}\,\overline{Z},\overline{W}) + \bar{\eta}(\overline{Y})(\bar{g}(\bar{\xi},\overline{Z})\bar{g}(\overline{Q}\,\overline{X},\overline{W})\\
&- \bar{g}(\overline{Q}\,\overline{X},\overline{Z})\bar{g}(\bar{\xi},\overline{W})) + \bar{\eta}(\overline{X})(\bar{g}(\overline{Q}\,\overline{Y},\overline{Z})\bar{g}(\bar{\xi},\overline{W})\\
&- \bar{g}(\bar{\xi},\overline{Z})\bar{g}(\overline{Q}\,\overline{Y},\overline{W}))) + \frac{k+m}{m+4}(\bar{g}(\bar{\phi}\,\overline{X},\overline{Z})\bar{g}(\bar{\phi}\,\overline{Y},\overline{W})\\
&- \bar{g}(\bar{\phi}\,\overline{Y},\overline{Z})\bar{g}(\bar{\phi}\,\overline{X},\overline{W}) + 2\bar{g}(\bar{\phi}\,\overline{X},\overline{Y})\bar{g}(\bar{\phi}\,\overline{Z},\overline{W})))\\
&+ \frac{k-4}{m+4}(\bar{g}(\overline{X},\overline{Z})\bar{g}(\overline{Y},\overline{W}) - \bar{g}(\overline{Y},\overline{Z})\bar{g}(\overline{X},\overline{W}))
\end{aligned} \tag{2.1}$$

$$- \frac{k}{m+4}(\overline{\eta}(\overline{Y})(\overline{g}(\overline{X},\overline{Z})\overline{g}(\overline{\xi},\overline{W}) - \overline{g}(\overline{\xi},\overline{Z})\overline{g}(\overline{X},\overline{W})) + \overline{\eta}(\overline{X})(\overline{g}(\overline{\xi},\overline{Z})\overline{g}(\overline{Y},\overline{W}) - \overline{g}(\overline{Y},\overline{Z})\overline{g}(\overline{\xi},\overline{W}))).$$

If we take a $\overline{\phi}$–basis $(\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n, \overline{\xi}, \overline{e}_{n+1}, \overline{e}_{n+2}, \dots, \overline{e}_r, \overline{\phi}\tilde{e}_1, \overline{\phi}\tilde{e}_2, \dots, \overline{\phi}\tilde{e}_n, \overline{\phi}\overline{e}_{n+1}, \dots, \overline{\phi}\overline{e}_r)$, $\overline{s}^\star$ is expressed by the following form

$$\begin{aligned}
\overline{s}^\star = & \sum_{i,j=1}^{n} \overline{g}(\overline{R}(\tilde{e}_i,\tilde{e}_j)\overline{\phi}\tilde{e}_j,\overline{\phi}\tilde{e}_i) + \sum_{i=1}^{n}\sum_{j=n+1}^{r} \overline{g}(\overline{R}(\tilde{e}_i,\overline{e}_j)\overline{\phi}\overline{e}_j,\overline{\phi}\tilde{e}_i) \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{g}(\overline{R}(\tilde{e}_i,\overline{\phi}\tilde{e}_j)\overline{\phi}^2\tilde{e}_j,\overline{\phi}\tilde{e}_i) \\
& + \sum_{i=1}^{n}\sum_{j=n+1}^{r} \overline{g}(\overline{R}(\tilde{e}_i,\overline{\phi}\overline{e}_j)\overline{\phi}^2\overline{e}_j,\overline{\phi}\tilde{e}_i) \\
& + \sum_{i=n+1}^{r}\sum_{j=1}^{n} \overline{g}(\overline{R}(\overline{e}_i,\tilde{e}_j)\overline{\phi}\tilde{e}_j,\overline{\phi}\overline{e}_i) \\
& + \sum_{i=n+1}^{r}\sum_{j=n+1}^{r} \overline{g}(\overline{R}(\overline{e}_i,\overline{e}_j)\overline{\phi}\overline{e}_j,\overline{\phi}\overline{e}_i) \\
& + \sum_{i=n+1}^{r}\sum_{j=1}^{n} \overline{g}(\overline{R}(\overline{e}_i,\overline{\phi}\tilde{e}_j)\overline{\phi}^2\tilde{e}_j,\overline{\phi}\overline{e}_i) \\
& + \sum_{i=n+1}^{r}\sum_{j=n+1}^{r} \overline{g}(\overline{R}(\overline{e}_i,\overline{\phi}\overline{e}_j)\overline{\phi}^2\overline{e}_j,\overline{\phi}\overline{e}_i) \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{g}(\overline{R}(\overline{\phi}\tilde{e}_i,\tilde{e}_j)\overline{\phi}\tilde{e}_j,\overline{\phi}^2\tilde{e}_i) \\
& + \sum_{i=1}^{n}\sum_{j=n+1}^{r} \overline{g}(\overline{R}(\overline{\phi}\tilde{e}_i,\overline{e}_j)\overline{\phi}\overline{e}_j,\overline{\phi}^2\tilde{e}_i) \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{g}(\overline{R}(\overline{\phi}\tilde{e}_i,\overline{\phi}\tilde{e}_j)\overline{\phi}^2\tilde{e}_j,\overline{\phi}^2\tilde{e}_i) \\
& + \sum_{i=1}^{n}\sum_{j=n+1}^{r} \overline{g}(\overline{R}(\overline{\phi}\tilde{e}_i,\overline{\phi}\overline{e}_j)\overline{\phi}^2\overline{e}_j,\overline{\phi}^2\tilde{e}_i) \\
& + \sum_{i=n+1}^{r}\sum_{j=1}^{n} \overline{g}(\overline{R}(\overline{\phi}\overline{e}_i,\tilde{e}_j)\overline{\phi}\tilde{e}_j,\overline{\phi}^2\overline{e}_i)
\end{aligned} \tag{2.2}$$

$$+\sum_{i=n+1}^{r}\sum_{j=n+1}^{r}\overline{g}(\overline{R}(\overline{\phi}\overline{e}_i,\overline{e}_j)\overline{\phi}\overline{e}_j,\overline{\phi}^2\overline{e}_i)$$
$$+\sum_{i=n+1}^{r}\sum_{j=1}^{n}\overline{g}(\overline{R}(\overline{\phi}\overline{e}_i,\overline{\phi}\tilde{e}_j)\overline{\phi}^2\tilde{e}_j,\overline{\phi}^2\overline{e}_i)$$
$$+\sum_{i=n+1}^{r}\sum_{j=n+1}^{r}\overline{g}(\overline{R}(\overline{\phi}\overline{e}_i,\overline{\phi}\overline{e}_j)\overline{\phi}^2\overline{e}_j,\overline{\phi}^2\overline{e}_i).$$

Thus, using (2.1) in the right hand members of (2.2), we get after some lengthy computation

$$\overline{s}^{\star}=\frac{4r-n+4}{r+2}\sum_{i=1}^{n}\overline{g}(\overline{Q}\tilde{e}_i,\tilde{e}_i)+\frac{4r+4}{r+2}\sum_{j=n+1}^{r}\overline{g}(\overline{Q}\overline{e}_j,\overline{e}_j) \tag{2.3}$$
$$-\frac{k+m}{r+2}\cdot r(2r+1)-\frac{k-4}{r+2}r\ .$$

On the other hand, by Lemma 1.1, we have

$$\sum_{j=n+1}^{r}\overline{g}(\overline{Q}\overline{e}_j,\overline{e}_j)=\frac{1}{2}\overline{S}-\sum_{i=1}^{n}\overline{g}(\overline{Q}\tilde{e}_i,\tilde{e}_i)-r. \tag{2.4}$$

Substituting (2.4) into (2.3), we get our result.

Lemma 2.2. *Let $\overline{M}^{2r+1}$ be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M^n is an anti–invariant submanifold normal to the structure vector field $\overline{\xi}$ of $\overline{M}^{2r+1}$, then we have*

$$4(r+1)(n-1)\sum_{i=1}^{n}\overline{g}(\overline{Q}\tilde{e}_i,\tilde{e}_i)=4(r+1)(r+2)S-4(r+1)(r+2)\sum_{B}(TrH_B)^2$$
$$+4(r+1)(r+2)\sum_{B}TrH_B{}^2-n(n-1)(6r+8)+n(n-1)\overline{S},$$

where S is the scalar curvature tensor of M^n.

PROOF. Taking a $\overline{\phi}$–basis, we find

$$\sum_{i=1}^{n}\overline{g}(\overline{Q}\tilde{e}_i,\tilde{e}_i)=\sum_{i,j=1}^{n}\overline{g}(\overline{R}(\tilde{e}_j,\tilde{e}_i)\tilde{e}_i,\tilde{e}_j)+\sum_{i=1}^{n}\overline{g}(\overline{R}(\overline{\xi},\tilde{e}_i)\tilde{e}_i,\overline{\xi})$$
$$+\sum_{i=1}^{n}\sum_{j=n+1}^{r}\overline{g}(\overline{R}(\overline{e}_j,\tilde{e}_i)\tilde{e}_i,\overline{e}_j)+\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{g}(\overline{R}(\overline{\phi}\tilde{e}_j,\tilde{e}_i)\tilde{e}_i,\overline{\phi}\tilde{e}_j) \tag{2.5}$$
$$+\sum_{i=1}^{n}\sum_{j=n+1}^{r}\overline{g}(\overline{R}(\overline{\phi}\overline{e}_j,\tilde{e}_i)\tilde{e}_i,\overline{\phi}\overline{e}_j).$$

Here, by (1.2) we have

$$\sum_{i,j=1}^{n} \overline{g}(\overline{R}(\tilde{e}_j,\tilde{e}_i)\tilde{e}_i,\tilde{e}_j) = S - \sum_B (TrH_B)^2 + \sum_B TrH_B{}^2.$$

Moreover, using (2.1) in any other members on the right hand side of (2.5), we get

$$\begin{aligned}(2.6)\qquad &\frac{1}{r+2}\sum_{i=1}^{n}\overline{g}(\overline{Q}\tilde{e}_i,\tilde{e}_i) + \frac{n}{r+2}\sum_{j=n+1}^{r}\overline{g}(\overline{Q}\overline{e}_j,\overline{e}_j) + S - \sum_B (TrH_B)^2 \\ &+ \sum_B TrH_B{}^2 - \frac{n}{2r+4}(4(n-r-1)+k(2r-n+3)) = 0.\end{aligned}$$

Substituting (2.4) into (2.6), we obtain our result.

Lemma 2.3. *Let $\overline{M}^{2r+1}$ be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M^{n+1} is an anti-invariant submanifold tangent to the structure vector field $\overline{\xi}$ of $\overline{M}^{2r+1}$, then we have*

$$\begin{aligned}&4(r+1)(n-1)\sum_{i=1}^{n}\overline{g}(\overline{Q}\tilde{e}_i,\tilde{e}_i) = 4(r+1)(r+2)S \\ &-4(r+1)(r+2)\sum_B (TrH_B)^2 + 4(r+1)(r+2)\sum_B TrH_B{}^2 + n(n-1)\overline{S} \\ &-2n(4r^2+3(3+n)r+4n+4).\end{aligned}$$

PROOF. Taking a $\overline{\phi}$–basis, we have (2.5). Here, by (1.2), we get

$$\begin{aligned}\sum_{i,j=1}^{n} \overline{g}(\overline{R}(\tilde{e}_j,\tilde{e}_i)\tilde{e}_i,\tilde{e}_j) =& S - \sum_B (TrH_B)^2 + \sum_B TrH_B{}^2 \\ &- 2\sum_{i=1}^{n}\overline{g}(\overline{R}(\overline{\xi},\tilde{e}_i)\tilde{e}_i,\overline{\xi}).\end{aligned}$$

From (2.1) and Lemma 1.1, we find

$$\sum_{i=1}^{n}\overline{g}(\overline{R}(\overline{\xi},\tilde{e}_i)\tilde{e}_i,\overline{\xi}) = \frac{2n(r+2)}{m+4}\ .$$

Thus, we have

$$\sum_{i,j=1}^{n} \overline{g}(\overline{R}(\tilde{e}_j, \tilde{e}_i)\tilde{e}_i, \tilde{e}_j) = S - \sum_B (TrH_B)^2 + \sum_B TrH_B{}^2 - \frac{4n(r+2)}{m+4} .$$

Moreover, using (2.1) in any other members on the right hand side of (2.5), we find

$$\begin{aligned} \frac{1}{r+2}\sum_{i=1}^{n}\overline{g}(\overline{Q}\tilde{e}_i, \tilde{e}_i) + \frac{n}{r+2}\sum_{j=n+1}^{r}\overline{g}(\overline{Q}\overline{e}_j, \overline{e}_j) + S - \sum_B (TrH_B)^2 \\ + \sum_B TrH_B{}^2 - \frac{n}{2r+4}(4(n+1) + k(2r-n+3)) = 0. \end{aligned} \tag{2.7}$$

Substituting (2.4) into (2.7), we obtain our result.

3. The results

Proposition 3.1. *Let M^{2r+1} be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M^n is an anti-invariant submanifold normal to the structure vector field $\overline{\xi}$ of $\overline{M}^{2r+1}$, then we have*

$$\begin{aligned} 4n(r+1)(r+2)\left(S + \sum_B TrH_B{}^2 - \sum_B (TrH_B)^2\right) = \\ n^2(n-1)(6r+8-\overline{S}). \end{aligned} \tag{3.1}$$

PROOF. By Lemma 1.2, Lemma 2.1 and Lemma 2.2, we have our result.

From (3.1) we get the following.

Proposition 3.2. *Let M^n be a minimal anti-invariant submanifold normal to the structure vector field $\overline{\xi}$ of a Sasakian manifold $\overline{M}^{2r+1}$ with vanishing contact Bochner curvature tensor. If the scalar curvature S of M^n satisfies $S \geq 0$, then the scalar curvature $\overline{S}$ of $\overline{M}^{2r+1}$ satisfies $\overline{S} \leq 2(3r+4)$.*

Theorem 3.1. *Let M^n be a minimal anti-invariant submanifold normal to the structure vector field $\overline{\xi}$ of a Sasakian manifold $\overline{M}^{2r+1}$ with vanishing contact Bochner curvature tensor. If the scalar curvature $\overline{S}$ of $\overline{M}^{2r+1}$ satisfies $\overline{S} \geq 2(3r+4)$ and the scalar curvature S of M^n satisfies $S \geq 0$, then M^n is totally geodesic, and $\overline{S} = 2(3r+4)$, $S = 0$.*

PROOF. By our assumption, the left hand side of (3.1) is non-negative and the right hand side non-positive. This completes the proof of the theorem.

Proposition 3.3. *Let $\overline{M}^{2r+1}$ be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M^{n+1} is an anti-invariant submanifold tangent to the structure vector field $\overline{\xi}$ of $\overline{M}^{2r+1}$, then we have*

$$
\begin{aligned}
&4(r+1)(r+2)n\sum_B TrH_B{}^2 + 4(r+1)(r+2)nS \\
&=4(r+1)(r+2)n\sum_B (TrH_B)^2 \\
&\quad + (n-1)(4r(r+1)-n^2)(\overline{S}-(8+6r)) \\
&\quad + 8(r+1)(r+2)(n^2+r(2r+1)(n-1)).
\end{aligned} \tag{3.2}
$$

PROOF. By Lemma 1.2, Lemma 2.1 and Lemma 2.3, we get our result.

Theorem 3.2. *Let M^{n+1} be a minimal anti-invariant submanifold tangent to the structure vector field $\overline{\xi}$ of a Sasakian manifold $\overline{M}^{2r+1}$ with vanishing contact Bochner curvature tensor. If the scalar curvature $\overline{S}$ of $\overline{M}^{2r+1}$ satisfies $\overline{S} \leq 2(3r+4)$ and the scalar curvature S of M^{n+1} satisfies $S \geq \frac{2(n^2+r(2r+1)(n-1))}{n}$, then M^{n+1} is totally geodesic, and $S = \frac{2(n^2+r(2r+1)(n-1))}{n}$, $\overline{S} = 2(3r+4)$.*

PROOF. From (3.2), we have the following relation

$$
\begin{aligned}
&4n(r+1)(r+2)\Big(\sum_B TrH_B{}^2 + S - \frac{2(n^2+r(2r+1)(n-1))}{n} \\
&\quad - \sum_B (TrH_B)^2\Big) = (n-1)(4r(r+1)-n^2)(\overline{S}-(8+6r)).
\end{aligned} \tag{3.3}
$$

This completes the proof of the Theorem.

From (3.3) we have the following

Proposition 3.4. *Let M^{n+1} be a minimal anti-invariant submanifold tangent to the structure vector field $\overline{\xi}$ of a Sasakian manifold $\overline{M}^{2r+1}$ with vanishing contact Bochner curvature tensor. If the scalar curvature S of M^{n+1} satisfies $S \geq \frac{2(n^2+r(2r+1)(n-1))}{n}$, then the scalar curvature $\overline{S}$ of $\overline{M}^{2r+1}$ satisfies $\overline{S} \geq 2(3r+4)$.*

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