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A note on radical and radical free topological groups

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Abstract. All groups considered in this paper are abelian. We prove that a connected topological group has sufficiently many real characters if and only if it is radical free and the radical of a locally compact connected group is a topological direct summand. We have also proved that a connected radical free group and its residual subgroups are divisible groups.

1. Introduction. Throughout this paper G stands for a topological abelian group and all the groups considered are abelian groups. Let M be a maximal open semigroup not containing the identity O of the group which is called a maximal O-proper open semigroup by WRIGHT [3]. Let $S(M) = \{x \in G : x + M \subset M\}$. Then $b(M) = s(M) \cup s(-M)$ is called a residual subgroup of G and the intersection T of all residual subgroups is called the radical of G. When $T = G$, G is called a radical group and when $T = (0)$, G is called a radical-free group. WRIGHT [3] has studied the properties of radical and radical free groups and with the help of these groups has described the structure of locally compact abelian groups. We prove that a connected topological group G has sufficiently many real characters (i.e. given any pair $x, y \in G$, $x \neq y$, there exists a real character f such that $f(x) \neq f(y)$) if and only if it is a radical-free group. We further show that connected radical-free groups and their residual subgroups are always divisible. The annihilator of a divisible subgroup D of a locally compact group G turns out to be an Honest subgroup of the dual X of G, containing the torsion subgroup of X. It is well-known (see Theorems (24.25) and (24.26), HEWITT and ROSS [1]) that a compact group is connected (totally-disconnected) if and only if its dual is a torsion-free group (torsion group). We observe in this note that analogues of these theorems for locally compact groups which are also well-known (see 24.18 and 24.19 HEWITT and ROSS [1]) could be put in the following elegant form: A locally compact group is connected (totally-disconnected) if and only if its dual is a radical-free (radical) group.

Lemma 1. *Let G be a topological group without proper open subgroups. Then the kernel of a non-trivial continuous homomorphism into the reals is a residual subgroup.*

PROOF. Let us suppose that f is a continuous homomorphism from G into the reals R and $\ker f = H$. Let M be the set of all elements of G which are mapped into the positive real numbers. Obviously M is an open semigroup not containing the identity zero of G. Let M' be an open semigroup properly containing M and not containing zero. Let $y \in M'$, $y \notin M$. Then $f(y) = 0$. For, clearly $f(y) \leq 0$. $f(y) < 0 \Rightarrow f(-y) > 0 \Rightarrow -y \in M \subset M' \Rightarrow 0 \in M'$, a contradition. Now M' is open implies that there exists an open symmetric neighbourhood U of 0 such that $y + U \subset M'$. For $u \in U$, if $f(u) \neq 0$, either $f(u) > 0$ or $f(u) < 0$. If $f(u) > 0$, then $f(-y+u) = f(-y) + f(u) = f(u) > 0$ i.e. $-y + u \in M \subset M'$. U is symmetric implies $y - u \in y + U \subset M'$. If $f(u) < 0$ then similarly we get that $-y-u, y+u \in M'$. In either case we get that $0 \in M'$, a contradiction again, which proves that $f(u) = 0$ for every $u \in U$. Hence $U \subset \ker f = H$ which implies that H is open contradicting our hypothesis. Thus we have proved that M is a maximal open semigroup not containing O in G. It is easy to see that $H = s(M) \cap s(-M)$ where $s(M) = \{x \in G : x + M \subset M\}$ which shows that H is a residual subgroup.

A continuous homomorphism from G into the reals R is called a real character of G. It is known as a consequence of the Pontryagin duality theory (See Corollary (24.35), p. 390, HEWITT and ROSS [1]) that a locally compact group has sufficiently many real characters if and only if the character group of G is connected. We now show that a similar result holds for all connected groups.

Theorem 2. *If G is a connected topological group then G has sufficiently many real characters if and only if G is radical-free.*

PROOF. Let G be a radical-free connected topological group. Since G is radical-free the intersection of all residual subgroups is the identity O of G. Hence corresponding to any non-zero element x in G there exists a residual subgroup $b(M)$ of G such that $x \notin b(M)$. $G/b(M)$ is continuously isomorphic to the reals by Theorem 5.2 and 5.4, WRIGHT [3]. Combining this continuous isomorphism with the natural map G to $G/b(M)$ we get a real character f on G such that $f(x) \neq 0$.

Conversely if G is a connected topological group having sufficiently many real characters of G, then corresponding to any x in G, $x \neq 0$, there exists a continuous homomorphism f from G into the reals such that $f(x) \neq 0$. By Lemma 1. above, the $\ker f = H$, is a residual subgroup not containing x. This shows that the intersection of all residual subgroups of G is the identity of G which proves that G is radical-free.

Definition 3. A subgroup H of a group G is called a honest subgroup if $nx \in H$ implies $x \in H$ for every $x \in G$ and positive integer n.

It is clear from this definition that an honest subgroup H always contains the torsion-subgroup and it is a pure subgroup (see. p. 14, Section 7, Definition, KAPLANSKY [2]).

Proposition 4. *The residual subgroups of topological groups are honest subgroups.*

PROOF. If $b(M)$ is a residual subgroup of a topological group G associated to a maximal O-proper open semigroup M, then by Lemma 3.2 and Theorem 3.3, WRIGHT [3], $G = -M \cup M \cup b(M)$ which is a disjoint union and so $nx \in b(M)$ implies $x \in b(M)$.

Proposition 5. A connected radical-free group is divisible.

PROOF. Since the only connected subgroups of the reals R are O and R, we see that any connected subgroup K of πR_f, f varying in any indexing set I and each R_f being the reals R, is such that $P_f(K)$ is either R or (O) where P_f is the projection corresponding to f in I, i.e. K is topologically isomorphic to a product of the reals and hence is divisible. Let Φ be the mapping from G into πR_f, $f \in G^*$ defined by $\Phi(x) = \{f(x)\}$, $f \in G^*$, $x \in G$. When G is a connected radical free group, it is easily seen from Theorem 2 above that Φ is one-to-one and hence is a continuous isomorphism into πR_f, $f \in G^*$. Hence $\Phi(G)$ is a connected subgroup of πR_f, $f \in G^*$. Thus we see that $\Phi(G)$ is divisible and therefore G itself is divisible, since Φ is an isomorphism.

Corollary 6. *Any residual subgroup of a connected radical-free topological group is divisible.*

PROOF. By Proposition 4, any residual subgroup H of a topological group is a honest subgroup and therefore is a pure subgroup of G. Proposition 5 now implies that H is a pure subgroup of the divisible group G and hence itself is divisible by, (c), p. 14, KAPLANSKY [2].

Following the notations of HEWITT and ROSS [1] we denote by

$$D^{(n)} = \{nx : x \in D\}$$

and

$$D(n) = \{x \in G : nx \in D\}$$

associated to a subgroup D of G and a positive integer n.

Proposition 7. *The annihilator of a divisible subgroup D of a locally-compact group G is a closed honest subgroup of the dual group X.*

PROOF. Let the annihilator $A(X, D)$ of D in X be denoted by H. Then $\chi \in A(X, D^{(n)}) \iff \chi(y) = 1$ for every $y \in D^{(n)} \iff \chi(y) = \chi(nx) = n\chi(x) = 1$ for $y = nx$, $x \in D \iff n\chi \in A(X, D) = H \iff \chi \in H(n)$, i.e. $A(X, D^{(n)}) = H(n)$. D is a divisible subgroup of G if and only if $D^{(n)} = D$ for every positive integer n. Hence $H = H(n)$ for every n, i.e. H is an honest subgroup of X.

Theorem 8. *The radical of a locally-compact group contains the smallest honest subgroup.*

PROOF. Let T be the radical of a locally compact group X and G the dual of X. Let D be the largest divisible subgroup of G and C be the connected component of G. Then by Theorem (24.24), HEWITT and ROSS [1] we have

$$C \subset D \subset \overline{D} \subset \bigcap_{n=1}^{\infty} \overline{G^{(n)}} = A(G, \Phi)$$

where Φ is the torsion subgroup of X. Hence

$$T = A(X, C) \supset A(V, D) = A(X, \overline{D}) = H \supset A(X, A(G, \Phi)) = \Phi.$$

Since D is the largest divisible subgroup, H is the smallest closed Honest subgroup by Proposition 7 above.

Lemma 9. *The annihilator of the connected component C of a locally compact group G in its dual X is the radical of X.*

PROOF. If the annihilator $A(X, C)$ of the connected component C of G in X is denoted by T, then by Theorem (24.17), HEWITT and ROSS [1], it follows that T is the subgroup of compact elements of X. Now by Theorem 8.10, WRIGHT [3] we see that T is the radical of the group X.

Theorem 10. *Let G be a locally compact group and X its dual group. Then G is connected (totally-disconnected) if and only if X is radical-free (radical) group.*

PROOF. G is connected $\iff G = C \iff T = A(x, C) = (O)$ and G is totally-disconnected $\iff C = (O) \iff T = A(X, C) = X$.

Theorem 11. *Let G be a locally-compact connected group and T be the radical of G. Then T is a topological direct summand of G.*

PROOF. Let X be the character group of G and K be the component of the identity of X. Then by Lemma 9, $K = A(X, T)$ and $T = A(G, K)$. By Theorem 10 above X is a locally-compact radical-free group and by Theorem 8.5, WRIGHT [3], K is an open subgroup topologically isomorphic to R^n and X is topologically isomorphic to $K + X/K$. By duality using (23.34) and Theorem (24.11), HEWITT and ROSS [1] we get that G is topologically isomorphic to $T + G/T$ i.e. T is a direct summand.

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