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On Faber Expansions

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To the memory of Professor Béla Barna

Let G be a Jordan domain in $\mathbf{C}$ with smooth boundary ∂G. By the theorem of Riemann on the conform equivalence of simply connected domains, there exists a unique holomorphic function

$$w = \varphi(z)$$

mapping $\bar{\mathbf{C}} \setminus \partial G$ onto $\bar{\mathbf{C}} \setminus \partial \mathbf{D}$ such that

$$0 < \lim_{z \to \infty} \frac{\varphi(z)}{z} < \infty$$

(as usual, $\mathbf{D}$ denotes the open unit disk on $\mathbf{C}$). Consider for any natural number n the Laurent expansion of $\varphi^n(z)$ with the center $z = \infty$:

$$\varphi^n(z) = \sum_{m=-\infty}^{n} b_m z^m =: p_n(\bar{G}, z) + \sum_{m=-\infty}^{-1} b_m z^m .$$

The polynomials $p_n(\bar{G}, z)$ are called Faber polynomials. Denote $A_c(\bar{G})$ the class of functions analytic on G and continuous on $\bar{G}$. The Faber expansion of $f \in A_c(\bar{G})$ is defined by

$$(1) \qquad f(z) \sim \sum_{m=0}^{\infty} a_m p_m(\bar{G}, z) \quad , \quad a_m := \frac{1}{2\pi i} \int\limits_{|\tau|=1} \frac{f(\varphi^{-1}(\tau))}{\tau^{m+1}} d\tau .$$

We shall investigate two problems for Faber expansions: the strong $(C,1)$ summability and a Bohr type inequality. Denote

$$s_m(f,\bar{G},z) := \sum_{i=0}^{m} a_i p_i(\bar{G},z).$$

We say that a complex–valued function f defined on $\bar{G}$ is Zygmund–continuous, if

$$\left| f(z_1) - 2f\left(\frac{z_1+z_2}{2}\right) + f(z_2) \right| \le c_f |z_1 - z_2| \quad (z_1, z_2 \in \bar{G}).$$

We shall prove the following

Theorem 1. *Let $\bar{G}$ be a Jordan domain with analytic boundary ∂G and let $f \in A_c(\bar{G})$ be Zygmund–continuous. Then*

$$\text{(2)} \qquad \max_{z\in\bar{G}} \frac{1}{n+1} \sum_{m=0}^{n} |f(z) - s_m(f,\bar{G},z)| = O\left(\frac{\ln n}{n}\right) \quad (n \ge 1).$$

Remark. This estimate is exact. In [1] it is proved that the function

$$g(z) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k} \sum_{\ell=2^{k-1}+1}^{2^k} \frac{1}{\ell} \left(z^{2^{k+2}-\ell} - z^{2^{k+2}+\ell} \right)$$

belongs to the class $A_c(\bar{G}) \cap \mathrm{Lip}(1,\bar{G})$, i.e.

$$|g(z_1) - g(z_2)| \le c_g |z_1 - z_2| \quad (z_1, z_2 \in \bar{G}),$$

further

$$\frac{1}{n+1} \sum_{m=0}^{n} |g(1) - s_m(g,\bar{G},1)| \ge \frac{1}{96} \frac{\log_2 n}{n+1}$$

holds for $n \ge 64$.

The proof requires some notions and lemmas. First, denote

$$s \mapsto z(s)$$

the natural (arc–lenght) parametrization of the boundary $\partial\bar{G}$ and let

$$s \mapsto \Theta(s)$$

denote the angle between the positive real axis and the tangent of ∂G at $z(s)$. Finally let $\omega(\Theta,\delta)$ be the modulus of continuity of $\Theta(s)$.

We shall use the following statements of ALPER (they are not given explicitly in [3] but the proofs easily give them in the present form).

Lemma 1. ([3]). *Let G be any Jordan domain with smooth boundary satisfying the condition*

$$\int_0^1 \frac{\omega(\Theta,\delta)}{\delta} \ln\frac{1}{\delta}\, d\delta < \infty. \tag{3}$$

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\psi'(e^{it})}{\psi(e^{it})-\psi(e^{ix})}-\frac{1}{e^{it}-e^{ix}}\right| dt \le c(\bar{G}) < \infty \tag{4}$$

and

$$\begin{aligned}(5)\quad & p_m(\bar{G},\psi(e^{ix})) = \\ &= e^{imx} + \frac{1}{2\pi i}\,\text{v.p.}\int_{-\pi}^{\pi}\left[\frac{\psi'(e^{it})}{\psi(e^{it})-\psi(e^{ix})}-\frac{1}{e^{it}-e^{ix}}\right] e^{imt}\, d(e^{it})^{\star}.\end{aligned}$$

Here

$$\psi := \varphi^{-1}.$$

Lemma 2. ([3]). *Let G be any Jordan domain with smooth boundary satisfying (3) and let $f \in A_c(\bar{G})$. Then the function*

$$f^{+}(z) := \frac{1}{2\pi i}\int_{|\tau|=1}\frac{f(\psi(\tau))}{t-\tau}d\tau \quad (|z|<1)$$

can be extended onto the closed unit disk and for every $w = e^{ix}$ we have

$$\begin{aligned}(6)\quad & \\ f^{+}(e^{ix}) = & f(\psi(e^{ix})) - \\ & -\frac{1}{2\pi i}\,\text{v.p.}\int_{-\pi}^{\pi}\left[\frac{\psi'(e^{it})}{\psi(e^{it})-\psi(e^{ix})}-\frac{1}{e^{it}-e^{ix}}\right] f(\psi(e^{it}))\, d(e^{it})\end{aligned}$$

and $f^{+}(e^{ix})$ has the Fourier series

$$f^{+}(e^{ix}) \sim \sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|\tau|=1}\frac{f(\psi(\tau))}{\tau^{m+1}}d\tau\right)e^{imx}.$$

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*v.p. denotes the principal value of the integral.

Further we have

$$\omega(f^+ \circ \exp, \delta) := \sup_{|x_1 - x_2| \le \delta} |f^+(e^{ix_1}) - f^+(e^{ix_2})| \le$$
$$\le c(\bar{G}) \sup_{\substack{|z_1 - z_2| \le \delta \\ z_1, z_2 \in \bar{G}}} |f(z_1) - f(z_2)| = c(\bar{G})\, \omega(f, \delta).$$

We mention further the following result of [4]:

Lemma 3. *Let h be any 2π–periodic continuous function on $\mathbf{R}$. Then*

$$\frac{1}{n} \sum_{m=n}^{2n-1} |s_m(h, x)| \le c\|h\|_{L^\infty(-\pi,\pi)} \quad (x \in \mathbf{R}) \tag{7}$$

holds with $c = \frac{\pi}{8} + 3 + \frac{4}{\sqrt{\pi}}$ *.

Lemma 4. *Let G and f be as in Lemma 2. Then*

$$\frac{1}{n} \sum_{m=n}^{2n-1} |s_m(f, \bar{G}, z)| \le c\, \|f\|_{L^\infty(\bar{G})} \,. \tag{8}$$

PROOF. Using (5) we get

$$|s_m(f, \bar{G}, \psi(e^{ix}))| \le s_m(f^+, x) +$$
$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\psi'(e^{it})}{\psi(e^{it}) - \psi(e^{ix})} - \frac{1}{e^{it} - e^{ix}} \right| \cdot |s_m(f^+, t)|\, dt,$$

where

$$s_m(f^+, x) := \sum_{\ell=0}^{m} \left(\frac{1}{2\pi i} \int_{|\tau|=1} \frac{f(\psi(\tau))}{\tau^{\ell+1}} d\tau \right) e^{i\ell x}.$$

Hence, taking (4) into account, we get

$$\frac{1}{n} \sum_{m=n}^{2n-1} |s_m(f, \bar{G}, \psi(e^{ix}))| \le c \max_{-\pi \le x \le \pi} \frac{1}{n} \sum_{m=n}^{2n-1} |s_m(f^+, x)|.$$

According to (6) and (7), further using the maximum modulus principle we obtain (8).

Now define for any $f \in A_c(\bar{G})$ the n–th best approximation by polynomials as follows

$$E_n(f, \bar{G}) := \inf\{\|f - p\|_{L^\infty(\bar{G})} : \ p \in P_n\},$$

where P_n denotes the set of algebraic polynomials of order $\le n$.

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*$s_m(h, x)$ denotes the m–th partial sum of the trigonometric Fourier series of h in this paper.

Lemma 5. *Let G and f satisfy the conditions of Lemma 2. Then*

$$\frac{1}{n}\sum_{m=n}^{2n-1}|f(z)-s_m(f,\bar{G},z)|\le cE_n(f,\bar{G}). \tag{9}$$

Proof. According to a theorem of L. Tonelli [5, p.406], for any $n\ge 0$ there exists a best approximating polynomial $p_n^\star$:

$$\|f-p_n^\star\|_{L^\infty(\bar{G})}=E_n(f,\bar{G}). \tag{10}$$

Then

$$\begin{aligned}\frac{1}{n}\sum_{m=n}^{2n-1}|f(z)-s_m(f,\bar{G},z)|\le&\frac{1}{n}\sum_{m=n}^{2n-1}|f(z)-p_n^\star(f,\bar{G},z)|+\\&+\frac{1}{n}\sum_{m=n}^{2n-1}|s_m(p_n^\star-f,\bar{G},z)|.\end{aligned}$$

Using (8) and (10) we get (9).

Lemma 6. ([6]). *Let G be any Jordan domain with analytic boundary and $f\in A_c(\bar{G})$. Then the r–th derivative $f^{(r)}$ of f, $r=0,1,\dots$ is Zygmund–continuous on $\bar{G}$ if and only if*

$$E_n(f,\bar{G})=O(n^{-r-1}),\quad n\ge 1.$$

We need the following corollary of Lemmas 5 and 6.

Lemma 7. *If G is a Jordan domain with analytic boundary, $f\in A_c(\bar{G})$ and $r\in\{0,1,2,\dots\}$. Then $f^{(r)}$ is Zygmund–continuous if and only if the strong de la Vallée–Poussin means of the Faber series of f converge to f in the order n^{-r-1} :*

$$\max_{z\in\bar{G}}\frac{1}{n}\sum_{m=n}^{2n-1}|f(z)-s_m(f,\bar{G},z)|=O(n^{-r-1})\quad (n\ge 1).$$

Now we can establish the

Prove Theorem 1. Since f is Zygmund–continuous, Lemma 7 states that

$$\frac{1}{n}\sum_{m=n}^{2n-1}|f(z)-s_m(f,\bar{G},z)|\le\frac{c}{n}.$$

Suppose that

$$2^{m_0-1} \le n < 2^{m_0},$$

then

$$\begin{aligned} &\frac{1}{n+1}\sum_{m=1}^{n} |f(z) - s_m(f,\bar{G},z)| \le \\ &\le \frac{1}{n+1}\sum_{m=1}^{m_0} 2^{m-1}\left(\frac{1}{2^{m-1}}\sum_{\ell=2^{m-1}}^{2^m-1} |f(z) - s_m(f,\bar{G},z)|\right) \le \\ &\le \frac{c}{n+1}\sum_{m=1}^{m_0} 2^{m-1}\frac{1}{2^{m-1}} = c\frac{m_0}{n+1} \le c\frac{\ln n}{n} \end{aligned}$$

as we asserted.

Now we present a Bohr type inequality with a consequence for Faber series. As it is well known ([7]), H. Bohr proved the following inequality

$$\max_{x\in\mathbf{R}}\left|\sum_{m=n+1}^{N}\frac{a_m}{im}e^{imx}\right| \le \frac{\pi}{2(n+1)}\max_{x\in\mathbf{R}}\left|\sum_{m=n+1}^{N}a_m e^{imx}\right|, \tag{11}$$

where $(a_m)_{m=n+1}^N$ are arbitrary complex numbers. The constant $\frac{\pi}{2(n+1)}$ is exact. We shall prove

Theorem 2. *Let G be any Jordan domain with smooth boundary satisfying (3). Then for arbitrary complex numbers $(a_m)_{n+1}^N$ we have*

$$\max_{z\in\bar{G}}\left|\sum_{m=n+1}^{N}\frac{a_m}{im}p_m(\bar{G},z)\right| \le c\max_{x\in\mathcal{R}}\left|\sum_{m=n+1}^{N}a_m e^{imx}\right|. \tag{12}$$

PROOF. From (5) we get

$$\begin{aligned} &\left|\sum_{m=n+1}^{N}\frac{a_m}{im}p_m(\bar{G},\psi(e^{ix}))\right| \le \left|\sum_{m=n+1}^{N}\frac{a_m}{im}e^{imx}\right| + \\ &+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\psi'(e^{it})}{\psi(e^{it})-\psi(e^{ix})} - \frac{1}{e^{it}-e^{ix}}\right|\cdot\left|\sum_{m=n+1}^{N}\frac{a_m}{im}e^{imt}\right|dt \quad (x\in\mathbf{R}) \end{aligned}$$

hence (4) implies

$$\left|\sum_{m=n+1}^{N}\frac{a_m}{im}p_m(\bar{G},\psi(e^{ix}))\right| \le c\left\|\sum_{m=n+1}^{N}\frac{a_m}{im}e^{imx}\right\|_{L^\infty(\mathbf{R})}.$$

Using the "ordinary" Bohr inequality and the maximum modulus principle, (12) follows.

Definition. Denote $A_{B^{(r)}}(\bar{G})$ the class of functions of the form

$$f(z) := a_0(f, \bar{G}) + (\beta_r \star g)(z)$$

where $g \in A_c(\bar{G})$,

$$(\beta_r \star g)(z) := \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \beta_r(t) \int_{\partial G} \frac{g(\psi(\varphi(\xi)e^{-it}))}{\xi - z}\, d\xi\, dt$$

and β_r denotes the Bernoulli–kernel:

$$\beta_r(t) := \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{e^{imt}}{(im)^r}.$$

This class was introduced by Dzjadyk [8, p.372]. Using the above notations we can write

$$A_{B^{(r)}}(\bar{G}) = \left\{\text{const.} + \beta_r \star g \;:\; g \in A_c(\bar{G})\right\}.$$

A theorem of DZJADYK [8] states that any $f \in A_{B^{(r)}}(\bar{G})$ has Faber series of the form

$$f(z) \sim a_0(f, \bar{G}) + \sum_{m=1}^{\infty} \frac{a_m(g, \bar{G})}{(im)^r} p_m(\bar{G}, z) \quad (z \in \bar{G}),$$

where, as usual

$$a_m(g, \bar{G}) := \frac{1}{2\pi i} \int_{|\tau|=1} \frac{g(\psi(\tau))}{\tau^{m+1}} d\tau.$$

It is easy to see that this series converge uniformly on $\partial\bar{G}$. We shall prove more:

Theorem 3. *Let G be any Jordan domain with smooth boundary satisfying (3), further let $f \in A_{B^{(r)}}(\bar{G})$, $r \geq 1$. Then*

$$\|f - s_n(f, \bar{G})\|_{L^\infty(\bar{G})} = \omega(g, \frac{1}{n})\, O\left(\frac{\ln n}{n^r}\right) \quad (n \geq 2). \tag{13}$$

PROOF. Apply once (12) and $r-1$ times the classical Bohr inequality to obtain

$$\left\| \sum_{m=n+1}^{N} \frac{a_m(g, \bar{G})}{(im)^r} p_m(\bar{G}, z) \right\|_{L^\infty(\bar{G})} \leq c(r) \left\| \sum_{m=n+1}^{N} a_m(g, \bar{G}) e^{imx} \right\|_{L^\infty(\mathbf{R})}.$$

Taking the limit $N \to \infty$ we get by

$$g^+(e^{ix}) - \sum_{m=0}^{n} a_m(g,\bar{G})e^{imx} = \lim_{N\to\infty} \sum_{m=n+1}^{\infty} a_m(g,\bar{G})e^{imx}$$

the estimate

$$\|f - s_n(f,\bar{G})\|_{L^\infty(\bar{G})} \le \frac{c(r)}{(n+1)^r} \max_{x\in\mathbf{R}} \left| g^+(e^{ix}) - \sum_{m=0}^{n} a_m(g,\bar{G})e^{imx} \right| .$$

From Lemma 2 it follows that

$$\max_{x\in\mathbf{R}} \left| g^+(e^{ix}) - \sum_{m=0}^{n} a_m(g.\bar{G})e^{imx} \right| = \omega\left(g, \frac{1}{n}\right) O(\ln n)$$

which gives the desired estimate.

Finally we shall prove

Theorem 4.

$$A_{B^{(1)}}(\bar{\mathbf{D}}) = \{f \,:\, f' \in A_c(\bar{\mathbf{D}})\} .$$

PROOF. Let $f \in A_{B^{(1)}}(\bar{\mathbf{D}})$, i.e.

$$f(z) = a_0(f,\bar{\mathbf{D}}) + \sum_{m=1}^{\infty} \frac{a_m(g,\bar{\mathbf{D}})}{im} z^m ,$$

where $g \in A_c(\bar{\mathbf{D}})$. We see that

$$f'(z) = -i \sum_{m=1}^{\infty} a_m(g,\bar{\mathbf{D}}) z^{m-1} ,$$

i.e.

$$zf'(z) = -i\left(g(z) - a_0(g,\bar{\mathbf{D}})\right) ,$$

hence $f'(z) \in A_c(\bar{\mathbf{D}})$. Suppose now that

$$f(z) = a_0(f,\bar{\mathbf{D}}) + \sum_{m=1}^{\infty} a_m(f,(\bar{\mathbf{D}})z^m$$

and

$$f'(z) = \sum_{m=1}^{\infty} m a_m(f,\bar{\mathbf{D}}) z^{m-1} \in A_c(\bar{\mathbf{D}}) .$$

Then

$$g(z) := izf'(z) = \sum_{m=1}^{\infty} ima_m(f,\bar{\mathbf{D}}) z^m \in A_c(\bar{\mathbf{D}}) ,$$

as we asserted.

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