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Extensions and ideals of rings

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Abstract. Being an ideal of a ring is not a transitive relation. For a ring A, consider a chain $J \triangleleft I \triangleleft A$. We examine conditions on J, I, A, I/J and A/I respectively which are necessary and sufficient for $J \triangleleft A$ to hold. The one–sided versions are also discussed.

1. Introduction and Preliminaries

Rings considered will not necessarily be associative and need not have an identity. Ideals, left ideals and right ideals will be denoted by $\triangleleft$, $\triangleleft_\ell$ and $\triangleleft_r$ respectively. For notational reasons, $A \triangleleft B$ will sometimes be denoted by $A \triangleleft_t B$. For a ring A, A^+ will denote the underlying group. The ring of integers will be denoted by Z. If A is a ring and $a \in A$, Za is the subgroup $Za = \{na \mid n \in Z\}$ of A^+. The Dorroh extensions of A (i.e. the usual embedding of A as an ideal in a ring with an identity) will be denoted by $D(A)$. This means $D(A)^+ = A^+ \oplus Z^+$ and the multiplication is given by

$$(a,n)(b,m) = (ab + ma + nb, nm).$$

We identify A with $\{(a,0) \mid a \in A\}$ and Z with $\{(0,n) \mid n \in Z\}$.

This research is motivated by the work of SANDS [7] and the following condition (F) which a class of rings $\mathcal{M}$ may satisfy and which is often encountered in the theory of (Kurosh–Amitsur) radicals of associative rings:

(F) $J \triangleleft I \triangleleft A$ and $I/J \in \mathcal{M}$ implies $J \triangleleft A$

(see, for example, [1], [2] and [5]). Recently SANDS [7] showed that in the variety of associative rings, a necessary and sufficient condition for (F) to hold is that all the rings in $\mathcal{M}$ should be quasi–semiprime (i.e. if $b \in A$ and $AbA = 0$, then $b = 0$). He also considered conditions on J, I, A and A/I respectively to ensure that $J \triangleleft A$. We extended these results to the not

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necessarily associative case and also investigate all the related conditions concerning one–sided ideals.

In the sequel, x, y and z will always be elements from the set $\{r, l, t\}$. A class of rings $\mathcal{M}$ is said to satisfy condition

$F(t, y, z)$ if $J \triangleleft I \triangleleft_y A$ and $I/J \in \mathcal{M}$ implies $J \triangleleft_z A$
$G(x, y, z)$ if $J \triangleleft_x I \triangleleft_y A$ and $J \in \mathcal{M}$ implies $J \triangleleft_z A$
$H(x, y, z)$ if $J \triangleleft_x I \triangleleft_y A$ and $I \in \mathcal{M}$ implies $J \triangleleft_z A$
$K(x, y, z)$ if $J \triangleleft_x I \triangleleft_y A$ and $A \in \mathcal{M}$ implies $J \triangleleft_z A$
$L(x, t, z)$ if $J \triangleleft_x I \triangleleft A$ and $A/I \in \mathcal{M}$ implies $J \triangleleft_z A$

If $X \in \{F, G, H, K, L\}$, then the following implications are obvious:

$$\begin{array}{ccccccc} & & & X(x,y,t) & & & \\ & & \swarrow & \downarrow & \searrow & & \\ X(x,y,r) & & & X(x,t,t) & & & X(x,y,l) \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & & X(x,t,r) & & X(x,t,l) & & \end{array}$$

Also $\mathcal{M}$ satisfies $X(x, y, t)$ if and only if $\mathcal{M}$ satisfies both $X(x, y, r)$ and $X(x, y, l)$.

Although the characterisations of conditions of the type $X(x, x, x)$ for $F \neq X \neq L$, $F(t, x, x)$ and $L(x, t, x)$ on a class of rings in the variety of associative rings have been settled by SANDS [7], we will restate them here (sometimes explicitly and sometimes as a special case of a more general result) for completeness and for the purpose of comparison.

2. On Condition $F(x, y, z)$

Using a construction of LEAVITT and VAN LEEUWEN [5], we start with

Lemma 2.1. *Let $R \neq 0$ be a ring. Then there exists a ring A and a chain $J \triangleleft I \triangleleft A$ with $I/J \simeq R$ and J is neither a left nor a right ideal of A (in fact, $AJ \cap JA \not\subseteq J$). The ring A is associative if R is associative and satisfies any one of the following three conditions:*
(1) $bR = 0 = Rb$ for some $0 \neq b \in R$
(2) $bR = 0 = R^2 b$ and $Rb \neq 0$ for some $0 \neq b \in R$
(3) $Rb = 0 = bR^2$ and $bR \neq 0$ for some $0 \neq b \in R$.

PROOF. Let $0 \neq b \in R$. Let A be the ring generated by the symbols u, v and the ring R over Z subject to $u^2 = v^2 = uc = cu = vc = cv = 0$

for all $c \in R$ and $uv = b = vu$. Let $J = \{nu \mid n \in Z\}$ and let $I = J + R$. Then $J \triangleleft I \triangleleft A$ and $I/J \simeq R$. Clearly $AJ \cap JA \not\subseteq J$.
Suppose R is associative. If (1) holds, then A as constructed above, is associative (this was the case in [5], Proposition 6). If (2) holds, choose $0 \neq eb \in Rb$. Construct the ring A as above, except let $uv = eb = vu$. Then A is an associative ring. If (3) holds, let $uv = be = vu$ where $0 \neq be \in bR$.

Proposition 2.2. *In the variety of all rings, a class $\mathcal{M}$ satisfies condition $F(t,y,z)$ if and only if $\mathcal{M} = \{0\}$.*

PROOF. If $0 \neq R \in \mathcal{M}$, then there is a ring A and a chain $J \triangleleft I \triangleleft A$ with $I/J \simeq R \in \mathcal{M}$, but J is not an ideal of A (Lemma 2.1).

Proposition 2.3. *In the variety of associative rings, a class of rings $\mathcal{M}$ satisfies condition $F(t,t,r)$ if and only if $b \in R \in \mathcal{M}$ and $bR = 0$ implies $b = 0$. $\mathcal{M}$ satisfies condition $F(t,t,\ell)$ if and only if $b \in R \in \mathcal{M}$ and $Rb = 0$ implies $b = 0$.*

PROOF. Suppose $\mathcal{M}$ satisfies condition $F(t,t,r)$ and let $b \in R \in \mathcal{M}$ with $bR = 0$ and $b \neq 0$. Lemma 2.1 (1) gives an associative ring A and a chain $J \triangleleft I \triangleleft A$ with $I/J \simeq R \in \mathcal{M}$ and J not a right ideal in A. This contradicts $F(t,t,r)$; hence $Rb \neq 0$. We now show $R^2 b = 0$. Let B be the following subset of the ring of 3×3 matrices over R:

$$B = \begin{bmatrix} R & R & R \\ Rb & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $bR = 0$, so is RbR and consequently B is a ring. Let

$$D = \begin{bmatrix} 0 & R & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and let } C = \begin{bmatrix} R & R & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $D \triangleleft C \triangleleft B$ and $C/D \simeq R \in \mathcal{M}$. By $F(t,t,r)$ we get $D \triangleleft_r B$. Hence, for $d, e \in R$

$$\begin{bmatrix} dey & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ ey & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in D, \quad \text{i.e.} \quad R^2 b = 0.$$

Using Lemma 2.1 (2), we once again obtain a contradiction; hence $b = 0$.

Conversely, assume every ring in $\mathcal{M}$ has zero left annihilator. Consider $J \triangleleft I \triangleleft A$ with $I/J \in \mathcal{M}$. Let $j \in J$ and $a \in A$. Then $ja + J \in I/J$ and for any $c \in I$, $(ja+J)(c+J) = (ja)c + J = j(ac) + J = 0$. By the assumption on the rings in $\mathcal{M}$, $ja + J = 0$, i.e. $ja \in J$ and $J \triangleleft_r A$ follows.
The proof of the second part, apart from taking the transposed of the above matrices, is similar.

We recall, an associative ring R is *quasi–semiprime* (cf. DE LA ROSA [4]) if $RbR = 0$ implies $b = 0$. This is easily seen to be equivalent to $bR = 0$ or $Rb = 0$ implies $b = 0$. Consequently we have

Corollary 2.4. (SANDS [7], Theorem 2). *A class of rings in the variety of associative rings satisfies condition $F(t,t,t)$ if and only if every ring in the class is a quasi-semiprime ring.*

Having considered the condition $bR = 0$ or $Rb = 0$ implies $b = 0$, one may ask for more information on $Rb = 0 = bR$ implies $b = 0$. This is given in

Proposition 2.5. *In the variety of associative rings, the following two conditions on a class of rings $\mathcal{M}$ are equivalent:*
(1) $b \in R \in \mathbf{M}$ and $Rb = 0 = bR$ implies $b = 0$.
(2) $J \triangleleft I \triangleleft A$ and $I/J \in \mathcal{M}$ implies J is a quasi-ideal in A (i.e. J is a subgroup of A and $AJ \cap JA \subseteq J$).

PROOF. Lemma 2.1 (1) gives (2) $\Rightarrow$ (1). Conversely, consider $J \triangleleft I \triangleleft A$ with $I/J \in \mathcal{M}$. Then

$$\frac{I}{J}\left(\frac{AJ \cap JA + J}{J}\right) = 0 = \left(\frac{AJ \cap JA + J}{J}\right)\frac{I}{J}$$

and by (1),

$$\frac{AJ \cap JA + J}{J} = 0, \text{ i.e. } AJ \cap JA \subseteq J.$$

For a wide range of classes $\mathcal{M}$, the above conditions coincide. Firstly, recall a class of rings $\mathcal{M}$ is *regular* if $0 \neq I \triangleleft A \in \mathcal{M}$, then there exists an ideal $J \triangleleft I$ such that $0 \neq I/J \in \mathcal{M}$. For example, any hereditary class is regular.

Corollary 2.6. *Let $\mathcal{M}$ be a regular class in the variety of associative rings. Then the following are equivalent :*
(1) $\mathcal{M}$ satisfies condition $F(t,t,t)$
(2) $\mathcal{M}$ consists of semiprime rings
(3) $\mathcal{M}$ consists of quasi-semiprime rings
(4) $b \in R \in \mathcal{M}$ and $Rb = 0 = bR$ implies $b = 0$.

PROOF. (1) $\Leftrightarrow$ (2) is well-known; (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is obvious. We show (4) $\Rightarrow$ (2) : Let $I \triangleleft A \in \mathcal{M}$ with $I^2 = 0$. If $I \neq 0$, there is an ideal $J \triangleleft I$ with $0 \neq I/J \in \mathcal{M}$ by the regularity of $\mathcal{M}$. Then $(I/J)^2 = 0$ and by (4) we have $I/J = 0$; a contradiction. Hence $I = 0$.

Finally, to complete this section, we have

Proposition 2.7. *For a class of rings $\mathcal{M}$ in the variety of associative rings, the following are equivalent:*
(1) $\mathcal{M}$ satisfies condition $F(t,y,z)$ with $y \neq t$

(2) $\mathcal{M} = \{0\}$.

PROOF. The same examples given by SANDS [7] to show the equivalence of the cases $F(t,r,r)$ and $F(t,l,l)$ will suffice: Let

$$A = \begin{bmatrix} R & R \\ D(R) & R \end{bmatrix},\ I = \begin{bmatrix} R & 0 \\ D(R) & 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} 0 & 0 \\ D(R) & 0 \end{bmatrix}.$$

Then $J \triangleleft I \triangleleft_\ell A$, $I/J \simeq R$ and $J \triangleleft_\ell A$ if and only if $J \triangleleft_r A$ if and only if $R = 0$. The transposed of the above takes care of the case $J \triangleleft I \triangleleft_r A$.

3. On Condition $G(x,y,z)$

For an associative ring R, PUCZYLOWSKI and SANDS (cf. SANDS [7]) constructed associative rings A_1 and A_2 with chains of ideals $J_i \triangleleft I_i \triangleleft A_i$ such that $J_i \simeq R$ and $J_1 \triangleleft_\ell A_1$ if and only if $R^2 = R$ if and only if $J_2 \triangleleft_r A_2$. These constructions, with the same properties bar the associativity, can be extended to the variety of all rings. In fact, the rings A_i are associative if and only if R is associative. Using these examples, the following result is obvious:

Proposition 3.1. *In the varieties of all rings or all associative rings, if a class of rings satisfies condition $G(x,y,z)$, then $R^2 = R$ for all rings in the class.*

The converse is not true for all choices of x, y and z. For completeness we state the known results:

Proposition 3.2. (SANDS [7]). *In the variety of associative rings, for $y \neq t$, the following conditions are equivalent for a class of rings $\mathcal{M}$:*
(1) $R^2 = R$ *for all* $R \in \mathcal{M}$
(2) $G(x,x,x)$
(3) $G(t,y,y)$
(4) $G(t,t,y)$
(5) $G(y,t,y)$

The remainder 18 conditions concerning $G(x,y,z)$ in the variety of associative rings are taken care of by:

Proposition 3.3. *In the variety of associative rings, a class of rings $\mathcal{M}$ satisfies any one of the conditions $G(x,y,z)$ not appearing in Proposition 3.2 if and only if $\mathcal{M} = \{0\}$.*

PROOF. In all cases, the necessity is obvious. Assuming any one of the mentioned conditions $G(x,y,z)$, we know from Proposition 3.1 that $R^2 = R$ for all $R \in \mathcal{M}$. Using the examples below as well as their transposures, straightforward and tedious verifications yield $R = 0$:

(1) $\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \triangleleft_\ell \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix} \triangleleft_r \begin{bmatrix} R & R \\ R & R \end{bmatrix}$ and $\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$ is a one–sided ideal in $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$ if and only if $R^2 = 0$.

(2) $\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \triangleleft \begin{bmatrix} D(R) & 0 \\ 0 & 0 \end{bmatrix} \triangleleft_\ell \begin{bmatrix} D(R) & D(R) \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \triangleleft_r \begin{bmatrix} D(R) & D(R) \\ 0 & 0 \end{bmatrix}$ if and only if $R = 0$.

(3) $\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \triangleleft_\ell \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix} \triangleleft \begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$ and $\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \triangleleft_r \begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$ if and only if $R^2 = 0$.

All conditions concerning $G(x, y, z)$ are quickly disposed of in the not necessarily associative ring case:

Proposition 3.4. *In the variety of all rings, a class of rings $\mathcal{M}$ satisfies condition $G(x, y, z)$ if and only if $\mathcal{M} = \{0\}$.*

PROOF. Assume $\mathcal{M}$ satisfies condition $G(x, y, z)$ and let $R \in \mathcal{M}$. By Proposition 3.1, we know $R^2 = R$. Let A be the ring with $A^+ = R^+ \oplus R^+ \oplus R^+$ and with (non–associative) multiplication defined by $(a, b, c)(d, e, f) = (ad, cd + fa, 0)$. Then

$$R \simeq \{(a, 0, 0) \mid a \in R\} \triangleleft \{(a, b, 0) \mid a, b \in R\} \triangleleft A$$

and $\{(a, 0, 0) \mid a \in R\}$ is an one–sided ideal of A if and only of $R^2 = 0$. Consequently $R = R^2 = 0$.

4. On Condition $H(x, y, z)$

For a ring I, $E(I_I)$ and $E({}_II)$ will be the rings of endomorphisms of the right I–module I_I and the left I–module ${}_II$ respectively. The elements of the former will be acting on the right and the elements of the latter on the left. A *double homothetism* of a ring I (cf. REDEI [6]) is a pair (λ, ρ) where $\lambda \in E(I_I)$ and $\rho \in E({}_II)$ such that for all $a, b \in I$, $a(\lambda b) = (a\rho)b$ and $(\lambda b)\rho = \lambda(b\rho)$.

Proposition 4.1. *Let $\mathcal{M}$ be a class of rings in the variety of all associative rings. Then $\mathcal{M}$ satisfies condition*
(1) $H(x, l, l)$ if and only if $\lambda J \subseteq J$ for all $J \triangleleft_x I \in \mathcal{M}$ and all $\lambda \in E(I_I)$
(2) $H(x, r, r)$ if and only if $J\rho \subseteq J$ for all $J \triangleleft_x I \in \mathcal{M}$and all $\rho \in E({}_II)$
(3) $H(x, t, l)$ if and only if $\lambda J \subseteq J$ for all $J \triangleleft_x I \in \mathcal{M}$ and all double homothetisms (λ, ρ) of I
(4) $H(x, t, r)$if and only if $J\rho \subseteq J$ for all $J \triangleleft_x I \in \mathcal{M}$ and all double homothetisms (λ, ρ) of I.

PROOF.

(1) Suppose $\mathcal{M}$ satisfies condition $H(x,\ell,\ell)$ and let $J \triangleleft_x I \in \mathcal{M}$. As in Sands [7], let A be the ring defined by $A^+ = I^+ \oplus E(I_I)^+$ with multiplication

$$(b_1,\lambda_1)(b_2,\lambda_2) = (b_1b_2 + \lambda_1 b_2, b_1\lambda_2 + \lambda_1 \circ \lambda_2)$$

where $b_1\lambda_2$ is the endomorphism of I_I defined by $(b_1\lambda_2)(c) = b_1\lambda_2(c)$ for all $c \in I$. Then A is an associative ring and $I \cong \{(b,0) \mid b \in I\} \triangleleft_\ell A$. By our assumption, $J \cong \{(j,0) \mid j \in J\} \triangleleft_\ell A$ holds; hence

$$(0,\lambda)(c,0) = (\lambda c, 0) \in \{(j,0) \mid j \in J\} \text{ for all } c \in J.$$

Thus, $\lambda J \subseteq J$ for all $\lambda \in E(I_I)$.

Conversely, consider $J \triangleleft_x I \triangleleft_\ell A$ with $I \in \mathcal{M}$. Let $a \in A$ and define $\lambda_a : I \to I$ by $\lambda_a(c) = ac$ for all $c \in I$. Then $\lambda_a \in E(I_I)$ and by the assumption, $aJ = \lambda_a(J) \subseteq J$, i.e. $J \triangleleft_\ell A$.

(2) is proved by the obvious left–right interchanges in (1).

(3) Assume condition $H(x,t,\ell)$ on $\mathcal{M}$. Let $J \triangleleft_x I \in \mathcal{M}$ and let (λ,ρ) be a double homothetism of I. Let B be any ring of double homothetisms of I which contains (λ,ρ) (such rings do exist – cf. Redei [6] or Sands [7]). Let A be the ring (cf. Sands [7]) defined by $A^+ = I^+ \oplus B^+$ and with multiplication

$$(b_1,(\lambda_1,\rho_1))(b_2,(\lambda_2,\rho_2)) = (b_1b_2 + \lambda_1 b_2 + b_1\rho_2, (\lambda_1 \circ \lambda_2, \rho_1 \circ \rho_2)).$$

Then A is an associative ring and

$$J \cong \{(j,(0,0)) \mid j \in J\} \triangleleft_x \{(b,(0,0)) \mid b \in I\} \cong I \triangleleft A.$$

By our assumption, $J \triangleleft_\ell A$ holds. Hence, for $c \in J$,

$$(\lambda c,(0,0)) = (0,(\lambda,\rho))(c,(0,0)) \in \{(j,(0,0)) \mid j \in J\},$$

i.e., $\lambda J \subseteq J$.

Conversely, consider $J \triangleleft_x I \triangleleft A$ with $I \in \mathcal{M}$. Let $a \in A$ and define $\lambda_a : I \to I$ by $\lambda_a(c) = ac$ and $\rho_a : I \to I$ by $(c)\rho_a = ca$ for all $c \in I$. Then (λ_a, ρ_a) is a double homothetism of I and by the assumption

$$aJ = \lambda_a(J) \subseteq J, \quad \text{i.e. } J \triangleleft_\ell A.$$

(4) is proved similarly.

Using Proposition 4.1 (3) and (4) we get

Corollary 4.2. (cf Rédei [6]). *A class of rings $\mathcal{M}$ in the variety of associative rings satisfies condition $H(x,t,t)$ if and only if $\lambda J \subseteq J$ and $J\rho \subseteq J$ for all $J \triangleleft_x I \in \mathcal{M}$ and all double homothetisms (λ, ρ) of I.*

The remaining 12 conditions in the variety of associative rings, can only occur in the trivial case:

Proposition 4.3. *Let $\mathcal{M}$ be a class of rings in the variety of associative rings. Then the following are equivalent:*
(1) $\mathcal{M}$ satisfies condition $H(x,\ell,z)$, $\quad z \neq \ell$
(2) $\mathcal{M}$ satisfies condition $H(x,r,z)$, $\quad z \neq r$
(3) $\mathcal{M} = \{0\}$.

Proof. (1) $\iff$ (3): Obviously $\mathcal{M} = \{0\} \Rightarrow \mathcal{M}$ satisfies $H(x,\ell,t) \Rightarrow \mathcal{M}$ satisfies $H(x,\ell,r)$. Assume condition $H(x,\ell,r)$ on $\mathcal{M}$ and let $R \in \mathcal{M}$. Then

$$\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \triangleleft \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \triangleleft_\ell \begin{bmatrix} D(R) & D(R) \\ 0 & 0 \end{bmatrix}$$

and by the assumption,

$$\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \triangleleft_r \begin{bmatrix} D(R) & D(R) \\ 0 & 0 \end{bmatrix}.$$

Hence $RD(R) = 0$, i.e. $R = 0$.
The equivalence of (2) and (3) is proved likewise.

Concerning the not necessarily associative case, firstly note that Proposition 4.3 holds also for this case. By $E(I^+)$ we denote the ring of endomorphisms of the group I^+.

Proposition 4.4. *Let $\mathcal{M}$ be a class of rings in the variety of all rings. Then the following conditions on $\mathcal{M}$ are equivalent:*
(1) $\lambda J \subseteq J$ for all $J \triangleleft_x I \in \mathcal{M}$ and all $\lambda \in E(I^+)$
(2) $H(x,y,y)$
(3) $H(x,t,z)$, $\quad z \neq t$.

Proof. Firstly note that if $\mathcal{M}$ satisfies condition $H(x,y,z)$, then $\lambda J \subseteq J$ for all $J \triangleleft_x I \in \mathcal{M}$ and all $\lambda \in E(I^+)$. Indeed, let A be the ring with $A^+ = I^+ \oplus E(I^+)^+$ and with multiplication defined by:

$$(b_1,\lambda_1)(b_2,\lambda_2) = (b_1 b_2 + \lambda_1 b_2 + \lambda_2 b_1, \lambda_1 \circ \lambda_2), \;\; b_i \in I,\, \lambda_i \in E(I^+).$$

Then $J \triangleleft_x I \cong \{(b,0) \mid b \in I\} \triangleleft A$ and by $H(x,y,z)$, for $z = r$ we get $(b,0)(0,\lambda) = (\lambda b, 0) \in \{(c,0) \mid c \in J\}$ for all $b \in J$, i.e. $\lambda J \subseteq J$. For $z = \ell$, we have $(0,\lambda)(b,0) = (\lambda b, 0) \in \{(c,0) \mid c \in J\}$ for all $b \in J$, i.e. $\lambda J \subseteq J$. Routine verifications, using the obvious endomorphisms in $E(I^+)$, show that (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

5. On condition $K(x, y, z)$

If $a \in A$ then the ideal (left, right ideal respectively) in A generated by a will be denoted by (a) ($(a)_\ell$, $(a)_r$ respectively). Recall, a ring A is a *left (right) duo ring* if every left ideal (right ideal respectively) is an ideal in A.

Lemma 5.1. *Let $\mathcal{M}$ be a class of rings.*
(1) *For $z \neq \ell$, if $\mathcal{M}$ satisfies condition $K(x, r, z)$ or $K(r, y, z)$, then every ring in $\mathcal{M}$ is a left duo ring.*
(2) *For $z \neq r$, if $\mathcal{M}$ satisfies condition $K(x, r, z)$ or $K(r, y, z)$, then every ring in $\mathcal{M}$ is a right duo ring.*

PROOF. Assume condition $K(x, \ell, z)$ on $\mathcal{M}$ and let $I \triangleleft_\ell A \in \mathcal{M}$. Then $I \triangleleft_x I \triangleleft_\ell A$ and $I \triangleleft A$ follows. If $\mathcal{M}$ satisfies condition $K(\ell, y, z)$ and $I \triangleleft_\ell A \in \mathcal{M}$, then $I \triangleleft_\ell A \triangleleft_y A$ and $I \triangleleft A$ follows.

Proposition 5.2. *Let $\mathcal{M}$ be a class of associative rings.*
(1) (SANDS [7], Theorem 4) *$\mathcal{M}$ satisfies condition $K(t, t, t)$ if and only if for all $a \in A \in \mathcal{M}$, $(a) = (a)^2 + Za$*
(2) *For $y \neq t$, the following conditions on $\mathcal{M}$ are equivalent:*
  (2.1) *$K(t, y, t)$*
  (2.2) *$K(y, t, t)$*
  (2.3) *$K(y, y, t)$*
  (2.4) *For all $a \in A \in \mathcal{M}$, $(a) = (a^2)_y + Za$.*
(3) *The following conditions on $\mathcal{M}$ are equivalent:*
  (3.1) *$K(\ell, r, t)$*
  (3.2) *$K(r, \ell, t)$*
  (3.3) *For all $a \in A \in \mathcal{M}$, $(a) = aAa + Za^2 + Za$.*

PROOF.
(2) We show the equivalences for $y = r$; the other case $y = \ell$ being similar.
  (2.1) $\Rightarrow$ (2.2). Consider $J \triangleleft_r I \triangleleft A \in \mathcal{M}$. Then $J \triangleleft J + JA \triangleleft_r A \in \mathcal{M}$ and from $K(t, r, t)$ we get $J \triangleleft A$.
  (2.2) $\Rightarrow$ (2.3). Consider $J \triangleleft_r I \triangleleft_r A \in \mathcal{M}$. By Lemma 5.1, A is a right duo ring; hence $I \triangleleft A$ and $J \triangleleft A$ follows from $K(r, t, t)$.
  (2.3) $\Rightarrow$ (2.4). Let $a \in A \in \mathcal{M}$. Then $(a^2)_r + Za \triangleleft_r (a)_r \triangleleft_r A$ and by $K(r, r, t)$ we have $(a^2) + Za \triangleleft A$. But $a \in (a^2)_r + Za \subseteq (a)$; hence $(a) = (a^2) + Za$.
  (2.4) $\Rightarrow$ (2.1). Consider $J \triangleleft I \triangleleft_r A \in \mathcal{M}$ and let $j \in J$ and $a \in A$. Then $ja \in (j) = (j^2)_r + Zj$. Without loss of generality, we may assume $ja = j^2b + kj^2 + nj$ for some $b \in A$, $k, n \in Z$. Since $j^2b = j(jb) \in J$, we get $ja \in J$. Likewise, $aj \in J$; hence $J \triangleleft A$.
(3) (3.1) $\Rightarrow$ (3.2). Consider $J \triangleleft_r I \triangleleft_\ell A \in \mathcal{M}$. Then $J \triangleleft_\ell JA + J \triangleleft_r A \in \mathcal{M}$ and by $K(\ell, r, t)$, $J \triangleleft A$ follows.

(3.2) $\Rightarrow$ (3.3). Let $a \in A \in \mathcal{M}$. Then $aA + Za^2 + Za \triangleleft_r (a)_r \triangleleft_\ell A$ and from $K(r, \ell, t)$, we get $aAa + Za^2 + Za \triangleleft A$. Since $a \in aAa + Za^2 + Za \subseteq (a)$, we have $(a) = aAa + Za^2 + Za$.

(3.3) $\Rightarrow$ (3.1). Consider $J \triangleleft_\ell I \triangleleft_r A \in \mathcal{M}$. Let $j \in J$, $a \in A$. Then $ja \in (j) = jAj + Zj^2 + Zj$. Hence $ja = jbj + kj^2 + nj$ for some $b \in A$, $k, n \in Z$. Since $jb \in I$, we have $jbj \in J$ and $ja \in J$ follows. Likewise $aj \in J$; thus $J \triangleleft A$.

Let us mention that examples of associative rings which has the property $K(t, t, t)$ were given by SZASZ [8, p. 197] and WIEGANDT [9, p. 300].

Proposition 5.3. *Let $\mathcal{M}$ be a class of associative rings.*

(1) (SANDS [7]) *$\mathcal{M}$ satisfies condition $K(r, r, r)$ if and only if for all $a \in A \in \mathcal{M}$, $(a)_r = (a^2)_r + Za$.*

(2) The following conditions on $\mathcal{M}$ are equivalent:

(2.1) $K(r, \ell, r)$

(2.2) $K(\ell, r, r)$

(2.3) For all $a \in A \in \mathcal{M}$, $(a)_r = aAa + Za^2 + Za$.

(3) The following conditions on $\mathcal{M}$ are equivalent:

(3.1) $K(t, \ell, r)$

(3.2) $K(\ell, t, r)$

(3.3) $K(\ell, \ell, r)$

(3.4) For all $a \in A \in \mathcal{M}$, $(a)_r \subseteq (a^2)_\ell + Za$.

(4) $\mathcal{M}$ satisfies condition $K(t, r, r)$ if and only if for all $a \in A \in \mathcal{M}$, $(a)_r = (a^2)_r + aAa + aAa^2A + Za$.

(5) $\mathcal{M}$ satisfies condition $K(r, t, r)$ if and only if for all $a \in A \in \mathcal{M}$, $(a)_r = (a^2)_r + aAa + aAaA + Za$.

(6) $\mathcal{M}$ satisfies condition $K(t, t, r)$ if and only if for all $a \in A \in \mathcal{M}$, $(a)_r \subseteq \langle a \rangle$ where $\langle a \rangle$ is the ideal in (a) generated by a.

PROOF.

(2) (2.1) $\Rightarrow$ (2.2). Consider $J \triangleleft_\ell I \triangleleft_r A \in \mathcal{M}$. Then $J \triangleleft_r AJ + J \triangleleft_\ell A$ and by $K(r, \ell, r)$, we get $J \triangleleft_r A$.

(2.2) $\Rightarrow$ (2.3). Let $a \in A \in \mathcal{M}$. Then $aAa + Za^2 + Za \triangleleft_\ell (a)_r \triangleleft_r A$ and by $K(\ell, r, r)$, $aAa + Za^2 + Za \triangleleft_r A$. Since $a \in aAa + Za^2 + Za \subseteq (a)_r$, we get $(a)_r = aAa + Za^2 + Za$.

(2.3) $\Rightarrow$ (2.1). Let $J \triangleleft_r I \triangleleft_\ell A \in \mathcal{M}$ and let $j \in J$, $a \in A$. Then $aj \in (j)_r = jAj + Zj^2 + Zj \subseteq J$; hence $J \triangleleft_r A$.

(3) (3.1) $\Rightarrow$ (3.2). If $J \triangleleft_\ell I \triangleleft A \in \mathcal{M}$, then $J \triangleleft AJ + J \triangleleft_\ell A$ and from $K(t, \ell, r)$ we get $J \triangleleft_r A$.

(3.2) $\Rightarrow$ (3.3). If $J \triangleleft_\ell I \triangleleft_\ell A \in \mathcal{M}$, $I \triangleleft A$ by Lemma 5.1. By $K(\ell, t, r)$ we get $J \triangleleft_r A$.

(3.3) $\Rightarrow$ (3.4). Let $a \in A \in \mathcal{M}$. Then $(a^2)_\ell + Za \triangleleft_\ell (a)_\ell \triangleleft_\ell A$. By $K(\ell, \ell, r)$ we have $(a^2)_\ell + Za \triangleleft_r A$ and from $a \in (a^2)_\ell + Za$ it follows that $(a)_r \subseteq (a^2)_\ell + Za$.

(3.4) $\Rightarrow$ (3.1). Let $J \triangleleft I \triangleleft_\ell A \in \mathcal{M}$ and let $j \in J$, $a \in A$. Then $ja \in (j)_r \subseteq (j^2)_\ell + Zj \subseteq J$; hence $J \triangleleft_r A$.

(4) Assume $K(t,r,r)$ on $\mathcal{M}$. Let $a \in A \in \mathcal{M}$. Then $(a^2)_r + aAa + aAa^2A + + Za \triangleleft (a)_r \triangleleft A$. By $K(t,r,r)(a^2)_r + aAa + aAa^2A + Za \triangleleft_r A$ from which the desired equality follows. The converse is easily seen to be valid, for if $J \triangleleft I \triangleleft_r A \in \mathcal{M}$, $j \in J$ and $a \in A$, then $ja \in (j)_r = (j^2)_r + jAj + + jAj^2A + Zj \subseteq J$.

(5) is proved similarly.

(6) The "if" part is obvious, so is the "only if" part when noting that if $J \triangleleft I \triangleleft A$, then $\langle j \rangle \subseteq j^2A + Aj^2 + jAj + (jA)^2 + (Aj)^2 + Aj^3A + + Aj(jA)^2 + (Aj^2jA + (Aj)^3) + Zj^2 + Zj \subseteq J$.

Proposition 5.3 has the obvious corresponding results for the cases $K(x,y,\ell)$ which we state for completeness:

Proposition 5.4. *Let $\mathcal{M}$ be a class of associative rings.*

(1) (Sands [7]) *$\mathcal{M}$ satisfies condition $K(\ell,\ell,\ell)$ if and only if for all $a \in A \in \mathcal{M}$, $(a)_\ell = (a^2)_\ell + Za$.*

(2) The following conditions on $\mathcal{M}$ are equivalent:

(2.1) $K(\ell,r,\ell)$

(2.2) $K(r,\ell,\ell)$

(2.3) For all $a \in A \in \mathcal{M}$, $(a)_\ell = aAa + Za^2 + Za$.

(3) The following conditions on $\mathcal{M}$ are equivalent:

(3.1) $K(t,r,\ell)$

(3.2) $K(r,t,\ell)$

(3.3) $K(r,r,\ell)$

(3.4) For all $a \in A \in \mathcal{M}$, $(a)_\ell \subseteq (a^2)_r + Za$.

(4) $\mathcal{M}$ satisfies condition $K(t,\ell,\ell)$ if and only if for all $a \in A \in \mathcal{M}$, $(a)_\ell = (a^2)_\ell + aAa + Aa^2Aa + Za$.

(5) $\mathcal{M}$ satisfies condition $K(\ell,t,\ell)$ if and only if for all $a \in A \in \mathcal{M}$, $(a)_\ell = (a^2)_\ell + aAa + AaAa + Za$.

(6) $\mathcal{M}$ satisfies condition $K(t,t,\ell)$ if and only if for all $a \in A \in \mathcal{M}$, $(a)_\ell \subseteq \langle a \rangle$ where $\langle a \rangle$ is the ideal in (a) by a.

For the not necessarily associative case, we only consider the one condition $K(t,t,t)$. As ususal, for a ring B the powers $B^{(n)}$ are defined inductively by $B^{(0)} = B$ and for $n > 0$, $B^{(n)} = B^{(n-1)} \cdot B^{(n-1)}$. Let us call a ring A an *Andrunakievič ring* if there is some $n \geq 0$ such that for every chain $J \triangleleft I \triangleleft A$, $\overline{J}^{(n)} \subseteq J$ (where $\overline{J}$ is the ideal in A generated by J). The smallest such n for an Andrunakievič ring A will be called the *index* of A. Every associate ring is an Andrunakievič ring of index at most 2 (cf. Andrunakievič [3]).
Corresponding to Proposition 5.2 (1), we have:

Proposition 5.5. *A class of rings $\mathcal{M}$ satisfies condition $K(t,t,t)$ if and only if every ring A in $\mathcal{M}$ is an Andrunakievič ring and if the index of A is n, then*

$$(a) = (a)^{(k)} + Za \text{ for all } a \in A \text{ where } k = \max\{n-1; 1\}.$$

PROOF. Suppose $\mathcal{M}$ satisfies condition $K(t,t,t)$ and let $A \in \mathcal{M}$. Then $J \triangleleft I \triangleleft A$ implies $J \triangleleft A$; hence A is an Andrunakievič ring with index 0. Then $k = 1$ and from $(a)^{(1)} + Za \triangleleft (a) \triangleleft A$ we get $(a) = (a)^{(1)} + Za$.

Conversely let $A \in \mathcal{M}$ be an Andrunakievič ring with index n and let $k = \max\{n-1; 1\}$. Consider $J \triangleleft I \triangleleft A$. Then $\overline{J}^{(n)} \subseteq J$ and $\overline{J} \subseteq I$. We now show that $J + \overline{J}^{(k)} \triangleleft A$. Indeed, if $b \in J + \overline{J}^{(k)}$ and $a \in A$, then $ab \in (b) = (b)^{(k)} + Zb$. Since $J + \overline{J}^{(k)} \subseteq J, (b) \subseteq \overline{J}$ and thus $(b)^k \subseteq \overline{J}^{(k)}$. Hence $ab \in J + \overline{J}^{(k)}$. Likewise $ba \in J + \overline{J}^{(k)}$; hence $J + \overline{J}^{(k)} \triangleleft A$. But $J \subseteq J + \overline{J}^{(k)}$; hence $J = J + \overline{J}^{(k)}$. Then $\overline{J}^{(k)} \subseteq \overline{J}^{(1)} = [J + \overline{J}^{(k)}]^{(1)} \subseteq$ $\subseteq J + \overline{J}^{(k+1)} \subseteq J$ since $\overline{J}^{(k+1)} \subseteq J$. Hence $J = J + \overline{J}^{(k)} \triangleleft A$.

6. On condition $L(x,y,z)$

These conditions, both for the associative and not necessarily associative ring cases are easily taken care of by the examples form SANDS [7] who proved the result for the cases $L(x,t,x)$.

Proposition 6.1. *Let $\mathcal{M}$ be a class of rings in the variety of all rings or all associative rings. Then $\mathcal{M}$ satisfies condition $L(x,t,z)$ if and only if $\mathcal{M} = \{0\}$.*

PROOF. Let $0 \neq R \in \mathcal{M}$. Then

$$\begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix} \triangleleft \begin{bmatrix} 0 & D(R) \\ 0 & 0 \end{bmatrix} \triangleleft \begin{bmatrix} R & D(R) \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix} \triangleleft_\ell \begin{bmatrix} R & D(R) \\ 0 & 0 \end{bmatrix}$$

if and only if $R = 0$.
Transposing, we have

$$\begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix} \triangleleft \begin{bmatrix} 0 & 0 \\ D(R) & 0 \end{bmatrix} \triangleleft \begin{bmatrix} R & 0 \\ D(R) & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix} \triangleleft_r \begin{bmatrix} R & 0 \\ D(R) & 0 \end{bmatrix}$$

if and only if $R = 0$.

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