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Continuous orthogonality spaces

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To the memory of Professor Béla Barna

Introduction

In this note we take further steps towards a more complete description of orthogonally additive mappings. It is known (c.f. [10]) for a real orthogonality space $(\boldsymbol{X}, \perp)$ in the RÄTZ sense (c.f. [6]) that assuming $\dim \boldsymbol{X} \geq 3$, there can exist a non–trivial even orthogonally additive mapping on $\boldsymbol{X}$ with values in an abelian group $(\boldsymbol{Y}, +)$ only if, $\boldsymbol{X}$ is an inner product space with the ordinary orthogonality $\perp$. This means that in this context the only problems left open are in the 2–dimensional case. Here we provide another approach to this case. Namely, we strenghten the crucial axiom (O4') of $\perp$ introducing the notion of a *continuous* orthogonality space. This concept will prove to be fruitful from two points of view. On the one hand, most of the known examples of orthogonality spaces are "continuous", and on the other hand we can prove more easily our above mentioned result, and what is more, also for $\dim \boldsymbol{X} = 2$. As a consequence, we reprove for the Birkhoff–James orthogonality (c.f. [3], [4]) on a real normed vector space the earlier results obtained under regularity conditions (c.f. [2], [8]), for $\dim \boldsymbol{X} \geq 3$ (c.f. [5]) or by complicated methods in the general case (c.f. [9]).

Throughout the paper, $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Q}$, $\mathbf{N}$ denote the set of real, nonnegative real, rational numbers and positive integers, respectively. Furthermore, $\operatorname{lin} V$ stands for the linear hull of a subset $V \subset \boldsymbol{X}$. The constant mapping with value c is denoted by $\underline{c}$. Finally, we use 0 for the zero vector, for the number zero and for the identity element of the group $\boldsymbol{Y}$, as well.

1. Preliminaries

Definition 1.1. (RÄTZ [6], Def. 1) Let $\boldsymbol{X}$ be a real vector space of dimension ≥ 2 and $\perp$ be a binary relation on $\boldsymbol{X}$ with the following properties:

(O1) *total for zero:* $x \perp 0,\ 0 \perp x$ for all $x \in \boldsymbol{X}$;
(O2) *independent:* if $x, y \in \boldsymbol{X} \setminus \{0\},\ x \perp y$, then x and y are linearly independent;
(O3) *homogeneous:* if $x, y \in \boldsymbol{X},\ x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbf{R}$;
(O4') *thalesian:* if P is a 2–dimensional linear subspace of $\boldsymbol{X}$, $x \in P$ and $\lambda \in \mathbf{R}_+$, then there exists $y \in P$ such that $x \perp y$ and $(x+y) \perp (\lambda x - y)$.

Then $\perp$ is said to be an *orthogonality relation* on $\boldsymbol{X}$ and $(\boldsymbol{X}, \perp)$ is called an *orthogonality space.*

Definition 1.2. Let $(\boldsymbol{X}, \perp)$ be an orthogonality space and $(\boldsymbol{Y}, +)$ an abelian group. The mappings A, Q, $F : \boldsymbol{X} \to \boldsymbol{Y}$ are said to be *additive, quadratic,* or *orthogonally additive,* respectively, if they satisfy the Cauchy–, the Jordan–von Neumann– or the conditional Cauchy functional equations:

$$A(x+y) = A(x) + A(y) \quad \text{for all } x, y \in \boldsymbol{X}, \tag{1.1}$$

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \quad \text{for all } x, y \in \boldsymbol{X}, \tag{1.2}$$

$$F(x+y) = F(x) + F(y) \quad \text{for all } x, y \in \boldsymbol{X} \text{ with } x \perp y. \tag{1.3}$$

Throughout this section $\boldsymbol{X}$ and $\boldsymbol{Y}$ will be an orthogonality space and an abelian group, respectively, and we shall use the following notations:

$$\begin{aligned}
\operatorname{Hom}(\boldsymbol{X}, \boldsymbol{Y}) &= \{A : \boldsymbol{X} \to \boldsymbol{Y} \mid A \text{ is additive}\},\\
\operatorname{Quad}(\boldsymbol{X}, \boldsymbol{Y}) &= \{Q : \boldsymbol{X} \to \boldsymbol{Y} \mid Q \text{ is quadratic}\},\\
\operatorname{Hom}_\perp(\boldsymbol{X}, \boldsymbol{Y}) &= \{F : \boldsymbol{X} \to \boldsymbol{Y} \mid F \text{ is orthogonally additive}\},\\
(\mathrm{o})\operatorname{Hom}_\perp(\boldsymbol{X}, \boldsymbol{Y}) &= \{D : \boldsymbol{X} \to \boldsymbol{Y} \mid D \text{ is odd orthogonally additive}\},\\
(\mathrm{e})\operatorname{Hom}_\perp(\boldsymbol{X}, \boldsymbol{Y}) &= \{E : \boldsymbol{X} \to \boldsymbol{Y} \mid E \text{ is even orthogonally additive}\}.
\end{aligned}$$

Theorem 1.3. (ACZÉL [1], Thm. 2) *For any $Q \in \operatorname{Quad}(\boldsymbol{X}, \boldsymbol{Y})$ there exists a function $B : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{Y}$ which is*
(i) symmetric: *$B(x,y) = B(y,x)$ for all $x, y \in \boldsymbol{X}$;*
(ii) biadditive: *$B(x, y+z) = B(x,y) + B(x,z)$ for all $x, y, z \in \boldsymbol{X}$;*
(iii) representative: *$4Q(x) = B(x,x)$ for all $x \in \boldsymbol{X}$.*
The mapping B is uniquely determined by Q and we call it the biadditive representation *of $4Q$.*

Lemma 1.4. (SZABÓ [10], Lemma 2.6) *Let $Q \in \operatorname{Quad}(\boldsymbol{X}, \boldsymbol{Y})$ and let B be the biadditive representation of $E = 4Q$. Then E is even and*
(i) $E \in \operatorname{Hom}_\perp(\boldsymbol{X}, \boldsymbol{Y}) \iff 2B(x,y) = 0$ for all $x, y \in \boldsymbol{X},\ x \perp y$;
(ii) $E \in \operatorname{Hom}_\perp(\boldsymbol{X}, \boldsymbol{Y}) \implies E(\lambda u) = E(\lambda v)$ for any $u, v \in \boldsymbol{X}$ such that $(u+v) \perp (u-v)$ and every $\lambda \in \mathbf{R}$;

(iii) $E(\lambda u) = E(\lambda v)$ *for some* $u, v \in \mathbf{X}$ *and every* $\lambda \in \mathbf{R} \Longrightarrow B(\lambda u, \mu u) = B(\lambda v, \mu v)$ *for all* $\lambda, \mu \in \mathbf{R}$.

Theorem 1.5. (RÄTZ [6], Thms. 5,6; SZABÓ [9], Thm. 1.8) *For any orthogonality space* $(\mathbf{X}, \perp)$ *and any abelian group* $(\mathbf{Y}, +)$, *we have*

(i) $(\mathrm{o})\,\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) = \mathrm{Hom}(\mathbf{X}, \mathbf{Y})$;

(ii) $(\mathrm{e})\,\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) \subset \mathrm{Quad}(\mathbf{X}, \mathbf{Y})$;

(iii) $\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) = \mathrm{Hom}(\mathbf{X}, \mathbf{Y}) \Longleftrightarrow (\mathrm{e})\,\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) = \{\underline{0}\}$.

Lemma 1.6. (SZABÓ, [10], Lemma 3.1) *If* $E \in (\mathrm{e})\,\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) \setminus \{\underline{0}\}$, *then* $2E \in (\mathrm{e})\,\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) \setminus \{\underline{0}\}$, *too.*

Definition 1.7. Let L be a real vector space of dimension ≥ 2. The binary relation $\vdash$ on L is said to be

(i) *symmetric,* if $x, y \in L,\ x \vdash y \Longrightarrow y \vdash x$;

(ii) *right additive,* if $x, y, z \in L,\ x \vdash y,\ x \vdash z \Longrightarrow x \vdash (y + z)$;

(iii) *right homogeneous,* if $x, y \in L, x \vdash y \Longrightarrow x \vdash \beta y$ for all $\beta \in \mathbf{R}$;

(iv) *right projective,* if $x, y \in L \Longrightarrow$ there is $\alpha \in \mathbf{R}$ with $x \vdash (y - \alpha x)$;

(v) *right unique,* if $x, y \in L,\ x \neq 0 \Longrightarrow$ there exists at most one $\alpha \in \mathbf{R}$ such that $x \vdash (y - \alpha x)$.

Analogously, one can define the corresponding "left sided" properties of $\vdash$. In the symmetric case the "left" and "right" attributes are omitted.

Remark 1.8. One can readily see, that any orthogonality relation $\perp$ on $\mathbf{X}$ is right projective. Also, an equivalent formulation can be given for the right uniqueness of $\perp$ as follows: if $u, v \in \mathbf{X} \setminus \{0\}$, $u \perp v$ and $y \in \mathrm{lin}\{u, v\}$, $u \perp y$, then $y = \beta v$ with some $\beta \in \mathbf{R}$. As to the other properties of $\perp$, we have

Theorem 1.9. (RÄTZ [7], Thm. 2.3, SZABÓ [10], Cor. 3.4) *If* $(\mathrm{e})\,\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) \neq \{\underline{0}\}$, *then the orthogonality* $\perp$ *is*

(i) symmetric;

(ii) additive;

(iii) unique.

Lemma 1.10. *Suppose that* $(\mathrm{e})\,\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) \neq \{\underline{0}\}$ *and* $P \subset S \subset \mathbf{X}$ *are linear subspaces with* $\dim P = 2$, $\dim S = 3$. *If* $u \in P \setminus \{0\}$, *then there exists vectors* $v \in P \setminus \{0\}$ *and* $w \in S \setminus P$ *such that* $u \perp v$, $u \perp w$ *and* $v \perp w$.

PROOF. By axiom (O4') there exists a vector $v \in P$ with $u \perp v$, $(u+v) \perp (u-v)$ and axiom (O2) ensures that $v \neq 0$. Now choose $x \in S \setminus P$ and $y \in \mathrm{lin}\{v, x\} \setminus \{0\}$ such that $v \perp y$. Finally, let $w \in \mathrm{lin}\{u, y\}$ with $u \perp w$, $(u + w) \perp (u - w)$, whence again $w \neq 0$. Since $w = \alpha u + \beta y$, the symmetry, additivity and homogeneity of $\perp$ implies that $v \perp w$.

Lemma 1.11. (Szabó [10], Lemma 3.2) *Suppose that B is the biadditive representation of a mapping $E \in (\mathrm{e})\operatorname{Hom}_{\perp}(\mathbf{X}, \mathbf{Y}) \setminus \{\underline{0}\}$. If $x, y \in \mathbf{X}$ are linearly independent and $2B(\lambda x, \mu y) = 0$ for all $\lambda, \mu \in \mathbf{R}$, then $x \perp y$.*

Corollary 1.12. (Szabó [10], Cor. 3.3) *If $(\mathrm{e})\operatorname{Hom}_{\perp}(\mathbf{X}, \mathbf{Y}) \neq \{\underline{0}\}$ and $x, y \in \mathbf{X}$ are such that $x \perp y$, $(\alpha x + y) \perp (\beta x - y)$ with some $\alpha, \beta \in \mathbf{R}$, then $(\alpha x - y) \perp (\beta x + y)$ holds, too.*

Corollary 1.13. *If $u, v \in \mathbf{X} \setminus \{0\}$, $u \perp v$ and there exists $E \in (\mathrm{e})\operatorname{Hom}_{\perp}(\mathbf{X}, \mathbf{Y}) \setminus \{\underline{0}\}$ such that $E(\lambda u) = E(\lambda v)$ for all $\lambda \in \mathbf{R}$, then $(u + v) \perp (u - v)$.*

Proof. Let B be the biadditive representation of $4E \neq \underline{0}$. Then by Lemma 1.4, part (i) and (iii), we have for all $\lambda, \mu \in \mathbf{R}$ that

$$\begin{gathered} 2B(\lambda[u+v], \mu[u-v]) = \\ = 2B(\lambda u, \mu u) - 2B(\lambda u, \mu v) + 2B(\lambda v, \mu u) - 2B(\lambda v, \mu v) = 0. \end{gathered}$$

Since clearly $u + v$ and $u - v$ are linearly independent, Lemma 1.11 implies $(u + v) \perp (u - v)$.

Corollary 1.14. *Suppose that $(\mathrm{e})\operatorname{Hom}_{\perp}(\mathbf{X}, \mathbf{Y}) \neq \{\underline{0}\}$. Let P be a 2–dimensional linear subspace of $\mathbf{X}$ and $u, v \in P \setminus \{0\}$ such that $u \perp v$ and $(u + v) \perp (u - v)$. If $x = \xi(mu + nv)$, $y = \alpha u + \beta v \in P$ with some $m, n \in \mathbf{N}$, $\xi, \alpha, \beta \in \mathbf{R}$, $\xi \neq 0$, then*

$$x \perp y \iff m\alpha + n\beta = 0.$$

Proof. Let B be the biadditive representation of a mapping $E \in (\mathrm{e})\operatorname{Hom}_{\perp}(\mathbf{X}, \mathbf{Y}) \setminus \{\underline{0}\}$, and $z = nu - mv$. By (O2) u and v are linearly independent, whence one can derive easily the linear independency of x and z, too.

Sufficiency. According to Lemma 1.4, for all $\lambda, \mu \in \mathbf{R}$ it follows that

$$\begin{aligned} 2B(\lambda x, \mu z) &= 2B(\lambda\xi[mu + nv], \mu[nu - mv]) = \\ &= 2B(\lambda\xi mu, \mu nu) - 2B(\lambda\xi mu, \mu mv) + 2B(\lambda\xi nv, \mu nu) - \\ &\quad - 2B(\lambda\xi nv, \mu mv) = 2mnB(\lambda\xi u, \mu u) - 2nmB(\lambda\xi v, \mu v) = 0. \end{aligned}$$

Thus Lemma 1.11 implies that $x \perp z$. Since $m\alpha + n\beta = 0$, we have

$$y = \alpha u - \frac{m}{n}\alpha v = \frac{\alpha}{n}(nu - mv) = \frac{\alpha}{n} z,$$

and so, by the homogeneity of $\perp$, $x \perp y$ follows.

Neccessity. Because of the uniqueness of $\perp$ and $x \perp z$, $x \perp y$, we have $y = \eta z$ (see Remark 1.8 above).

2. Continuous orthogonality spaces

Definition 2.1. The real vector space $\mathbf{X}$ of dimension ≥ 2 is said to be a *continuous orthogonality space,* if it is equipped with a *continuous orthogonality relation* $\perp$, i.e. one having axioms (O1), (O2), (O3) and being

(O4") *continuously thalesian:* if $P \subset \mathbf{X}$ is a 2-dimensional linear subspace and $x \in P$, then there exist $y \in P$ with $x \perp y$ and continuous functions $\xi, \eta : [-1,1] \to \mathbf{R}$ such that $\xi(\pm 1) = \pm 1$ and

$$(x + [\xi(\tau)x + \eta(\tau)y]) \perp (x - [\xi(\tau)x + \eta(\tau)y]) \quad \text{for all } \tau \in [-1,1].$$

Proposition 2.2. *Every continuous orthogonality space* $(\mathbf{X}, \perp)$ *is an ordinary orthogonality space as well, or more precisely*

$$\text{(O3) and (O4")} \Longrightarrow \text{(O4')}.$$

PROOF. Let $P \subset \mathbf{X}$ be a 2-dimensional linear subspace, $x \in P$ and $\lambda \in \mathbf{R}_+$. Then applying axiom (O4") for $(1+\lambda)x$, we obtain $y \in P$ and continuous functions ξ, η on $[-1,1]$ with the above properties. Since $-1 \leq (1-\lambda)/(1+\lambda) \leq 1$, we can choose $\tau_\lambda \in [-1,1]$ such that $\xi(\tau_\lambda) = (1-\lambda)/(1+\lambda)$. Thus axiom (O4") implies that

$$2\left(x + \frac{\eta(\tau_\lambda)}{2}y\right) = \left((1+\lambda)x + \left[\frac{1-\lambda}{1+\lambda}(1+\lambda)x + \eta(\tau_\lambda)y\right]\right) \perp$$
$$\perp \left((1+\lambda)x - \left[\frac{1-\lambda}{1+\lambda}(1+\lambda)x + \eta(\tau_\lambda)y\right]\right) = 2\left(\lambda x - \frac{\eta(\tau_\lambda)}{2}y\right).$$

Thus, by the homogeneity of $\perp$, (O4') holds with $[\eta(\tau_\lambda)/2]y$.

Next we give several examples of continuous orthogonality spaces.

Example 2.3. (RÄTZ [6], Exa. A) The *trivial orthogonality* on $\mathbf{X}$ defined by (O1) and for $x, y \in \mathbf{X} \setminus \{0\}$ $x \perp y \iff x, y$ are linearly independent. Here (O4") is satisfied e.g. with $\xi = \mathrm{Id}_{[-1,1]}$ and an arbitrary positive continuous η.

Example 2.4. (RÄTZ [6], Exa. B) The *ordinary orthogonality* on an inner product space $(\mathbf{X}, \langle \cdot, \cdot \rangle)$ defined by $x \perp y \iff \langle x, y \rangle = 0$. Here the continuous functions ξ, η for (O4") can be chosen always to be $\xi = \mathrm{Id}_{[-1,1]}$ and $\eta : \tau \to \sqrt{1-\tau^2}$.

Example 2.5. (RÄTZ [6], Exa. C) The *Birkhoff–James orthogonality* on a normed linear space $(\mathbf{X}, \|\cdot\|)$ defined by $x \perp y \iff \|x + \beta y\| \geq \|x\|$ for all $\beta \in \mathbf{R}$. The crucial axiom (O4") can be checked in several steps by a method used in [9] for proving Lemma 2.1 and Theorems 2.2, 2.3, as follows:

i) Let $P \subset \mathbf{X}$ be a linear subspace, $\dim P = 2$ and $x \in P$. Since $\perp$ satisfies (O4'), there is a vector $y \in P$ with $\|y\| = \|x\|$ and $x \perp y$. For $x = 0$ the functions $\xi = \mathrm{Id}_{[-1,1]}$ and $\eta = \underset{\cdot}{0}$ work, thus we may assume that $x \neq 0$. Let the continuous function $K :]-1,1[\times \mathbf{R} \times \mathbf{R} \to \mathbf{R}_+$ be defined by

$$K(\tau, \gamma, \alpha) = \|(2-\alpha)x + \alpha(\tau x + \gamma y)\|.$$

and consider for all $\tau \in]-1,1[$ the sets

$$\Gamma_*(\tau) = \{\gamma \mid K(\tau,\gamma,\alpha) \geq K(\tau,\gamma,1) \text{ for all } \alpha \leq 1\},$$
$$\Gamma^*(\tau) = \{\gamma \mid K(\tau,\gamma,\alpha) \geq K(\tau,\gamma,1) \text{ for all } \alpha \geq 1\}.$$

Using the convexity of the functions $\alpha \to K(\tau,\gamma,\alpha)$ and the limits $\lim_{\alpha\to\pm\infty} K(\tau,\gamma,\alpha) = +\infty$ for all $\tau \in]-1,1[,\ \gamma \in \mathbf{R}$, one can verify along the lines of proof of [9], Lemma 2.1 that for every $\tau \in]-1,1[$

— $0 \in \Gamma_*(\tau),\ 4 \in \Gamma^*(\tau)$ are closed subsets of $\mathbf{R}$;

— $\Gamma_*(\tau) \cup \Gamma^*(\tau) = \mathbf{R}$;

— $\gamma_* \in \Gamma_*(\tau)$ and $\gamma^* \in \Gamma^*(\tau) \Longrightarrow \gamma_* \leq \gamma^*$.

ii) Let now the function $\gamma :]-1,1[\to [0,4]$ be well defined by

$$\gamma(\tau) = \sup \Gamma_*(\tau) = \inf \Gamma^*(\tau) \quad \text{for all } \tau \in]-1,1[\ .$$

The continuity of γ can be proved again by a technique used for [9], Theorem 2.2. In the sequel, we show that the limits $\lim_{\tau\to\pm1}\gamma(\tau)$ also exist. Consider e.g. the case $\tau \nearrow 1$.

We need the following implication: if $0 < \tau_1 < \tau_2 < 1$ and $\gamma_1 \in \Gamma^*(\tau_1)$, then $\gamma_2 = \gamma_1(1+\tau_2)/(1+\tau_1) \in \Gamma^*(\tau_2)$. Indeed, for any $\alpha_2 \geq 1$ we have

$$(2-\alpha_2)x + \alpha_2(\tau_2 x + \gamma_2 y) = \lambda[(2-\alpha_1)x + \alpha_1(\tau_1 x + \gamma_1 y)]$$

with scalars $\lambda = 1 + \alpha_2(\tau_2 - \tau_1)/(1+\tau_1) \geq (1+\tau_2)/(1+\tau_1)$ and $\alpha_1 = [\alpha_2(1+\tau_2)/(1+\tau_1)]/\lambda \geq 1$. Then

$$K(\tau_2,\gamma_2,\alpha_2) = \lambda K(\tau_1,\gamma_1,\alpha_1) \geq \frac{1+\tau_2}{1+\tau_1} K(\tau_1,\gamma_1,1) = K(\tau_2,\gamma_2,1),$$

i.e. $\gamma_2 \in \Gamma^*(\tau_2)$. In particular, $\gamma(\tau_2) \leq 2\gamma(\tau_1)/(1+\tau_1)$.

Now consider a strictly monotone increasing sequence of positive numbers τ_n converging to 1 with $\lim_{n\to\infty}\gamma(\tau_n) = \liminf_{\tau\nearrow1}\gamma(\tau)$.

Then by the above implication we have $\gamma(\tau) \leq 2\gamma(\tau_n)/(1+\tau_n)$ for all $n \in \mathbf{N}$, $\tau_n < \tau < 1$. Thus

$$\limsup_{\tau \nearrow 1} \gamma(\tau) \leq \lim_{n\to\infty} 2\gamma(\tau_n)/(1+\tau_n) = \liminf_{\tau \nearrow 1} \gamma(\tau),$$

what was to be proved.

The case $\tau \searrow -1$ can be verified in an analogous way and these limits make it possible to extend γ continuously to $[-1,1]$ by $\gamma(\pm 1) = \lim_{\tau\to\pm 1} \gamma(\tau)$.

iii) Finally, (O4") will be satisfied with the continuous functions $\xi = \mathrm{Id}_{[-1,1]}$ and $\eta = \gamma$. To check this, one should only consider the definition of $\perp$ and the function K (see also [9], Thm. 2.3). We remark that with a fixed y the functions ξ and η are essentially unique, i.e. $-1 \leq \xi(\tau) \leq 1$ and $\eta(\tau)$ is uniquely determined by $\xi(\tau)$ for all $\tau \in [-1,1]$.

Example 2.6. Let $\mathbf{X} = \mathbf{R}^2$ and $r : \mathbf{R} \to \mathbf{R}$ be a positive, π–periodic and continuously differentiable function. Define the *orthogonality* on $\mathbf{X}$ by $x \perp y \iff$

$$x = \lambda(r(\varphi)\cos\varphi; r(\varphi)\sin\varphi),$$
$$y = \mu(r'(\varphi)\cos\varphi - r(\varphi)\sin\varphi; r'(\varphi)\sin\varphi + r(\varphi)\cos\varphi)$$

for some λ, μ, $\varphi \in \mathbf{R}$. The axioms (O1), (O2), and (O3) are clearly fulfilled and the axiom (O4") also holds for $x = \lambda_x(r(\varphi_x)\cos\varphi_x;\ r(\varphi_x)\sin\varphi_x)$ with

$$y = \lambda_x(r'(\varphi_x)\cos\varphi_x - r(\varphi_x)\sin\varphi_x;\ r'(\varphi_x)\sin\varphi_x + r(\varphi_x)\cos\varphi_x)$$

and functions $\xi = \alpha \circ \varphi$, $\eta = \beta \circ \varphi$ where $\alpha, \beta : \mathbf{R} \to \mathbf{R}$ are defined, using the abbreviation $S_x(\varphi) = r'(\varphi_x + \varphi)\sin\varphi + r(\varphi_x + \varphi)\cos\varphi$, by

$$\alpha(\varphi) = -2\frac{S_x(\varphi)[r'(\varphi_x)\sin\varphi - r(\varphi_x)\cos\varphi]}{r(\varphi_x + \varphi)\cdot r(\varphi_x)} - 1,$$
$$\beta(\varphi) = 2\frac{S_x(\varphi)\cdot\sin\varphi}{r(\varphi_x + \varphi)}$$

and $\varphi : [-1,1] \to \mathbf{R}$, $\varphi(\tau) = \varphi_0 \cdot (1-\tau)/2$ with $0 < \varphi_0 < \pi$ such that $\cot\varphi_0 = r'(\varphi_x)/r(\varphi_x)$.

Remark 2.7. We do not know whether there exists at all an orthogonality space which is not continuous.

3. Existence of an equivalent inner product

Lemma 3.1. *Suppose that* $(\mathbf{X}, \perp)$ *is a continuous orthogonality space such that* (e) $\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) \neq \{\underline{0}\}$ *with some abelian group* $(\mathbf{Y}, +)$. *Then for any 2–dimensional linear subspace* $P \subset \mathbf{X}$ *and* $u \in P \setminus \{0\}$ *there exists a unique inner product* $\langle \cdot, \cdot \rangle_P^u : P \times P \to \mathbf{R}$ *such that*

$$\langle u, u\rangle_P^u = 1 \text{ and } x \perp y \iff \langle x, y\rangle_P^u = 0 \quad \text{for all } x, y \in P.$$

PROOF. Let $P \subset \mathbf{X}$ be a 2–dimensional linear subspace, $u \in P \setminus \{0\}$. By (O4"), there exist $v \in P\setminus\{0\}$ and contiuous functions $\xi, \eta : [-1, 1] \to \mathbf{R}$ such that $u \perp v$, $\xi(\pm 1) = \pm 1$ and

$$(3.1) \qquad (u + [\xi(\tau)u + \eta(\tau)v]) \perp (u - [\xi(\tau)u + \eta(\tau)v])$$

for all $\tau \in [-1, 1]$. We may and do assume that $(u + v) \perp (u - v)$ with respect to the equality $\eta(\tau)v = [\eta(\tau)/\eta(\tau_0)][\eta(\tau_0)v]$, where $\tau_0 \in [-1, 1]$ is such that $\xi(\tau_0) = 0$ and so by axiom (O2) $\eta(\tau_0) \neq 0$.

We are going to show that $\xi^2(\tau) + \eta^2(\tau) = 1$ for all $\tau \in [-1, 1]$. For if this equality were not satisfied, then taking into account the continuity of the functions ξ, η and $\xi(\pm 1) = \pm 1$, there would exist an open interval $]\tau_1, \tau_2[\subset [-1, 1]$ such that $\xi^2(\tau) + \eta^2(\tau) \neq 1$ whenever $\tau \in]\tau_1, \tau_2[$ and at least one of the functions ξ and η is not constant on $]\tau_1, \tau_2[$. Then there are numbers $\tau_0 \in]\tau_1, \tau_2[$ and $m_0, n_0 \in \mathbf{N}$ such that $\eta(\tau_0) = [n_0/m_0][1 + \xi(\tau_0)]$ or $\eta(\tau_0) = [n_0/m_0][1 - \xi(\tau_0)]$. Indeed,

— if $\xi(\tau) = \xi_0 \neq -1$ on $]\tau_1, \tau_2[$, then $\tau \to \eta(\tau)/[1 + \xi_0]$ is a non–constant function on $]\tau_1, \tau_2[$ and so it takes on a non–zero rational number as a value: $\eta(\tau_0)/[1 + \xi(\tau_0)] = n_0/m_0 \in \mathbf{Q}$;

— if $\xi(\tau) = \xi_0 \neq 1$ on $]\tau_1, \tau_2[$, then with the same argument, we have $\eta(\tau_0)/[1 - \xi(\tau_0)] = n_0/m_0 \in \mathbf{Q}$

— if ξ is not constant on $]\tau_1, \tau_2[$, then we may assume that $\xi(\tau) \neq \pm 1$ on $]\tau_1, \tau_2[$, and because of the axiom (O2), one of the continuous functions $\tau \to \eta(\tau)/[1 \pm \xi(\tau)]$ is not constant on $]\tau_1, \tau_2[$, making it possible to apply one of the above arguments.

Now let e.g. $\eta(\tau_0) = [n_0/m_0][1 + \xi(\tau_0)]$. This means by (3.1) that

$$\frac{1 + \xi(\tau_0)}{m_0}(m_0 u + n_0 v) \perp \frac{1}{m_0}([1 - \xi(\tau_0)]m_0 u - [1 + \xi(\tau_0)]n_0 v).$$

Now, with respect to Corollary 1.14, we have

$$[1 - \xi(\tau_0)]m_0^2 - [1 + \xi(\tau_0)]n_0^2 = 0$$

whence

$$\xi(\tau_0) = \frac{m_0^2 - n_0^2}{m_0^2 + n_0^2}, \quad \eta(\tau_0) = \frac{2m_0 n_0}{m_0^2 + n_0^2},$$

i.e. $\xi^2(\tau_0) + \eta^2(\tau_0) = 1$. This contradiction proves our assertion $\xi^2 + \eta^2 = 1$ on $[-1, 1]$.

Now one can easily see that each vector $x = \alpha_x u + \beta_x v \in P \setminus \mathrm{lin}\{v\}$ can be written into one of the forms:

$$x = \lambda_x([1 + \xi(\tau_x)]u \pm \eta(\tau_x)v)$$

with some $\lambda_x \in \mathbf{R}$ and $\tau_x \in [-1, 1]$. Thus by (3.1) and Corollary 1.12 we have

$$x \perp ([1 - \xi(\tau_x)]u \mp \eta(\tau_x)v)$$

and so by the uniqueness of $\perp$, we have $x \perp y = \alpha_y u + \beta_y v \in P$ if, and only if, with some $\lambda_y \in \mathbf{R}$

$$y = \lambda_y([1 - \xi(\tau_x)]u \mp \eta(\tau_x)v),$$

i.e. $\alpha_x\alpha_y + \beta_x\beta_y = \lambda_x\lambda_y([1 - \xi^2(\tau_x)] - \eta^2(\tau_x)) = 0$. Hence $\langle x, y\rangle_P^u = \alpha_x\alpha_y + \beta_x\beta_y$ is an inner product on P possessing all of the required properties.

Uniqueness. If $\langle\cdot,\cdot\rangle_P^*$ is another such inner product on P, then

$$0 = \langle u + v, u - v\rangle_P^* = \langle u, u\rangle_P^* - \langle v, v\rangle_P^* = 1 - \langle v, v\rangle_P^* ,$$

and so

$$\langle x, y\rangle_P^* = \alpha_x\alpha_y\langle u, u\rangle_P^* + \beta_x\beta_y\langle v, v\rangle_P^* = \langle x, y\rangle_P^u .$$

Remark 3.2. For $\dim \mathbf{X} = 2$ the Lemma just proved turns into a final result: If (e) $\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) \neq \{\underline{0}\}$, then $\mathbf{X}$ is an inner product space with the usual orthogonality $\perp$. Although, for $\dim \mathbf{X} \geq 3$ this theorem has already been proved in a more general context (c.f. [10], Thm. 4.2), we are now in a position to offer a possibly simpler proof for the continuous case.

Theorem 3.3. *Suppose that $(\mathbf{X}, \perp)$ is a continuous orthogonality space and $(\mathbf{Y}, +)$ is an abelian group. If (e) $\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) \neq \{\underline{0}\}$, then $\mathbf{X}$ is an inner product space for some $\langle\cdot,\cdot\rangle : \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ such that*

$$x \perp y \iff \langle x, y\rangle = 0 \quad \text{for all } x, y \in \mathbf{X}.$$

PROOF. Let $E \in$ (e) $\mathrm{Hom}_\perp(X, Y) \setminus \{\underline{0}\}$ be fixed with its biadditive representation B and define a functional $\rho : \mathbf{X} \to \mathbf{R}_+$ as follows:

Let $\rho(0) = 0$ and assign to each $x \in \mathbf{X} \setminus \{0\}$ a positive real number $\rho(x)$ with the aid of a fixed vector $u \in \mathbf{X} \setminus \{0\}$ and the linear subspace $P_x = \mathrm{lin}\{u, x\}$:

— for $\dim P_x = 1$ let $\rho(x) = |\alpha|$, where $x = \alpha u$ for some $\alpha \in \mathbf{R}$;

— for $\dim P_x = 2$ let $\rho(x) = \sqrt{\langle x, x\rangle_{P_x}^u}$, where $\langle\cdot,\cdot\rangle_{P_x}^u : P_x \times P_x \to \mathbf{R}$ is the unique inner product defined by Lemma 3.1.

Clearly $\rho(u) = 1$ and $\rho(\lambda x) = |\lambda|\rho(x)$ whenever $x \in \mathbf{X}$, $\lambda \in \mathbf{R}$. Furthermore, for any $x \in \mathbf{X}$, we have either $x = \pm\rho(x)u$ or $(x + \rho(x)u) \perp (x - \rho(x)u)$, therefore $E(x) = E(\rho(x)u) = E(\rho(y)u) = E(y)$ for all $x, y \in \mathbf{X}$, $\rho(x) = \rho(y)$. In what follows we show that the desired inner product can be defined by

$$(3.2) \qquad \langle x, y\rangle = \frac{1}{4}\left(\rho^2(x+y) - \rho^2(x-y)\right) \quad \text{for all } x, y \in \mathbf{X}.$$

To prove that (3.2) defines an inner product on $\mathbf{X}$, it suffices to show that the restriction of $\langle\cdot,\cdot\rangle$ to any 2–dimensional linear subspace $P \subset \mathbf{X}$ is an inner product on P. The case $u \in P$ is trivial. When $u \notin P$, then $S = \operatorname{lin}\{u, P\}$ is 3–dimensional and so by Lemma 1.10 we can choose vectors $v, w \in S$ such that $u \perp v$, $u \perp w$, $v \perp w$, $\rho(v) = \rho(w) = 1$. Since $E(\lambda v) = E(\lambda w)$ for all $\lambda \in \mathbf{R}$, by Corollary 1.13 we have $(v+w) \perp (v-w)$. This means for the inner product $\langle\cdot,\cdot\rangle_Q^v$ on $Q = \operatorname{lin}\{v, w\}$ that

$$0 = \langle v+w, v-w\rangle_Q^v = \langle v, v\rangle_Q^v - \langle w, w\rangle_Q^v = 1 - \langle w, w\rangle_Q^v .$$

Therefore

$$\langle \beta_x v + \gamma_x w, \beta_y v + \gamma_y w\rangle_Q^v = \beta_x\beta_y + \gamma_x\gamma_y$$

whenever $x = \beta_x v + \gamma_x w$, $y = \beta_y v + \gamma_y w \in Q$. Now for $z = \beta v + \gamma w \in Q$, $\beta^2 + \gamma^2 = 1$, it follows that $u \perp z$ because of the additivity of $\perp$. On the other hand $(v+z) \perp (v-z)$ since $\langle v+z, v-z\rangle_Q^v = (1+\beta)(1-\beta) - \gamma^2 = 0$. Thus $E(\lambda u) = E(\lambda v) = E(\lambda z)$ for all $\lambda \in \mathbf{R}$ whence by Corollary 1.13 we have $(u+z) \perp (u-z)$, and so

$$0 = \langle u+z, u-z\rangle_{P_z}^u = \langle u, u\rangle_{P_z}^u - \langle z, z\rangle_{P_z}^u = 1 - \langle z, z\rangle_{P_z}^u .$$

Finally, for $x = \alpha_x u + \beta_x v + \gamma_x w \in S \setminus \operatorname{lin}\{u\}$,

$$\begin{aligned}
\rho^2(x) &= \rho^2\left(\alpha_x u + \sqrt{\beta_x^2 + \gamma_x^2}\left[\frac{\beta_x}{\sqrt{\beta_x^2+\gamma_x^2}}v + \frac{\gamma_x}{\sqrt{\beta_x^2+\gamma_x^2}}w\right]\right) = \\
&= \rho^2\left(\alpha_x u + \sqrt{\beta_x^2+\gamma_x^2}\, z\right) = \\
&= \left\langle \alpha_x u + \sqrt{\beta_x^2+\gamma_x^2}\, z,\ \alpha_x u + \sqrt{\beta_x^2+\gamma_x^2}\, z\right\rangle_{P_x}^u = \\
&= \alpha_x^2\langle u, u\rangle_{P_z}^u + (\beta_x^2+\gamma_x^2)\langle z, z\rangle_{P_z}^u = \alpha_x^2 + \beta_x^2 + \gamma_x^2
\end{aligned}$$

holds, i.e., ρ defines by (3.2) an inner product on $P \subset S$ (Notice that $P_x = P_z$).

In the rest of the proof we show that for $x, y \in \mathbf{X}$ we have $\langle x, y\rangle = 0$ if, and only if, $x \perp y$. We may suppose that $x, y \neq 0$ and so the linear independency of x, y.

First suppose that $\langle x, y\rangle = 0$. Then for any $\lambda, \mu \in \mathbf{R}$ we have

$$\rho\left(\frac{\lambda}{2}x + \frac{\mu}{2}y\right) = \rho\left(\frac{\lambda}{2}x - \frac{\mu}{2}y\right).$$

This implies that

$$\begin{aligned} &B\left(\frac{\lambda}{2}x, \frac{\lambda}{2}x\right) + 2B\left(\frac{\lambda}{2}x, \frac{\mu}{2}y\right) + B\left(\frac{\mu}{2}y, \frac{\mu}{2}y\right) = \\ &= B\left(\frac{\lambda}{2}x + \frac{\mu}{2}y, \frac{\lambda}{2}x + \frac{\mu}{2}y\right) = E\left(\frac{\lambda}{2}x + \frac{\mu}{2}y\right) = \\ &= E\left(\frac{\lambda}{2}x - \frac{\mu}{2}y\right) = B\left(\frac{\lambda}{2}x - \frac{\mu}{2}y, \frac{\lambda}{2}x - \frac{\mu}{2}y\right) = \\ &= B\left(\frac{\lambda}{2}x, \frac{\lambda}{2}x\right) - 2B\left(\frac{\lambda}{2}x, \frac{\mu}{2}y\right) + B\left(\frac{\mu}{2}y, \frac{\mu}{2}y\right) \end{aligned}$$

whence for all $\lambda, \mu \in \mathbf{R}$

$$B(\lambda x, \mu y) = 4B\left(\frac{\lambda}{2}x, \frac{\mu}{2}y\right) = 0.$$

Thus Lemma 1.11 gives $x \perp y$.

Conversely, if $x \perp y$, then we can choose $z \in \text{lin}\{x, y\}$ such that $\rho(z) = 1$ and $\langle x, z\rangle = 0$. By the assertion just proved $x \perp z$ follows and so, because of the (right) uniqueness of $\perp$, we have $y = \mu z$, i.e., $\langle x, y\rangle = 0$.

Corollary 3.4. *Let $(\mathbf{X}, \|\cdot\|)$ be a real normed linear space of dimension ≥ 2 with the Birkhoff–James orthogonality $\perp$, and $(\mathbf{Y}, +)$ an abelian group. If* $(\mathrm{e})\,\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) \neq \{\underline{0}\}$, *then $\mathbf{X}$ is an inner product space and by [6], Theorem 9,*

$$(\mathrm{e})\,\mathrm{Hom}_\perp(\mathbf{X}, \mathbf{Y}) = \{a \circ \|\cdot\|^2 \mid a \in \mathrm{Hom}(\mathbf{R}, \mathbf{Y})\}.$$

References

[1] J. Aczél, The general solution of two functional equations by reduction to functions additive in two variables and with the aid of Hamel bases, *Glasnik Mat.-Fiz.-Astr.* **20** (1965), 65–73.

[2] S. GUDDER and D. STRAWTHER, Orthogonally additive and orthogonally increasing functions on vector spaces, *Pacific J. Math.* **58** (1975), 427–436.

[3] R. C. JAMES, Orthogonality and linear functionals in normed vector spaces, *Trans. Amer. Math. Soc.* **61** (1947), 265–292.

[4] R. C. JAMES, Inner product in normed linear spaces, *Bull. Amer. Math. Soc.* **53** (1947), 559–566.

[5] J. LAWRENCE, Orthogonality and additive functions on normed linear spaces, *Colloquium Math.* **49** (1985), 253–255.

[6] J. RÄTZ, On orthogonally additive mappings, *Aequationes Math.* **28** (1985), 35–49.

[7] J. RÄTZ, On orthogonally additive mappings, II, *Publicationes Math. Debrecen* **35** (1988), 241–249.

[8] K. SUNDARESAN, Orthogonality and nonlinear functionals on Banach spaces, *Proc. Amer. Math. Soc.* **34** (1972), 187–190.

[9] Gy. SZABÓ, On mappings, orthogonally additive in the Birkhoff–James sense, *Aequationes Math.* **30** (1986), 93–105.

[10] Gy. SZABÓ, On orthogonality spaces admitting nontrivial even orthogonally additive mappings. (to appear in Acta Math. Hungar.)

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