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New characterizations of the arithmetic-geometric mean of Gauss and other well-known mean values

By HIROSHI HARUKI (Waterloo)

Dedicated to Professor Z. Daróczy on his 50th birthday

1. Introduction and statement of the results

Throughout this paper, a mean value of two positive real numbers a, b denoted by $M(a, b)$, is defined to be a real number which satisfies the following postulates:

(P_1) $M : R^+ \times R^+ \to R$;

(P_2) $M(a, b) = M(b, a)$ (symmetry property);

(P_3) $M(a, a) = a$ (reflexivity property).

See [2].

We consider the complete elliptic integral of the first kind

$$I(a,b) \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}, \tag{1}$$

where a, b are arbitrary positive real numbers. Throughout this paper a, b denote arbitrary positive real numbers.

The above $I(a, b)$ is closely related to the arithmetic-geometric mean of Gauss of a, b denoted by $G(a, b)$ (see [5], [7] and [9]). Indeed, the following equality holds in $R^+ \times R^+$:

$$G(a,b) = \frac{1}{I(a,b)}. \tag{2}$$

It is well-known that $f(a,b) = G(a,b)$ satisfies the following functional equation in $R^+ \times R^+$:

$$f\left(\frac{a+b}{2}, \sqrt{ab}\right) = f(a,b), \tag{3}$$

where $f : R^+ \times R^+ \to R$ and f is an unknown function. Hence, by (2) $f(a,b) = I(a,b)$ satisfies (3). Throughout this paper, for the sake of simplicity, we denote $\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$ by r.

The purpose of Section 3 is to prove the following theorem:

Theorem 1. *Let $f : R^+ \times R^+ \to R$. If f can be represented by the form, containing some function p, in $R^+ \times R^+$*

$$f(a,b) = \frac{1}{2\pi}\int_0^{2\pi} p(r)\, d\theta, \tag{4}$$

where $p : R^+ \to R$ and $p''(x)$ is continuous in R^+, then the only solution of (3) is given by

$$f(a,b) = A\, I(a,b) + B = A\frac{1}{G(a,b)} + B,$$

where A, B are arbitrary real constants.

Before proceeding to Theorem 2 we require the additional condition on p that p be strictly monotonic in R^+. We set (see [4])

$$M(a,b\,;p) \overset{\text{def}}{=} p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r)\, d\theta\right). \tag{5}$$

Throughout this paper, for the sake of simplicity, we write $M(a,b)$ for $M(a,b\,;p)$, i.e.,

$$M(a,b) \overset{\text{def}}{=} p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r)\, d\theta\right).$$

If we set $p(r) = \frac{1}{r}$, $p(r) = \log r$, $p(r) = \frac{1}{r^2}$, $p(r) = r^2$ in (5), by using (1), (2) and the three definite integrals

$$\frac{1}{2\pi}\int_0^{2\pi} \log\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta = \log\frac{a+b}{2},$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{1}{ab},$$

$$\frac{1}{2\pi}\int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta = \frac{a^2+b^2}{2},$$

we obtain $M(a,b) = G(a,b)$, $M(a,b) = \frac{a+b}{2}$, $M(a,b) = \sqrt{ab}$, $M(a,b) = \sqrt{\frac{a^2+b^2}{2}}$, respectively.

The purpose of Section 4 is to prove the following theorem:

Theorem 2.

(i) *$M(a,b) = G(a,b)$ holds for all positive real numbers a, b iff $p(r) = A\frac{1}{r} + B$ where $A(\neq 0)$, B are arbitrary real constants.*

(ii) *$M(a,b) = \frac{a+b}{2}$ holds for all positive real numbers a, b iff $p(r) = A\log r + B$ where $A(\neq 0)$, B are arbitrary real constants.*

(iii) *$M(a,b) = \sqrt{ab}$ holds for all positive real numbers a, b iff $p(r) = A\frac{1}{r^2} + B$ where $A(\neq 0)$, B are arbitrary real constants.*

(iv) *$M(a,b) = \sqrt{\frac{a^2+b^2}{2}}$ (the root–mean–square of a, b) holds for all positive real numbers a, b iff $p(r) = Ar^2 + B$ where $A(\neq 0)$, B are arbitrary real constants.*

(v) *There exists no $p(r)$ such that $M(a,b) = \frac{2ab}{a+b}$ (the harmonic mean of a, b) holds for all positive real numbers a, b.*

To prove Theorem 1 and Theorem 2 the lemma in Section 2 plays an important part.

Remark. About means see [1], pp. 234–244, [2]–[8] and [10]–[13].

2. Lemma

Lemma. *Let $p : R^+ \to R$. We assume that $p''(x)$ is continuous in R^+. If we set*

$$(6)\qquad f(a,b) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} p\left(\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\right)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} p(r)\,d\theta,$$

then

$$(i)\quad f_a(c,c) = f_b(c,c) = \frac{1}{2}p'(c),$$

$$(ii)\quad f_{aa}(c,c) = f_{bb}(c,c) = \frac{3cp''(c) + p'(c)}{8c};$$

$$(iii)\quad f_{ab}(c,c) = f_{ba}(c,c) = \frac{cp''(c) - p'(c)}{8c},$$

where c is an arbitrary positive real number.

PROOF. Throughout the proof we apply differentiation under the integral sign which can be done because $p(r)$ is of class C^2 in $R^+ \times R^+$ with respect to a, b.

Proof of (i). By (6) we obtain

$$f_a(a,b) = \frac{1}{2\pi}\int_0^{2\pi} p'(r)\frac{a}{r}\cos^2\theta\, d\theta. \tag{7}$$

Setting $a = b = c$ in (7) and using $\sqrt{c^2(\cos^2\theta + \sin^2\theta)} = c$ yields

$$f_a(c,c) = \frac{1}{2\pi}\int_0^{2\pi} p'(c)\frac{c}{c}\cos^2\theta\, d\theta = \frac{1}{2\pi}p'(c)\int_0^{2\pi}\cos^2\theta\, d\theta =$$
$$= \frac{1}{2\pi}p'(c)\pi = \frac{1}{2}p'(c).$$

Similarly we can prove that

$$f_b(c,c) = \frac{1}{2}p'(c).$$

Proof of (ii). By (7) we obtain

$$f_{aa}(a,b) = \frac{1}{2\pi}\int_0^{2\pi}\left(p''(r)\frac{a^2}{r^2}\cos^2\theta + p'(r)\frac{r^2 - a^2\cos^2\theta}{r^3}\right)\cos^2\theta\, d\theta\,.$$

Setting $a = b = c$ in the above equality yields

$$f_{aa}(c,c) = \frac{1}{2\pi}\left(p''(c)\int_0^{2\pi}\cos^4\theta\,d\theta + \frac{p'(c)}{c}\int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta\right) = $$
$$= \frac{1}{2\pi}\left(p''(c)\frac{3\pi}{4} + \frac{p'(c)}{c}\frac{\pi}{4}\right) = $$
$$= \frac{3cp''(c) + p'(c)}{8c}.$$

Similarly we can prove that

$$f_{bb}(c,c) = \frac{3cp''(c) + p'(c)}{8c}.$$

Proof of (iii). We can prove (iii) in a similar manner to that of (ii).
Q.E.D.

3. Proof of Theorem 1

Applying $\frac{\partial^2}{\partial a^2}$ to both sides of (3), using the Chain Rule for differentiation for two real variables and observing that $f_{ab}(a,b) = f_{ba}(a,b)$ yields

$$\begin{aligned}(8)\qquad &\frac{1}{4}f_{aa}\left(\frac{a+b}{2},\sqrt{ab}\right) + \frac{\sqrt{b}}{2\sqrt{a}}f_{ab}\left(\frac{a+b}{2},\sqrt{ab}\right) + \\ &+\frac{b}{4a}f_{bb}\left(\frac{a+b}{2},\sqrt{ab}\right) - \frac{\sqrt{b}}{4a\sqrt{a}}f_b\left(\frac{a+b}{2},\sqrt{ab}\right) = f_{aa}(a,b).\end{aligned}$$

Setting $a = b = c$, where c is an arbitrarily fixed positive real number, in (8) yields

$$(9)\qquad \frac{3}{4}f_{aa}(c,c) - \frac{1}{2}f_{ab}(c,c) - \frac{1}{4}f_{bb}(c,c) + \frac{1}{4c}f_b(c,c) = 0.$$

Substituting $f_{aa}(c,c)$, $f_{a,b}(c,c)$, $f_{bb}(c,c)$, $f_b(c,c)$ in Lemma in Section 2 into (9) and simplifying the resulting equality yields

$$p''(c) + \frac{2}{c}p'(c) = 0.$$

Since c was an arbitrarily fixed positive real number, we can replace c by a positive real variable x in the above equality. So we obtain in R^+

$$p''(x) + \frac{2}{x}p'(x) = 0.$$

Solving the above differential equation yields in R^+

$$p(x) = A\frac{1}{x} + B,$$

and so

$$p(r) = A\frac{1}{r} + B, \tag{10}$$

where A, B are real constants satisfying $A \neq 0$. Substituting (10) into (4) and using (1) yields in $R^+ \times R^+$

$$f(a,b) = A\,I(a,b) + B. \tag{11}$$

Direct substitution of (11) shows that (11) is a solution of our original functional equation (3).

4. Proof of Theorem 2

Remark. To prove Theorem 2 (i) we shall apply Theorem 1.

Proof of (i). We have only to prove the "only if" part. By hypothesis

$$M(a,b) = G(a,b) \tag{12}$$

holds in $R^+ \times R^+$. By (5), (12) we obtain in $R^+ \times R^+$

$$p(G(a,b)) = \frac{1}{2\pi}\int_0^{2\pi} p(r)d\theta. \tag{13}$$

Since $G(a,b)$ is a solution of the functional equation (3), $P(G(a,b))$ is also a solution of (3). Furthermore, (13) holds. Hence, by Theorem 1 we obtain in $R^+ \times R^+$

$$p(G(a,b)) = A\,I(a,b) + B = A\frac{1}{G(a,b)} + B, \tag{14}$$

where A, B are real constants satisfying $A \neq 0$. If we set $b = a$ in (14), then we obtain in R^+

$$p(G(a,a)) = A\frac{1}{G(a,a)} + B. \tag{15}$$

Since $G(a,a) = a$, by (15) we have in R^+

$$p(a) = A\frac{1}{a} + B,$$

or

$$p(r) = A\tfrac{1}{r} + B. \qquad \text{Q.E.D.}$$

Proof of (ii). We have only to prove the "only if" part. By hypothesis

$$M(a,b) = \frac{a+b}{2} \tag{16}$$

holds in $R^+ \times R^+$. By (5), (16) we obtain in $R^+ \times R^+$

$$p\left(\frac{a+b}{2}\right) = \frac{1}{2\pi}\int_0^{2\pi} p(r)\,d\theta\,. \tag{17}$$

If we set in $R^+ \times R^+$

$$f(a,b) = \frac{1}{2\pi}\int_0^{2\pi} p(r)\,d\theta\,, \tag{18}$$

by (17), (18) we have in $R^+ \times R^+$

$$f(a,b) = p\left(\frac{a+b}{2}\right). \tag{19}$$

Applying $\frac{\partial^2}{\partial a^2}$ to both sides of (19) and setting $a = b = c$, where c is an arbitrarily fixed positive real number in the resulting equality yields

$$f_{aa}(c,c) = \frac{1}{4}p''(c). \tag{20}$$

By (18) and by Lemma (ii) in Section 1 we obtain

$$f_{aa}(c,c) = \frac{3cp''(c) + p'(c)}{8c}. \tag{21}$$

Substituting (21) in (20) and simplifying the resulting equality yields

$$p''(c) + \frac{1}{c}p'(c) = 0,$$

and so we obtain in R^+

$$p''(x) + \frac{1}{x}p'(x) = 0.$$

Solving the above differential equation yields in R^+

$$p(x) = A \log x + B,$$

and so

$$p(r) = A \log r + B,$$

where A, B are real constants satisfying $A \neq 0$. Q.E.D.

Proof of (iii). Since we can prove (iii) by using a similar argument to that in (ii), we omit the proof.

Proof of (iv). Since we can prove (iv) by using a similar argument to that in (ii), we omit the proof.

Proof of (v). The proof is by contradiction. Assume contrary. Then there exists a $p(r)$ such that

$$M(a,b) = \frac{2ab}{a+b} \tag{22}$$

holds for all positive real numbers a, b. By (5), (22) we obtain in $R^+ \times R^+$

$$p\left(\frac{2ab}{a+b}\right) = \frac{1}{2\pi}\int_0^{2\pi} p(r)\,d\theta\,. \tag{23}$$

Starting with (23) and using a similar argument to that in (ii) yields the differential equation for p in R^+

$$p''(x) + \frac{5}{x}p'(x) = 0\,.$$

Solving the above differential equation yields in R^+

$$p(x) = A\frac{1}{x^4} + B\,,$$

and so

$$p(r) = A\frac{1}{r^4} + B = A\frac{1}{(a^2\cos^2\theta + b^2\sin^2\theta)^2} + B, \tag{24}$$

where A, B are real constants and $A \neq 0$. Substituting (24) into (5) yields

$$M(a,b) = \left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^2}\right)^{-\frac{1}{4}}. \tag{25}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^2} = \frac{1}{2ab}\left(\frac{1}{a^2} + \frac{1}{b^2}\right), \tag{26}$$

by (22), (25), (26) we obtain for all positive real numbers

$$\frac{2ab}{a+b} = \left(\frac{1}{2ab}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right)^{-\frac{1}{4}}$$

and so

$$a = b.$$

This is a contradiction. Q.E.D.

Remark. The mean $M(a,b) = \frac{a^2+b^2}{a+b}$ for all positive real numbers a, b is said to be the antiharmonic mean of a, b. As a generalization of the above mean and the arithmetic mean of a, b we consider the mean $M(a,b) = \frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ where n is a positive integer. In a similar way to Theorem 2(v) we can prove the following result: If $n > 1$, there exists no $p(r)$ such that $M(a,b) = \frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ holds for all positive real numbers a, b.

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HIROSHI HARUKI
DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO
CANADA
N2L 3GI

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