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Base three just touching covering systems

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Abstract. Let $\mathcal{A} = \{0, a_1, a_2\}$ where $a_1 \equiv 1 \pmod 3$ and $a_2 \equiv 2 \pmod 3$. Let $\mathcal{B} = \mathcal{A} - \mathcal{A} = \pm\{0, a_1, a_2, a_2 - a_1\}$. We say $\mathcal{A}$ is a Just Touching Covering System (JTCS) if every integer is expressible in the form $c_n 3^n + c_{n-1}3^{n-1} + \cdots + c_1 3 + c_0$ where the c_i are in $\mathcal{B}$ and n is a nonnegative integer. We prove $\mathcal{A}$ is a JTCS iff a_1 and a_2 are relatively prime.

1. Introduction

The material of this introduction and the conjecture we prove comes from KÁTAI's paper [1]. Consider a triple of numbers $\mathcal{A} = \{0, a_1, a_2\}$ where $a_1 \equiv 1 \pmod 3$ and $a_2 \equiv 2 \pmod 3$. We define a function $F_{\mathcal{A}}$ on the integers by the formula $F_{\mathcal{A}}(x) = (x - a)/3$ where a is in $\mathcal{A}$ and x and a are congruent modulo 3. Let m be the larger of the absolute values of a_1 and a_2. Define $I_{\mathcal{A}}$ to be the "interval" $[-m/2, m/2] \cap Z$. We have the following fact from [1]:

Fact 1: For any integer x there is a positive integer n such that $F^n_{\mathcal{A}}(x)$ is in $I_{\mathcal{A}}$. If x is in $I_{\mathcal{A}}$, then so is $F_{\mathcal{A}}(x)$.

This fact is established in a straightforward fashion by considering the inequalities involved. We picture a directed graph on the integers where x is connected to $F_{\mathcal{A}}(x)$ by an arrow. The above fact says that there is a path from any integer x into the interval $I_{\mathcal{A}}$. For numbers in $I_{\mathcal{A}}$, repeated application of $F_{\mathcal{A}}$ eventually leads either to 0 or to a periodic number. A periodic number is a number fixed by some $F^n_{\mathcal{A}}$. See Figure 1.

Figure 1. The directed graph defined by the function $F_{\mathcal{A}}$ where $\mathcal{A} = \{0, 10, 29\}$ and $I_{\mathcal{A}} = [-14, 14] \cap Z$

We will also need the following fact from [1]:

Fact 2: For the special case of $\mathcal{A} = \{0, a, -a\}$ where 3 does not divide a, all the elements of $I_{\mathcal{A}}$ are periodic. The directed graph described above, restricted to $I_{\mathcal{A}}$, consists of a collection of loops (one of which contains only the element zero).

In this special case $F_{\mathcal{A}}$ and $I_{\mathcal{A}}$ will be denoted by F_a and I_a. We have an inverse for F_a on the interval I_a which we will denote by T_a. The function T_a consists of tripling (modulo a, so that the image is in I_a). It is easily shown that the length of a loop is a factor of the smallest positive integer n where $3^n \equiv 1 \pmod{a}$. Fact 2 is illustrated in Figure 2.

If the triple $\mathcal{A} = \{0, a_1, a_2\}$ has no periodic numbers (i.e. all numbers connect to 0), we say $\mathcal{A}$ is a *number system*. This is equivalent to having every integer x expressible as $c_n 3^n + c_{n-1} 3^{n-1} + \ldots + c_1 3 + c_0$ where the c_i are in $\mathcal{A}$ and n is a nonnegative integer. This is easily seen by analyzing the path from x to $0 : F^i_{\mathcal{A}}(x) \equiv c_i \pmod{3}$.

A necessary, but not sufficient, condition for $\mathcal{A}$ to be a number system is that a_1 and a_2 be relatively prime: if k divides a_1 and a_2 then k must divide any x expressible as $c_n 3^n + c_{n-1} 3^{n-1} + \ldots + c_1 3 + c_0$ where the c_i are in $\mathcal{A}$.

Figure 2. $\mathcal{A} = \{0, -20, 20\}$ yields this graph on the integers of $I_{\mathcal{A}} = [-10, 10]$. Since 3^4 is congruent to $1 \pmod{20}$, the length of the loops are factors of 4.

To examine how sufficient the "relatively prime" condition is, we make the following definition. Consider the set $\mathcal{B} = \mathcal{A} - \mathcal{A}$. This consists of $0, a_1, a_2, a_3 = a_2 - a_1$ and their opposites. We say $\mathcal{A}$ is a Just Touching Covering System (JTCS) if every integer is expressible as $c_n 3^n + c_{n-1} 3^{n-1} + \ldots + c_1 3 + c_0$ where the c_i are in $\mathcal{B}$ and n is some nonnegative integer. (The origin of the terminology is found in [1]. There $\mathcal{A}$ is termed a JTCS if $\lambda(H + n \cap H + m) = 0$ for all distinct integers n and m, where λ is the Lebesgue measure and H is the set of real numbers expressible in the form $\sum_{i=1}^{\infty} c_i 3^{-i}$ with c_i in $\mathcal{A}$. The equivalence with the definition used in this paper is established in [2] and [3].) Another way of expressing this is by looking at a directed graph where the connections are created not by a single $F_{\mathcal{A}}$ as above, but by three functions: F_{a_1}, F_{a_2} and F_{a_3}. If x is any integer, $F_{a_i}(x)$ is the sole integral element of $\{x/3, (x - a_i)/3, (x + a_i)/3\}$. The integer x will be expressible in the polynomial form above if and only if application of the three F_{a_i} in some combination eventually leads to 0. We will prove the following theorem, conjectured by Kátai in [1]:

Theorem 1. *$\mathcal{A} = \{0, a_1, a_2\}$ is a JTCS iff a_1 and a_2 are relatively prime.*

Of course the "only if" is obvious, as it is for number systems. Before proving the theorem we make a slight change in the notation. In the sequel, we will assume a_1 and a_2 are relatively prime.

2. Notation and outline of proof

Recall that $\mathcal{B} = \{0, a_1, a_2, a_3, -a_1, -a_2, -a_3\}$ where $a_3 = a_2 - a_1$, $(a_1, a_2) = 1$, $a_1 \equiv 1 \pmod 3$ and $a_2 \equiv 2 \pmod 3$. We will be more concerned with the size of these numbers than with their class modulo 3. Notice that the sum of any two elements of $\mathcal{B}$ which are congruent modulo 3 is again an element of $\mathcal{B}$. Also notice that the difference of any two nonzero elements of opposite modularity is also an element of $\mathcal{B}$ (with the exception of differences of the form $x - (-x)$). Using these two facts we prove

Lemma 1. *If $b_1 < b_2 < b_3$ are the positive elements of $\mathcal{B}$, then $b_3 = b_1 + b_2$ and $b_1 \equiv b_2 \pmod 3$.*

Proof. If b_1 and b_2 are not congruent modulo 3, then $b_2 - b_1$ is in $\mathcal{B}$ by the above remarks. This is a positive integer smaller than b_2 and different from b_1. (Recall we are assuming throughout that a_1 and a_2 are relatively prime, so that $b_2 \neq 2b_1$.) This contradicts the definition of b_2. Therefore we may assume b_1 and b_2 are congruent modulo 3. By the above remarks, their sum $b_1 + b_2$ must be in $\mathcal{B}$ as well, and therefore must be b_3. $\square$

In future we describe $\mathcal{B}$ by its triple (b_1, b_2, b_3). Any such triple (b_1, b_2, b_3) of pairwise relatively prime positive integers which are not divisible by 3, with $b_1 < b_2 < b_3$, $b_1 \equiv b_2 \pmod 3$, and $b_3 = b_1 + b_2$, will be referred to as being in *required format*. For convenience of notation, we will also say the triple $(1, 1, 2)$ is in required format even though it does not fit the description just stated.

We will henceforth denote the three maps F_{a_i} with regard to this new notation as the maps F_{b_1}, F_{b_2}, and F_{b_3}. By repeated application of F_{b_i} we get a path from any integer into the interval $I_{b_i} = [-b_i/2, b_i/2] \cap Z$. Recall that, if x is in I_{b_i}, $T_{b_i}(x)$ is the unique element in $\{3x, 3x + b_i, 3x - b_i\}$ which lies in I_{b_i}. (If $x = \pm b_i/2$, x is fixed by T_{b_i}.) Because of the cyclical structure (Fact 2), $T_{b_i}(x)$ for x in I_{b_i} is equal to $F^n_{b_i}(x)$ for some n. Of course n depends on x since the cycles that result from the action of F_{b_i}

might be of different lengths. The important point is that in the directed graph defined by $\mathcal{B}$ there is a path leading from x in I_{b_i} to $T_{b_i}(x)$.

We have $I_{b_1} \subset I_{b_2} \subset I_{b_3}$. Every integer is of course connected by a path to an element of I_{b_3} by repeated application of F_{b_3}. Notice that for x in I_{b_3}, $F_{b_i}(x)$ is in I_{b_3} for $i = 1, 2, 3$. Therefore in determining if every integer leads to 0 the action of the F_{b_i} outside of I_{b_3} will never be used. Furthermore it is clear that any integer can eventually be connected to an element of I_{b_1} by applying F_{b_1} repeatedly, so that for $\mathcal{A}$ to be a JTCS we neeed only find paths to zero from those integers in I_{b_1}.

Consider a triple (b_1, b_2, b_3) in required format as defined above. We will refer to the directed graph created by connecting each x in I_{b_3} to the integers $F_{b_1}(x), F_{b_2}(x)$ and $F_{b_3}(x)$ as the *path system* of the triple (b_1, b_2, b_3). The theorem will be proven using two techniques. We first establish the existence of connections for the path system of a triple (d_1, d_2, d_3) where the d_i are smaller than the b_i. This reduction to a simpler system continues until no further reduction is possible. The reduction method is given by Lemma 2 in the next section, and is proven in Sections 4 through 7. The second technique will explicity find the paths leading to 0 in the system that we have reduced to. We first show there is a path connecting any two nonzero elements of I_{b_1}; this is Lemma 10 in Section 8. Finally, we show in Lemma 18 in Section 9 that there is at least one nonzero element of I_{b_1} which connects to 0. Combined with Lemma 10 this shows all elements of I_{b_1} connect to 0 as needed to prove Theorem 1.

3. Reduction to a smaller system

The key lemma for the reduction part of the proof is:

Lemma 2. *Consider the path system of a triple (b_1, b_2, b_3) in required format. By composing some of the paths from this path system we are able to derive the path system for a triple of numbers (d_1, d_2, d_3) in required format, where $d_1 < b_1$ and $d_3 \geq b_1$. This can be accomplished except when $\boldsymbol{r} = b_1 - d_1$ is divisible by three and $3\boldsymbol{r} < b_1$, in which case the connection from x to $F_{d_3}(x)$ might not exist for some x in I_{d_3}.*

Notice that, since $d_3 \geq b_1$, the map F_{b_1} will connect every element of I_{b_3} to an element in I_{d_3} of the derived path system.We may then apply the lemma a second time, to the derived system. This reduction may continue until we reach a system (c_1, c_2, c_3) for which $c_1 = 1$ (where F_{c_1} will obviously connect every element to $I_{c_1} = \{0\}$) or until we have a path system where $b_1 = d_1 + \boldsymbol{r}$, with $\boldsymbol{r}$ divisible by 3 and $3\boldsymbol{r} < b_1$. We will begin the proof of Lemma 2 in the following section where we explain how to find the numbers (d_1, d_2, d_3).

4. The derived triple

We begin by describing the triple (d_1, d_2, d_3) that will be derived from (b_1, b_2, b_3). Recall a triple of positive integers (d_1, d_2, d_3) is in required format if $(d_1, d_2, d_3) = (1, 1, 2)$ or if $d_1 < d_2 < d_3$ where $d_3 = d_1 + d_2$, the d_i are pairwise relatively prime and not divisible by 3, and $d_1 \equiv d_2 \pmod 3$.

Lemma 3. *Given the triple (b_1, b_2, b_3), write $b_2 = kb_1 + r$ where r and k are positive integers and r is less than b_1. Set $s = b_1 - r$. For the triple (d_1, d_2, d_3) that is derived from (b_1, b_2, b_3) there are two possibilities:*

1) *If neither r nor s is divisible by 3 then $(d_1, d_2, d_3) = (r, s, b_1)$ if $r < s$ and $(d_1, d_2, d_3) = (s, r, b_1)$ if $s < r$. In the case $r = s$, we have $(d_1, d_2, d_3) = (1, 1, 2)$.*

2) *If 3 divides exactly one of $\{r, s\}$ and d_1 represents the element not divisible by 3, $(d_1, d_2, d_3) = (d_1, b_1, d_1 + b_1)$.*

In either case the resulting triple is in required format, $d_1 < b_1$, and $d_3 \geq b_1$.

Remark. The bold $\boldsymbol{r}$ in Lemma 2 will in fact turn out to be either r or s.

Proof. First notice that at most one of r and s is divisible by 3 since their sum is b_1. Therefore there are only two cases as described. The inequalities are obvious, so we need only to prove that the triple (d_1, d_2, d_3) is in required format.

There are only two non-obvious things to check in Case 1. First, the numbers r and s are relatively prime. Since $r + s = b_1$, if any two share a factor then all three do. But if r and b_1 share a factor then b_1 and b_2 do, which is a contradiction. Second, r and s are congruent modulo 3. If they were not, then b_1 would be divisible by 3 which is not the case. Notice that, because they are relatively prime, $r = s$ only if $r = s = 1$ and $b_1 = 2$.

In Case 2, note first that d_1 and b_1 are congruent modulo 3 since $b_1 - d_1$ is the element of $\{r, s\}$ which is divisible by 3. Therefore, none of the numbers in (d_1, d_2, d_3) are divisible by 3. As in Case 1, we know d_1 and b_1 are relatively prime. Therefore d_1, d_2, and d_3 are pairwise relatively prime. The other requirements for the format are obvious. $\square$

5. Arranging the interval

To prove Theorem 1 it will be useful to display the elements of I_{b_3} in a particular array. If x is in I_{b_3} then exactly one of $\{x + b_1, x - b_2\}$ is also in I_{b_3}. This is true because $[-b_3/2, b_3/2]$ and $[x - b_2, x + b_1]$ are both intervals of length b_3 containing x. Therefore one of the endpoints of $[x - b_2, x + b_1]$ lies in $[-b_3/2, b_3/2]$ and, being integral, therefore lies in I_{b_3}. There is one exception: if b_3 is even and $x = (b_2 - b_1)/2$, then both $x + b_1 = b_3/2$ and $x - b_2 = -b_3/2$ are in I_{b_3}.

The array of the elements of I_{b_3} is defined as follows and will be referred to as an array of *Type 1* for the triple (b_1, b_2, b_3). (An example is presented in Figure 3 below.) Begin with the smallest multiple of b_1 in I_{b_3}. Repeatedly add b_1 to this number as long as possible to generate the first column of the array. When you reach an x such that $x + b_1$ is not in I_{b_3} but $x - b_2$ is, use $x - b_2$ to head the second column.

−60	−47	−54	−61	−48	−55	−62	−49	−56	−63	−50
−40	−27	−34	−41	−28	−35	−42	−29	−36	−43	−30
−20	**−7**	−14	−21	**−8**	−15	−22	**−9**	−16	−23	**−10**
0	13	**6**	**−1**	12	**5**	**−2**	11	**4**	**−3**	**10**
20	33	26	19	32	25	18	31	24	17	30
40	53	46	39	52	45	38	51	44	37	50
60			59			58			57	

−57	−44	−51	−58	−45	−52	−59	−46	−53	−60
−37	−24	−31	−38	−25	−32	−39	−26	−33	−40
−17	**−4**	−11	−18	**−5**	−12	−19	**−6**	−13	−20
3	16	**9**	**2**	15	**8**	**1**	14	**7**	**0**
23	36	29	22	35	28	21	34	27	20
43	56	49	42	55	48	41	54	47	40
63			62			61			60

Figure 3. The Type 1 array for $(b_1, b_2, b_3) = (20, 107, 127)$. Elements of $I_{b_1} = [-10, 10] \cap Z$ are in bold.

Begin adding b_1 again to form the second column. Continue the process to generate the remaining columns. In the case where b_3 is even, the number $-b_3/2$ will not appear in the array as it has been described. (When you reach $x = b_3/2$ you subtract b_2 to get $(b_1 - b_2)/2$ at the top of the next column.) In this situation, we will insert $-b_3/2$ above the element $(b_1-b_2)/2$. This does not disrupt the pattern since $(b_1-b_2)/2 = -b_3/2+b_1$.

With this description, all the numbers in I_{b_3} are included in the array exactly once: If x is in the array, then $x+mb_1-nb_2 = x$ implies $mb_1 = nb_2$. Then b_1 divides n and b_2 divides m so that $m+n$ is at least b_3. Therefore the first b_3 elements entered into the array are all different. If we include $-b_3/2$ as noted above in the case where b_3 is even, we see that the array will list each element of I_{b_3} once. Notice that each column of the array contains a single element of I_{b_1}, unless b_1 is even, in which case one column will contain both $-b_1/2$ and $b_1/2$. Therefore there are b_1 columns in the array.

If one continues the pattern after numbers begin to repeat, the first repeated numbers are the numbers in the first column of the array (the multiples of b_1 in I_{b_3}). This is clear since if x is in the first column of the array, then the numbers in column b_1+1 are of the form $x-b_1b_2+nb_1$ for some positive integer n. These are of course themselves multiples of b_1. Thus the array can extended indefinitely to the left or right by continuing the construction. With this in mind we state a clear but oft-used lemma:

Lemma 4. *If x and y are in I_{b_3} and $y = x - jb_2 + nb_1$ for nonnegative integers j and n, then y is j columns to the right of x in the Type 1 array for the triple (b_1, b_2, b_3). (If $j > b_1$ we assume the array has been extended as described in the preceding paragraph.)*

Proof. The proof is obvious. For a given j, there are only certain values of n which give elements of I_{b_3}. The numbers obtained by using such n are precisely the elements which are j columns to the right of x. $\square$

We need to see how elements of I_{b_1} in adjacent columns are related. Let r, s and k be as defined above in Lemma 3.

Lemma 5. *Let x be an element of I_{b_1} in the Type 1 array for the triple (b_1, b_2, b_3). The element of I_{b_1} which lies in the column to the right* *of x is either $x-r$ or $x+s$. (If there are two elements of I_{b_1} in the column to the right of x, they are $x-r$ and $x+s$.)*

PROOF. Since $r+s=b_1$, one of $\{x-r,\ x+s\}$ is in I_{b_1} (by an argument similar to the first paragraph of this section). Moreover, $x-r=x-b_2+kb_1$ and $x+s=x-b_2+(k+1)b_1$ are in the column to the right of x (if they are in I_{b_3}) by Lemma 4. If there are two I_{b_1} elements to the right of x, since $x+s$ and $x-r$ cannot be $-b_1/2$ and $b_1/2$ respectively (since s and r are positive), the two elements must be $x-r=-b_1/2$ and $x+s=b_1/2$. □

Remark. It follows from Lemma 5 and its proof that if you take the I_{b_1} elements from the Type 1 array generated by the triple (b_1,b_2,b_3) and form the Type 1 array corresponding to the triples (r,s,b_1) or (s,r,b_1), elements in adjacent columns in the original array become consecutive elements in the derived array. It may be necessary to reverse the order in which the numbers are written depending on the relative sizes of r and s. See Figure 4.

−7	−6	−5	−4	−10	−9	−8	−7
0	1	2	3	−3	−2	−1	0
7	8	9	10	4	5	6	7

Figure 4. If we take the elements of I_{b_1} from Figure 3 in the order in which they occur, we obtain the array for the triple $(d_1,d_2,d_3)=(7,13,20)$.

We also display the elements of I_{b_3} in an array of *Type 2*, described as follows. Again begin with the smallest multiple of b_1 in I_{b_3}. To generate the first column we subtract b_2 as many times as possible. (Of course that is either once or not at all, since $2b_2>b_3$.) We then start the next column by adding b_1 to the bottom of the first column. Continue by subtracting b_2 if possible, and beginning the next column by adding b_1. Notice each column has only one or two elements. As in the array of Type 1, all elements of I_{b_3} are eventually included, provided we insert $b_3/2$ above the element $(b_1-b_2)/2$ in case b_3 is even. It is clear each column contains exactly one element of I_{b_2} (except possibly when b_2 is even, a single column contains both $\pm b_2/2$). We have the following analog of Lemma 5:

Lemma 6. *Let x be an element of I_{b_2} in the Type 2 array for the triple (b_1, b_2, b_3). Then the element of I_{b_2} in the column to the right of x is either $x + b_1$ or $x + b_1 - b_2$. (If there are two elements of I_{b_2} in the column to the right of x, one of them is either $x + b_1$ or $x + b_1 - b_2$.)*

Proof. The element of I_{b_2} in the column to the right of x is in the form $x + b_1 - tb_2$ for some nonnegative integer t. Clearly $t < 2$ since $x + (b_1 - b_2) - b_2 < x - b_2 \leq -b_2/2$. □

6. Establishing links

Recall the assertion of Lemma 2. Using the paths that exist from x to $F_{b_1}(x)$, $F_{b_2}(x)$, and $F_{b_3}(x)$ for any x in I_{b_3}, we must find paths from x to $F_{d_1}(x)$, $F_{d_2}(x)$, and $F_{d_3}(x)$ for any x in I_{d_3}. Where convenient, we will also use the fact that if x is in I_{b_i}, then x is connected by a path to $T_{b_i}(x)$ (defined in Section 2). This is true because $T_{b_i}(x)$ is $F^n_{b_1}(x)$ for some positive integer n.

Lemma 7. *Assume there are paths from any x in I_{b_3} to the elements $F_{b_i}(x)$ for $i = 1, 2, 3$. Arrange the numbers in I_{b_3} in the array of Type 1 (Type 2) for the triple (b_1, b_2, b_3). Let x be any number in the array which is not divisible by 3. Then there are paths from x to the elements of I_{b_1} (I_{b_2}) lying in the adjacent columns.*

In the case where x is in the far left or right column of the array, one of the adjacent columns will be the column at the opposite end of the array, since the array may be continued ad infinitum after the pattern begins to repeat.

Proof. The proof uses the fact that you can home in on the I_{b_1} or I_{b_2} elements. We show the details for Type 1. Assume $x \equiv b_3 \pmod 3$, the other case being similar. First use the path from x to $F_{b_3}(x) = (x - b_3)/3$. Then connect to $F^j_{b_1}(F_{b_3}(x))$ where j is chosen so that the result is in I_{b_1}. Finally connect to $y = T^{j+1}_{b_1}(F^j_{b_1}(F_{b_3}(x)))$ which is in I_{b_1}. Since F_{b_1} consists of dividing by 3 (after possibly adding or subtracting b_1) and T_{b_1} consists of tripling (followed possibly by adding or subtracting b_1) we see that y is of the form $x - b_3 + t'b_1$. We may rewrite this in the form $x - b_2 + tb_1$ which by Lemma 4 is in the column to the right of x. If we substitute F_{b_2}

for F_{b_3} in the initial step and continue as before, we connect to an element of the form $x+b_2-tb_1$ which must lie in the column to the *left* of x. This works because b_2 and b_3 are opposites modulo 3.

In the case where b_1 is even and a column contains two elements of I_{b_1}, namely $b_1/2$ and $-b_1/2$, we may link to both of these elements. In fact there is a path from $b_1/2$ to $-b_1/2$ given by the following: $b_1/2$ connects to $F_{b_2}(b_1/2)=(b_1/2+b_2)/3$. This uses the fact that $b_1/2$ is opposite to b_1 (and b_2) modulo 3. This connects to $T_{b_3}((b_1/2+b_2)/3)=b_1/2+b_2+tb_3$ where t is the unique number in $\{-1,0,1\}$ that gives an element in I_{b_3}. It is clear that $t=-1$, and we have connected to $-b_1/2$. There is a path back via the same maps.

For the paths to I_{b_2} elements in the Type 2 array, we use similar arguments. The connections are given by the paths from x to $T_{b_2}^{j+1}(F_{b_2}^{j}(F_{b_3}(x)))$ and $T_{b_2}^{j+1}(F_{b_2}^{j}(F_{b_1}(x)))$. Here j is chosen so that $F_{b_2}^{j}$ brings us into I_{b_2}. In the case where b_2 is even we get from $b_2/2$ to $-b_2/2$ (and back) via the maps F_{b_1}, followed by T_{b_3}. $\square$

The proofs that follow are simplified by the following:

Lemma 8. *Assume there are paths from any x in I_{b_3} to the elements $F_{b_i}(x)$ for $i=1,2,3$. Consider the function F_d where d is some integer not divisible by 3. If we have paths from any $x\equiv b_1 \pmod 3$ to $F_d(x)$, then we have paths from any x to $F_d(x)$.*

Proof. If $x\equiv 0 \pmod 3$, then $F_d(x)=x/3=F_{b_1}(x)$. If $x\equiv -b_1 \pmod 3$, then by assumption there is a path from $-x$ to $F_d(-x)$. Since F_d is an odd function, $F_d(-x)=-F_d(x)$. The path from $-x$ to $-F_d(x)$ implies a path from x to $F_d(x)$ since the F_{b_i} are odd functions. $\square$

As an application of the connections described in Lemma 7, we have:

Lemma 9. *Assume there are paths from any x in I_{b_3} to the elements $F_{b_i}(x)$ for $i=1,2,3$. If x is in I_{b_2}, there is a path from x to $F_{2b_1}(x)$.*

Proof. By Lemma 8, we need only find paths from x to $F_{2b_1}(x)$ for $x\equiv b_1 \pmod 3$. Assume we have arrayed the elements of I_{b_3} in the Type 2 array. By Lemma 7, x connects to the I_{b_2} element in the column to its right. By Lemma 6, this element is either $x+b_1$ or $x+b_1-b_2$. In the first case we use the path from $x+b_1$ to $F_{b_1}(x+b_1)=(x+b_1+b_1)/3=F_{2b_1}(x)$.

In the latter case, we connect to $F_{b_3}(x+b_1-b_2) = (x+b_1-b_2+b_3)/3 = (x+2b_1)/3 = F_{2b_1}(x)$. $\square$

7. The proof of Lemma 2

We divide the proof into two cases, depending on the form of the triple (d_1, d_2, d_3). Notation is as in the statement of Lemma 3. We are assuming that for any x in I_{b_3}, we have paths from x to $F_{b_i}(x)$ for $i = 1, 2, 3$. Also assume the elements of I_{b_3} have been arranged in the Type 1 array for the triple (b_1, b_2, b_3).

Case 1. Here r and s are not divisible by 3, d_1 is the smaller of r and s, d_2 is the larger, and d_3 is b_1. This will also include the case where $r = s = 1$ and $b_1 = 2$, i.e. $(d_1, d_2, d_3) = (1, 1, 2)$. An example is found in Figure 3 above where $(b_1, b_2, b_3) = (20, 107, 127)$ and $(d_1, d_2, d_3) = (7, 13, 20)$.

Proof of Lemma 2 in Case 1. Let x be any element in $I_{d_3} = I_{b_1}$. We must show there is a path from x to the images of x under the maps F_{d_i} for $i = 1, 2, 3$. As noted in Lemma 8 above we only need to find such paths in the case where $x \equiv b_1 \pmod 3$. Since $F_{d_3} = F_{b_1}$, we only need to show that there are paths from x to $F_r(x) = (x + r)/3$ and $F_s(x) = (x + s)/3$. (Here we use the fact that r and s are congruent to $-b_1$ modulo 3.)

By Lemma 6 we know there is a path from x to the elements in I_{b_1} which lie in the columns adjacent to x. By Lemma 5, the element of I_{b_1} to the right of x is either $x - r$ or $x + s$. Since we have paths from $x - r$ to $F_{b_1}(x - r) = (x - r + b_1)/3 = (x + s)/3$, and from $x + s$ to $F_{b_1}(x + s) = (x + s)/3$, in either case we have a path from x to $F_s(x)$. It also follows from Lemma 5 that the element of I_{b_1} to the left of x is either $x + r$ or $x - s$. Since we have paths from $x + r$ to $F_{b_1}(x + r) = (x + r)/3$ and from $x - s$ to $F_{b_1}(x - s) = (x - s + b_1)/3 = (x + r)/3$, we have a path from x to $F_r(x)$.

Case 2. Here d_1 is the one of r and s which is not divisible by three, $d_2 = b_1$, and $d_3 = d_1 + b_1$. An example is presented in Figure 5.

Proof of Lemma 2 in Case 2. Let x be in I_{d_3}. We need to show x is connected by paths to $F_{d_1}(x)$, $F_{d_2}(x)$, and $F_{d_3}(x)$. Obviously there is a path from x to $F_{d_2}(x) = F_{b_1}(x)$. The case where x is in I_{b_1} will handled in Part 1 below. In Part 2 we deal with those x in I_{d_3} which are

−44	−29	−36	−43	−28	−35	−42	−27	−34	−41	−26	−33
−22	**−7**	−14	−21	**−6**	−13	−20	**−5**	−12	−19	**−9**	**−11**
0	15	**8**	**1**	16	**9**	**2**	17	**10**	**3**	18	**11**
22	37	30	23	38	31	24	39	32	25	40	33
44			45			46			47		

−40	−47	−32	−39	−46	−31	−38	−45	−30	−37	−44
−18	−25	**−10**	−17	−24	**−9**	−16	−23	**−8**	−15	−22
4	**−3**	12	**5**	**−2**	13	**6**	**−1**	14	**7**	**0**
26	19	34	27	20	35	28	21	36	39	22
	41			42			43			44

Figure 5. The Type 1 array of the triple (22,73,95) with I_{b_1} elements in bold. Here $(d_1, d_2, d_3) = (7, 22, 29)$ which is the array pictured in the third table.

−14	−8	−9	−10	−11	**−12**	**−13**	**−14**
−7	−1	−2	−3	−4	−5	−6	−7
0	6	5	4	3	2	1	0
7	**13**	**12**	11	10	9	8	7
14							**14**

In the derived array, the elements which are not in I_{b_1} (from Part 2 in Case 2 in the proof of Lemma 2) are in bold.

not in I_{b_1}. As noted in Lemma 8, we only need to establish the existence of such paths for $x \equiv b_1 \pmod 3$. For such x, $F_{d_1}(x) = (x - d_1)/3$ and $F_{d_3}(x) = (x + d_3)/3 = (x + d_1 + b_1)/3$.

Part 1: x is in I_{b_1}. For x in I_{b_1} we know we have a path to the elements of I_{b_1} in the adjacent columns. As in Case 1 the possible values of these elements are $x - s$ or $x + r$ on the left and $x + s$ or $x - r$ on the right. Whether r or s is d_1, we see that x connects to $x - d_1$ or $x - d_1 + b_1$ on one side, and to $x + d_1$ or $x + d_1 - b_1$ on the other side. We can then find paths to the images of these elements under F_{b_1} or F_{2b_1} (by Lemma 9, since the elements are in I_{b_1} and hence in I_{b_2}.) The appropriate images are displayed in the table below. Using one side we have a path from x to

$F_{d_1}(x)$ and using the other side a path from x to $F_{d_3}(x)$.

One Side	*Other Side*
$F_{b_1}(x-d_1)=(x-d_1)/3$	$F_{b_1}(x+d_1)=(x+d_1+b_1)/3$
$F_{b_1}(x-d_1+b_1)$ $=(x-d_1+b_1-b_1)/3=(x-d_1)/3$	$F_{2b_1}(x+d_1-b_1)$ $=(x+d_1-b_1+2b_1)/3$ $=(x+d_1+b_1)/3$

Part 2: x is in I_{d_3} but not in I_{b_1}. Since $2b_1 > d_3$, two elements of I_{d_3} cannot differ by $2b_1$. Therefore the I_{b_1} element in the same column as x is either $x+b_1$ or $x-b_1$.

First assume $x+b_1$ is in I_{b_1}. The I_{b_1} elements in the adjacent columns are either $(x+b_1)-s$ or $(x+b_1)+r$ on the left side, and $(x+b_1)+s$ or $(x+b_1)-r$ on the right side. Whether r or s is d_1, we have paths from x to $x+d_1$ or $x+d_3$ on one side, and to $x+b_1-d_1$ or $x+2b_1-d_1$ on the other side. The table displays the connections from there. We have paths from x to $F_{d_3}(x)$ using one column and to $F_{d_1}(x)$ using the other column:

One Side	*Other Side*
$F_{b_1}(x+d_1)=(x+d_1+b_1)/3$ $=(x+d_3)/3$	$F_{b_1}(x+b_1-d_1)=(x+b_1-d_1-b_1)/3$ $=(x-d_1)/3$
$F_{b_1}(x+d_3)=(x+d_3)/3$	$F_{2b_1}(x+2b_1-d_1)$ $=(x+2b_1-d_1-2b_1)/3=(x-d_1)/3$

Next assume $x-b_1$ is in I_{b_1}. Then the I_{b_1} elements in the adjacent columns are either $(x-b_1)-s$ or $(x-b_1)+r$ on the left side and $(x-b_1)+s$ or $(x-b_1)-r$ on the right side. Whether r or s is d_1, we have paths from x to $x-d_3$ or $x-d_1$ on one side, and to $x-b_1+d_1$ or $x-2b_1+d_1$ on the other side. The table displays the connections from there:

One Side	*Other Side*
$F_{b_1}(x-d_3)=(x-d_3+b_1)/3$ $=(x-d_1)/3$	$F_{2b_1}(x-b_1+d_1)$ $=(x-b_1+d_1+2b_1)/3$ $=(x+b_1+d_1)/3$
$F_{b_1}(x-d_1)=(x-d_1)/3$	$X-2b_1+d_1:$ Special Case

We have paths from x to $F_{d_1}(x)$ using one column and to $F_{d_3}(x)$ using the other column, assuming that there are no elements of I_{b_1} of the form $x - 2b_1 + d_1$.

A difficulty arises if we have $x - 2b_1 + d_1$ in I_{b_1} where x is in I_{d_3}. This is a problem since $x - 2b_1 + d_1 \equiv 0 \pmod 3$ and little can be done with it. In fact the existence of such an element forces the reduction process to terminate. The resulting situation is handled in Sections 8 and 9. If $x - 2b_1 + d_1$ is in I_{b_1}, then $x \geq -b_1/2 + 2b_1 - d_1 = 3b_1/2 - d_1$. Since x is in I_{d_3}, we known that $x \leq (b_1 + d_1)/2$. Therefore this problem can only arise if $3b_1/2 - d_1 \leq (b_1 + d_1)/2$. That is, $2b_1 \leq 3d_1$ or, writing $d_1 = b_1 - \boldsymbol{r}$, $b_1 \geq 3\boldsymbol{r}$. As remarked earlier this $\boldsymbol{r}$ is either r or s as used above, and is the one of r or s which is congruent to 0 modulo 3. Since 3 does not divide b_1, we in fact have $b_1 > 3\boldsymbol{r}$. This is the situation referred to in the statement of Lemma 2, which is now proven. □

In the remaining sections of the paper we will assume that the reductions have been applied so that we are now in the special situation just described, where $b_1 = d_1 + \boldsymbol{r}$, 3 divides $\boldsymbol{r}$ and $3\boldsymbol{r} < b_1$ (or equivalently $2\boldsymbol{r} < d_1$). We will no longer emphasize this $\boldsymbol{r}$ since the old r and s will no longer be used.

8. Paths connecting I_{b_1} elements

In the situation that remains we assume we have the path system for the triple (b_1, b_2, b_3) provided by the maps F_{b_i} for $i = 1, 2, 3$. Moreover we have shown in the proof of Lemma 2 that for any x in I_{b_1} there is a path from x to $F_{d_1}(x)$. Here $d_1 = b_1 - r$ where 3 divides r and $3r < b_1$. In this section we will show that there is a path from any nonzero element of I_{b_1} to any other.

We use only the maps F_{d_1} (referred to simply as F) and T_{b_1} (referred to as T). (For a result independent of Lemma 2, it is possible to use $F = F_{b_1}$ without substantial modifications.) If x is in I_{b_1}, recall there is a path from x to $T(x)$.

Let x be any element of I_{b_1} which is not divisible by 3. By Lemma 7, we know that there are paths from x to the elements of I_{b_1} in the adjoining columns, as they appear in the Type 1 array for the triple (b_1, b_2, b_3). As noted in the Remark of Section 5, if we write the elements of I_{b_1} in their own Type 1 array, given by the triple (r, d_1, b_1), we see that x connects to the elements adjacent to it (one step above or below it). If x is at the bottom of a column, x will connect to the number at the top of the column on the right. If x is at the top of a column, x will connect to the number at the bottom of the column on the left. In this section we will work only with the Type 1 array for the triple (r, d_1, b_1). An example is presented in Figure 6 at the end of this section.

What makes this case work is fact that any two elements in a column are congruent modulo 3 since they differ by a multiple of r. Assuming the array begins as usual with the column of multiples of r, elements of the second column will be congruent to $-d_1$ modulo 3, elements of the third column will be congruent to $-2d_1 \equiv d_1 \pmod 3$, elements of the fourth column are 0 modulo 3 and so on. In particular, we see that all the elements in columns 2 and 3 are connected to each other and connect to the bottom of column 1 and the top of column 4. We have a similar result for the elements of columns 5 and 6, columns 8 and 9, etc. These pairs of columns, consisting of numbers not divisible by 3 which are joined together by paths, will be called *blocks*. Columns 2 and 3 will be referred to as block 1, and so on to the right. Within a block, elements in the left column are congruent to $-d_1$ modulo 3 and elements in the right column are congruent to d_1 modulo 3. Since there are r columns in the array, there are $r/3$ blocks.

The main result of this section is the following:

Lemma 10. *There is a path from any nonzero element of I_{b_1} to any other.*

It will suffice to show that there is a path from an element of any one block to an element of any other block. This will demonstrate that all numbers not divisible by three will be connected by paths. That will imply that *any* two numbers will be connected (since if 3 divides x, there is a path from x to $F(x) = x/3$ and a path back from $x/3$ to $T(x/3) = x$). We begin our proof with a computational lemma.

Lemma 11. 1) *If z is at the top of a column in the array, $T(z) = 3z + b_1$. If z is at the bottom of a column then $T(z) = 3z - b_1$.*

2) *If 3 divides some number a, then $F(x + a) = F(x) + a/3$.*

Proof. 1) Recall $T(z)$ is the single element of $\{3z, 3z - b_1, 3z + b_1\}$ which is in I_{b_1}. (If b_1 is even, recall that T fixes both $b_1/2$ and $-b_1/2$.) If

z is at the top of the column, $z - r$ is not in I_{b_1} and so $z < -b_1/2 + r$. Then $3z < -3b_1/2 + 3r < -b_1/2$, since $3r < b_1$. Therefore $T(z) = 3z + b_1$. The proof is similar for z at the bottom of a column. Part 2 is obvious. □

Let M be the positive integer such that 3^M divides r and 3^{M+1} does not. A block B is said to be of *depth k* if there is a pair of numbers (x, y) in B where x is in the left column of B (hence congruent to $-d_1$ modulo 3), and y is in the right column (hence congruent to d_1 modulo 3), such that $F^i(x) \equiv -d_1 \pmod 3$ and $F^i(y) \equiv d_1 \pmod 3$ for $i = 1, \ldots, k$. *Saying B is of depth k does not imply it is not of a higher depth as well.* F^0 will denote the identity map. We have the following lemma:

Lemma 12. 1) *If x and $x' = x + tr$ are in the same column, $F^i(x') = F^i(x) + tr/3^i$ for $i = 0, \ldots, M$. We have $F^i(x') \equiv F^i(x) \pmod 3$ for $i = 0. \ldots, M - 1$.*

2) *If x and $x' = x \pm 3^i d_1 + tr$ are 3^i columns apart $(0 < i \leq M)$, then $F^j(x') = F^j(x) \pm 3^{i-j} d_1 + tr/3^j$ for $j = 0, \ldots, i$. We have $F^j(x') \equiv F^j(x)$ $\pmod 3$ for $j = 0, \ldots, i - 1$. (Here t is some integer. If x' is outside the array, we extend the array using the repetition of the pattern.)*

Proof. We apply Lemma 11 (2) repeatedly to prove both assertions. □

An important consequence of Lemma 12 (1) is that it does not matter which pair (x, y) you select from the block B in order to establish that it has depth k, provided k is less than M. Therefore we will always assume (x, y) is a *middle pair* of B, that is, x is at the bottom of the left column and $y = x - d_1$ is at the top of the right column. (In the special case $x = b_1/2$, we will still use $y = x - d_1$ though it is the second number in its column.) As a result of the following lemma, we will not need to compute depths of M or more, so that the middle pair will always suffice.

Lemma 13. *If B is of depth $M - 1$, then B is of depth M.*

Proof. Let (x, y) be the middle pair of B. Then $F^i(x) \equiv -d_1$ $\pmod 3$ and $F^i(y) \equiv d_1 \pmod 3$ for $i = 1, \ldots, M - 1$. For any $x' = x + tr$ in the same column as x, we have $F^i(x') \equiv F^i(x) \pmod 3$ for $i = 1, \ldots, M-1$ and $F^M(x') = F^M(x) + tr/3^M$ by Lemma 12. Now if $F^M(x)$ is not congruent to $-d_1$ modulo 3, then $F^M(x')$ will be, for either $t = -1$ or -2. This is because $r/3^M$ is not divisible by 3. (Recall x is at the bottom of the column. Because $2r < d_1$ there are at least three elements in any column.) One proceeds in the same way with y, using the fact that y is at the top of the column, and choosing $t = 1$ or 2 if necessary, for $y' = y + tr$. (In the exceptional case where y is the second element, t is either 1 or -1). This allows us to find a pair (x', y') which demonstrates that B has depth M. $\square$

Lemma 14. *If B is of depth k and not of depth $k+1$ (where $k < M-1$ by Lemma 13), let (x, y) be the middle pair of B. Then $(F^i(x), F^i(y))$ is a middle pair for $i = 1, \ldots, k$. The pair $(F^{k+1}(x),\ F^{k+1}(y))$ is congruent modulo 3 to either $(0, -d_1)$ or $(d_1, 0)$.*

Proof. We know $(F^i(x),\ F^i(y))$ is congruent modulo 3 to $(-d_1, d_1)$ for $i = 1, \ldots, k$. Notice that $F(y) = (y - d_1)/3 = (x - 2d_1)/3 = (x + d_1)/3 - d_1 = F(x) - d_1$. Therefore, $F(x)$ is at the bottom of a column and $F(y)$ at the top of the column to its right. Because $(F(x), F(y))$ is congruent modulo 3 to $(-d_1, d_1)$, $(F(x)F(y))$ is a middle pair of some (other) block. We can continue to apply F and see that each $(F^i(x), F^i(y))$ is again a middle pair for $i = 1, \ldots, k$. By assumption, $(F^{k+1}(x), F^{k+1}(y))$ is not congruent modulo 3 to $(-d_1, d_1)$. Since $(F^k(x),\ F^k(y))$ is a middle pair, $F^{k+1}(y) = F^{k+1}(x) - d_1$. Therefore $F^{k+1}(x)$ is at the bottom of a column and $F^{k+1}(y)$ is at the top of the column to its right. This proves the lemma. $\square$

If B is as described in Lemma 14, with $(F^{k+1}(x),\ F^{k+1}(y))$ congruent modulo 3 to $(0, -d_1)$ we will say B is of *maximal depth k rightwards*. If $(F^{k+1}(x), F^{k+1}(y))$ is congruent modulo 3 to $(d_1, 0)$ we will say B is of *maximal depth k leftwards*. By Lemma 13, we presume $k < M - 1$. We give the obvious interpretation to "maximal depth 0 (rightwards or leftwards)". *Every* block has "depth 0" and "maximal depth 0" means "not depth 1".

The following lemma establishes the depth pattern of the various blocks.

Lemma 15. 1) *If B is of depth k $(k < M)$ then so are the blocks 3^{k+1} columns from B.*

2) *Two blocks of depth k $(k < M)$ are a multiple of 3^{k+1} columns apart.*

3) *The blocks of depth $M - 1$ are all 3^M columns apart.*

4) *Assume $k < M-1$. If B is of maximal depth k leftwards (rightwards), then the block 3^{k+1} columns to the left (right) of B is of depth $k+1$.*

Proof. 1) Let (x, y) be the middle pair for B, a block of depth k. Let B' be a block 3^{k+1} columns away from B. Let (x', y') be the middle pair of B' with $x' = x \pm 3^{k+1}d_1 + tr$ and $y' = y \pm 3^{k+1}d_1 + tr$. By Lemma 12, $F^j(x') \equiv F^j(x) \pmod 3$ and $F^j(y') \equiv F^j(y) \pmod 3$ for $j = 1, \dots, k$. This shows that B' has depth k.

2) Let B and B' be blocks n columns apart, of depth k with middle pairs (x, y) and $(x' = x + a, y' = y + a)$, where $a = \pm nd_1 + tr$. We know $F^j(x') \equiv F^j(x) \equiv -d_1 \pmod 3$ and $F^j(y') \equiv F^j(y) \equiv d_1 \pmod 3$ for $j = 0, \dots, k$. Repeatedly apply this argument starting with $j = 0$: Since $F^j(x') = F^j(x) + a/3^j$, and $F^j(x') \equiv F^j(x) \pmod 3$, we have that 3 divides $a/3^j$. Then by Lemma 11 (2) we have $F^{j+1}(x') = F^{j+1}(x) + a/3^{j+1}$. In the final step, $j = k$ and we get have $F^{k+1}(x') = F^{k+1}(x) + a/3^{k+1}$. Since 3^{k+1} divides a and r, 3^{k+1} must divide n.

3) Follows from 1) and 2).

4) We demonstrate the "leftwards" case. Let (x, y) be the middle pair of B. Let B' be the block which is 3^{k+1} columns to the left of B. Consider (x', y'), the middle pair of B' with $x' = x+3^{k+1}d_1+tr$ and $y' = y+3^{k+1}d_1+tr$. By 1) B' is of depth k. By the "leftwards" part of the assumption we know that $F^{k+1}(x) \equiv d_1 \pmod 3$ and $F^{k+1}(y) \equiv 0 \pmod 3$. Then by Lemma 12, $F^{k+1}(x') = F^{k+1}(x) + d_1 + tr/3^{k+1}$ and $F^{k+1}(y') = F^{k+1}(y) + d_1 + tr/3^{k+1}$. Because 3^{k+2} divides r, we see that $F^{k+1}(x') \equiv 2d_1 \equiv -d_1 \pmod 3$ and $F^{k+1}(y') \equiv d_1$ modulo 3. In other words B' has depth $k+1$. □

We now see how the depth determines the path connections.

Lemma 16. 1) *Let x in I_{b_1} be such $F^i(x) \equiv -d_1 \pmod 3$ for $i = 0, \dots, k$. Then there is a path from x to an element 3^k columns to the left of x.*

2) *Let y in I_{b_1} be such $F^i(y) \equiv d_1 \pmod 3$ for $i = 0, \dots, k$. Then there is a path from y to an element 3^k columns to the right of y.*

3) *If B is of depth k with $k \le M$, there are paths from B to the blocks 3^k columns away on either side of B.*

Proof. 1) Assume x is in the left column of a block B, where $F^i(x) \equiv -d_1 \pmod 3$ for $i = 1, \dots, k$. The proof is by induction on k. If $k = 1$, we know $F(x) \equiv -d_1 \pmod 3$ and is therefore in the left column of its block. There is then a path from $F(x)$ to the element at the bottom of the column to its left (a column of numbers divisible by 3). Write this element as $F(x) + d_1 - jr = (x + d_1)/3 + d_1 - jr$. By Lemma 11, $T(F(x) + d_1 - jr) = (x + d_1) + 3d_1 - 3jr - b_1 = x + 3d_1 - (3j + 1)r$. We therefore have a path from x to the block three columns to the left.

Now assume the inductive hypothesis. Let x be such that $F^i(x) \equiv -d_1 \pmod 3$ for $i = 1, \ldots, k$, with $k > 1$. We know that $F(x)$ connects to $F(x)+3^{k-1}d_1-jr$ for some j by the inductive hypothesis. This is congruent modulo 3 to $F(x)$, i.e. to $-d_1$ modulo 3. We may thus assume it is at the bottom of its column. Then x connects to $T(F(x) + 3^{k-1}d_1 - jr) = (x + d_1) + 3^k d_1 - 3jr - b_1 = x + 3^k d_1 - (3j + 1)r$ as needed.

2) The proof is similar to 1) but uses tops of columns instead of bottoms.

3) This follows by applying 1) and 2) to a pair (x, y) that establishes the depth of B. □

Lemma 17. 1) *Let $k < M - 1$. If B is of maximal depth k leftwards (rightwards) then there is a path from B to the block 3^{k+1} columns to the left (right) of B. That block has depth $k + 1$. There is also a path back.*

2) *If B is of depth $M - 1$, there are paths to the blocks 3^M columns on either side of B (and paths back).*

Proof. 1) We prove the case of the "leftward" connection. The proof is by induction. Assume $k = 0$. If (x, y) denotes the middle pair for B of maximal depth 0 leftwards, then $F(x)$ is congruent to d_1 modulo 3 by definition. We connect $F(x)$ to the element at the bottom of the column to its left (a column of elements congruent to $-d_1$ modulo 3 in the same block as $F(x)$). Write this element as $F(x) + d_1 - jr = (x + d_1)/3 + d_1 - jr$. Then by Lemma 11, $T(F(x) + d_1 - jr) = (x + d_1) + 3d_1 - 3jr - b_1 = x + 3d_1 - (3j + 1)r$. We can therefore find a path from x to the block 3 columns away to the left of x.

Now assume the inductive hypothesis and let B be maximal depth k (leftwards) with $k > 0$. Let (x, y) be the middle pair of B. Let B' denote the block containing $(F(x), F(y))$ as its middle pair (Lemma 14). Then B' is of maximal depth $k - 1$ (leftwards). By the inductive hypothesis, $F(x)$ connects to $F(x) + 3^k d_1 - jr$ for some j. This is congruent modulo 3 to $F(x)$, i.e. to $-d_1$ modulo 3. We may thus assume it is at the bottom of its column. Then x connects to $T(F(x)+3^k d_1 - jr) = (x+d_1)+3^{k+1}d_1 - 3jr - b_1 = x+3^{k+1}d_1-(3j+1)r$ as needed. By Lemma 15, the block 3^{k+1} columns to the left of B has depth $k+1$. By Leamma 16 (3), there is a path back to B.

2) By Lemma 13, B is of depth M. By Lemma 16 (3) there are paths from B to the blocks 3^M columns on either sider of B. By Lemma 15 (3), these blocks are of depth $M-1$, and therefore of depth M. Then there are paths back to B. □

We are finally ready to prove Lemma 10.

Proof of Lemma 10. By Lemma 17 (1), there is a path from any block of "maximal depth k" to a block of greater depth. There is also a path back. Thus there are paths from all blocks to and from the blocks of depth $M-1$. By Lemma 15 (3) and Lemma 17 (2) the blocks of depth $M-1$ are all connected. □

9. Elements of I_{b_1} connecting to 0

In this section we will show:

Lemma 18. *Given the path system for the triple* (b_1, b_2, b_3), *there is at least one nonzero element of* I_{b_1} *connected by a path to 0.*

The proof is divided into three cases depending on how b_2 is expressed in terms of b_1. In this section we assume only that we have the path system on I_{b_3} corresponding to the maps F_{b_1}, F_{b_2}, and F_{b_3} referred to as F_1, F_2 and F_3. Recall also that if x is in I_{b_i} there is a path from x to $T_i(x)$ where T_i is the inverse of F_i on I_{b_i}. By Lemma 9, if x is in I_{b_2} there is a path from x to $F_{2b_1}(x)$.

In each case we will use the simple fact that *any multiple of* b_1 *is connected by a path to* 0 (under the action of F_1). Notice r as used in the remainder of the paper is not necessarily the r from the preceding sections. In particular r may be negative in Case 1.

Case 1. $b_2 = nb_1 + r$ where r in I_{b_1} is not divisible by 3 and n is a positive integer.

Proof of Case 1. If $r \equiv b_1 \pmod 3$, then $F_2(r) = (r-b_2)/3 = -nb_1/3$. If $r \equiv -b_1 \pmod 3$, then $F_3(r) = (r-b_3)/3 = -(n+1)b_1/3$. In either case, r in I_{b_1} is connected by a path to 0. □

	A			B			C			D	
−54	−55	−56	−57	−58	−59	−60	−61	−62	−63	−64	−65
0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11
54	53	52	51	50	49	48	47	46	45	44	43
	max	0 R		max	1 R		max	0 L		max	0 R

	E			F			G			H	
−66	−67	−68	−69	−70	−71	−72	−73	−74	−75	−76	−77
−12	−13	−14	−15	−16	−17	−18	−19	−20	−21	−22	−23
42	41	40	39	38	37	36	35	34	33	32	31
	max	2 R		max	0 L		max	0 R		max	1 L

	I			J			K			L	
−78	−79	−80	−81	−28	−29	−30	−31	−32	−33	−34	−35
−24	−25	−26	−27	26	25	24	23	22	21	20	19
30	29	28	27	80	79	78	77	76	75	74	73
	max	0 L	81	max	0 R		max	1 L		max	0 L

	M			N			O			P	
−36	−37	−38	−39	−40	−41	−42	−43	−44	−45	−46	−47
18	17	16	15	14	13	12	11	10	9	8	7
72	71	70	69	68	67	66	65	64	63	62	61
	max	0 R		max	2 L		max	0 L		max	0 R

	Q			R		
−48	−49	−50	−51	−52	−53	−54
6	5	4	3	2	1	0
60	59	58	57	56	55	54
	max	1 L		max	0 L	

Figure 6. The triple $(r, d_1, b_1) = (54, 109, 163)$. $M = 3$ in this example. "Max 1 R" signifies maximum depth 1 rightwards. By Lemma 13, a block of depth 2 is also of depth 3. Block E might seem, as marked, of maximum depth 2 rightwards if we use the middle pair $(41, -68)$. If however we use the pair $(-67, -14)$ the depth is 3. Below the array we show the connections between blocks. The lines represent equivalence, i.e. paths going in both directions. You see the depth 3 connection between blocks E and N.

Case 2. $b_2 = b_1 + r$ where r in I_{b_1} is divisible by 3.

Proof of Case 2. Of course r is positive here. Consider the sequence $\{c_k\}$ defined by $c_1 = b_1$, $c_k = 3c_{k-1} - b_3$. We have that $c_k < c_{k-1}$ iff $c_{k-1} < b_3/2$. Since $b_1 < b_3/2$ it follows that the sequence is decreasing. Notice also that $F_3(c_k) = c_{k-1}$. Therefore there is a path from any element of the sequence to b_1 and hence to 0. Let c_j be the last positive number in the sequence. If either c_j or c_{j+1} is in I_{b_1} then we are done. Otherwise consider $c_{j+1} + b_1$. Since $F_2(c_{j+1} + b_1) = (3c_j - b_3 + b_1 + b_2)/3 = c_j$, we also have a path from $c_{j+1} + b_1$ to zero. Since $c_j > b_1/2$ and $c_{j+1} < -b_1/2$ by assumption, we have

$$b_1/2 = -b_1/2 + b_1 > c_{j+1} + b_1 = 3c_j - b_2 > 3b_1/2 - b_2 = b_1/2 - r \geq 0$$

so that $c_{j+1} + b_1$ is in I_{b_1} as needed. □

Case 3. $b_2 = nb_1 \pm r$ where r in I_{b_1} is divisible by 3. Here $n > 1$ and r is positive. (Possible values of n are then $4, 7, 10$, etc., because b_1 and b_2 are congruent modulo 3.)

Proof of Case 3. We use the sequence of elements $\{b_1 - r, b_1 - 3r, \dots, b_1 - 3^s r\}$ where s is the unique positive integer such that $b_1 - 3^s r$ is in I_{b_1}: $b_1 - 3^s r$ is in $[-b_1/2, b_1/2]$ if r is in $[b_1/(2\cdot 3^s), b_1/(2\cdot 3^{s-1})]$. This must hold for some $s \geq 1$ and s is unique since the endpoints of the interval are not integral.

Each element of the sequence is in $[-b_1/2, b_1 - r]$. Since $b_1 < b_2/2$, they are all elements of I_{b_2}. Then $F_{2b_1}(b_1 - 3^t r) = (b_1 - 3^t r + 2b_1)/3 = b_1 - 3^{t-1} r$, so that there is a path from each element to the term preceding it. To finish the proof notice that in the case $b_2 = nb_1 + r$, we have $F_3(b_1 - r) = (b_1 - r + b_3)/3 = (2b_1 - r + nb_1 + r)/3 = (2 + n)b_1/3$. In case $b_2 = nb_1 - r$, we have $F_2(b_1 - r) = (b_1 - r - b_2)/3 = (b_1 - r - nb_1 + r)/3 = (1 - n)b_1/3$. □

With this final case we have completed the proof of Theorem 1.

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