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Quasilinear system of equations with quadratic growth in unbounded domains

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Dedicated to Professor Kátai Imre on his 50th birthday

Introduction

The aim of this paper is to generalize the results of [4], i.e. to obtain some existence results for certain quasilinear system of differential equations. The authors of paper [4] considered the following equation

$$-\sum_{i,j=1}^{n} D_i[a_{ij}(x,u)D_j u] + f(x,u,Du) = 0$$

in an unbounded domain of $\mathbf{R}^n$ with a Dirichlet boundary condition. Here, the coefficients a_{ij} and f satisfy the Carathéodory conditions and besides that the nonlinearity f satisfies two types of assumptions (assumptions (i), (ii) resp.) having quadratic growth in its third variable:

(i) $$f(x,\eta,\xi) = a_0(x)\eta + g(x,\eta,\xi)$$

where

$$|g(x,\eta,\xi)| \leq \varrho(x) + b(|\eta|)\left[k(x)|\xi| + |\xi|^2\right],$$
$$\alpha_0 \leq a_0(x) \leq \beta_0, \quad \alpha_0 > 0,$$

and $b : \mathbf{R}^+ \to \mathbf{R}^+$ is an increasing function, $\varrho \in L^2(\Omega) \cap L^\infty(\Omega)$, $k \in L^p(\Omega) \cap L^\infty(\Omega)$ for some $1 \leq p < +\infty$;

(ii) $$|f(x,\eta,\xi)| \leq b(|\eta|)\left[h(x) + k_1|\xi| + k_2|\xi|^2\right], h \in L^1(\Omega) + L^2(\Omega)$$

$k_1, k_2 \in \mathbf{R}^+$, and b is as in (i).

It is to be mentioned that nonlinear equations in unbounded domains with strongly nonlinear lower order terms have been considered e.g. in [8] – [11] (see also the references there).

In [2] there has been considered a system of equations in a bounded domain of $\mathbf{R}^n$. In this work the nonlinear term may have also quadratic growth with respect to the gradient. In the present paper, an analogous system will be treated in an unbounded domain of $\mathbf{R}^n$, and the conditions on the equations are in some sense more special than in [2] and [4]. We shall consider the following system of n equations:

$$(0.1)\qquad \begin{gathered} -\sum_{i,j=1}^{n} D_i[a_{ij}(x,u)D_j u^\nu] + a_0^\nu u^\nu + f^\nu(x,u,Du) = 0 \quad \text{in} \\ \Omega \subset \mathbf{R}^n,\ \nu = 1,2,\dots,M, \quad u = 0 \quad \text{on} \quad \partial\Omega, \\ u = (u^1,\dots,u^M). \end{gathered}$$

1. Assumptions

We define a vector valued Sobolev space $H_0^1(\Omega,\mathbf{R}^M)$ by

$$H_0^1(\Omega,\mathbf{R}^M) := \overset{1}{H_0^1}(\Omega)x \times \dots \times \overset{M}{H_0^1}(\Omega) \quad \text{with the norm}$$

$$\|u\|_{H_0^1(\Omega,\mathbf{R}^M)} := \left\{\sum_{\nu=1}^{M} \|u^\nu\|_{H_0^1(\Omega)}\right\}^{\frac{1}{2}}, \quad \text{where} \quad u = (u^1,\dots,u^M),$$

$H_0^1(\Omega)$ denotes the usual Sobolev space which can be obtained as the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_{H_0^1(\Omega)} = \left\{\sum_{j=1}^{n} \|D_j u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}},$$

similarly

$$L^\infty(\Omega,\mathbf{R}^M) := \overset{1}{L^\infty}(\Omega) \times \dots \times \overset{M}{L^\infty}(\Omega) \quad \text{with the norm}$$

$$\|u\|_{L^\infty(\Omega,\mathbf{R}^M)} := \left\{\sum_{\nu=1}^{M} \|u^\nu\|_{L^\infty(\Omega)}\right\}^{\frac{1}{2}}.$$

Now we formulate assumptions on a_{ij} and f. Let the coefficients $a_{ij} : \Omega \times \mathbf{R}^n \to \mathbf{R}$, $(1 \le i,\ j \le n)$ be functions satisfying the Carathéodory conditions, i.e. $a_{ij}(x, \eta)$ are measurable in x for all η belonging to $\mathbf{R}^M$ and continuous in η, for a.e. fixed x in $\mathbf{R}^n$. $a_0^\nu : \Omega \to \mathbf{R}$ is a measurable function. Next, we suppose that

$$(A_1)\quad \begin{cases} \exists \alpha > 0,\ \alpha_0 > 0 \text{ and } \beta_0 > 0 \text{ such that for a.e. } \ x \text{ in } \Omega, \\ \forall \eta \in \mathbf{R}^M, \quad \forall \xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n \\ \sum\limits_{i,j=1}^{n} a_{ij}(x,\eta)\xi_i\xi_j \ge \alpha|\xi|^2, \quad \alpha_0 \le a_0^\nu(x) \le \beta_0 \quad (\nu = 1, \dots, M); \end{cases}$$

$$(A_2)\qquad \begin{cases} \exists \beta > 0 \quad \text{such that for a.e. } x \text{ in } \Omega, \quad \forall \eta \in \mathbf{R}^M, \\ |a_{ij}(x,\eta)| \le \beta. \end{cases}$$

Furthermore, we consider the nonlinear function f^ν defined as follows:

$$f^\nu(x,u,Du) := f^{0\nu}(x,u,Du) + Q(x,u,Du)Du^\nu, \quad 1 \le \nu \le M.$$

Function $f^{0\nu} : \Omega \times \mathbf{R}^M \times \mathbf{R}^{Mn} \to \mathbf{R}$ satisfies the Carathéodory conditions and verifies majoration

$$(A_3)\qquad \begin{cases} \text{for a.e.} \quad x \in \Omega, \forall \eta \in \mathbf{R}^M, \quad \forall \Xi (= (\xi^1, \dots, \xi^M)) \in \mathbf{R}^{Mn}, \\ |f^{0\nu}(x,\eta,\Xi)| \le \varrho_1(x) + b(|\eta|)k_1(x)|\xi^\nu|. \end{cases}$$

The coefficient $Q : \Omega \times \mathbf{R}^M \times \mathbf{R}^{Mn} \to \mathbf{R}^n$ also satisfies the Carathéodory conditions and verifies majorations

$$(A_4)\qquad \begin{cases} \text{for a.e. } x \in \Omega, \quad \forall \eta \in \mathbf{R}^M, \quad \forall \Xi \in \mathbf{R}^{Mn}, \\ |Q(x,\eta,\Xi)| \le b(|\eta|)\varrho_2(x)[1 + |\Xi|_{\mathbf{R}^{Mn}}]. \end{cases}$$

$\varrho_1, \varrho_2 \ge 0$ and belong to the space $L^2(\Omega) \cap L^\infty(\Omega)$. The function $k_1 \ge 0$ belongs to the space $L^p(\Omega) \cap L^\infty(\Omega)$ for some $p > 2$, b is a positive monotone function from $\mathbf{R}^+$ to $\mathbf{R}^+$.

Remark 1. The quasilinear system (0.1) is of particular type. The principal part has diagonal from and the term f^ν is a sum of two terms as mentioned above, such that f^ν grows quadratically in its third variable (grad u).

Remark 2. We observe that the hypotheses of [4] are more general in comparison to the hypotheses $(A_3), (A_4)$ for a single equation. At the same time, the hypotheses considered for the system of equations in section 4 of [2] are quite analogous except that the domain Ω in our case is unbounded and so the constants can't belong to any $L^p(\Omega)$. Therefore, we made the restriction that k_1 belongs to $L^p(\Omega) \cap L^\infty(\Omega)$.

2. Existence theorem

Theorem 2.1. *Under the assumptions $(A_1)-(A_4)$ there exists a function u belonging to $H_0^1(\Omega, \mathbf{R}^M) \cap L^\infty(\Omega, \mathbf{R}^M)$ such that*

$$\sum_{i,j=1}^{n} \langle a_{ij}(x,u) D_j u^\nu, D_i v^\nu \rangle + \langle a_0^\nu u^\nu, v^\nu \rangle = -\langle f^\nu, v^\nu \rangle \quad i.e.$$

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,u) D_j u^\nu D_i v^\nu dx + \int_\Omega a_0^\nu u^\nu v^\nu dx = \tag{2.1}$$

$$= -\int_\Omega f^\nu(x,u,Du) v^\nu dx\,, \quad \nu = 1, 2, \dots, M$$

holds for all $v = (v^1, \dots, v^M)$ belonging to the space $H_0^1(\Omega, \mathbf{R}^M) \cap L^\infty(\Omega, \mathbf{R}^M)$.

To prove the theorem we proceed as follows: let x_0 be a fixed point of $\mathbf{R}^n$, $B(x_0, \mu) = \{x \in \mathbf{R}^n : |x - x_0| < \mu\}$ a ball of radius $\mu \in \mathbf{N}$, and consider the following problem in $\Omega_\mu = \Omega \cap B(x_0, \mu)$:

$$\begin{cases} u_\mu \in H_0^1(\Omega_\mu, \mathbf{R}^M) \cap L^\infty(\Omega_\mu, \mathbf{R}^M)\,, \quad u_\mu = (u_\mu^1, \dots, u_\mu^M)\,, \\ \sum_{i,j=1}^{n} \int_{\Omega_\mu} a_{ij}(x, u_\mu) D_j u_\mu^\nu D_i v^\nu dx + \int_{\Omega_\mu} a_0^\nu u_\mu^\nu v^\nu dx + \\ \quad + \int_{\Omega_\mu} f^\nu(x, u_\mu, Du_\mu) v^\nu dx = 0\,, \quad 1 \le \nu \le M\,; \\ \forall v \in H_0^1(\Omega_\mu, \mathbf{R}^M) \cap L^\infty(\Omega_\mu, \mathbf{R}^M). \end{cases} \tag{2.2}$$

Since Ω_μ is bounded, we can use the results of [2], which proves the existence of solutions u_μ of (2.2) such that

$$\|u_\mu^\nu\|_{L^\infty(\Omega_\mu)} \le \|\varrho_1\|_{L^\infty(\Omega_\mu)} / \alpha_0 \tag{2.3}$$

holds. Further, we suppose that $\bar{u}_\mu$ is the zero extension of u_μ outside Ω_μ. Thus $\bar{u}_\mu \in H_0^1(\Omega, \mathbf{R}^M)$ and

$$\|\bar{u}_\mu^\nu\|_{L^\infty(\Omega)} \le \|\varrho_1\|_{L^\infty(\Omega)} / \alpha_0 \tag{2.4}$$

Then for all $v \in H_0^1(\Omega, \mathbf{R}^M) \cap L^\infty(\Omega, \mathbf{R}^M)$ with compact support supp $v \subset \Omega_\mu$ if $\mu \ge \mu_0$ and we have

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x, \bar{u}_\mu) D_j \bar{u}_\mu^\nu D_i v^\nu dx + \int_\Omega a_0^\nu \bar{u}_\mu^\nu v^\nu dx =$$

$$= -\int_\Omega f^{0\nu}(x, \bar{u}_\mu, D\bar{u}_\mu) v^\nu dx - \int_\Omega Q(x, \bar{u}_\mu, D\bar{u}_\mu) D\bar{u}_\mu^\nu v^\nu dx\,. \tag{2.5}$$

We simplify the notations: let $C_1 := b(\sqrt{M}\|\varrho_1\|_{L^\infty(\Omega)}/\alpha_0)$. From (2.4) we have $\|\bar{u}_\mu\|_{L^\infty(\Omega,\mathbf{R}^M)} \le \sqrt{M}\|\varrho_1\|_{L^\infty(\Omega)}/\alpha_0$. Moreover, define $\bar{U}_\mu := |\bar{u}_\mu|^2_{\mathbf{R}^M} = \sum_{\nu=1}^{M}(\bar{u}^\nu_\mu)^2$, then $D_j\bar{U}_\mu = 2\sum_{\nu=1}^{M}\bar{u}^\nu_\mu D_j\bar{u}^\nu_\mu$. Since $|D\bar{u}_\mu|^2_{\mathbf{R}^{Mn}} = \sum_{\mu=1}^{M}|D\bar{u}^\nu_\mu|^2_{\mathbf{R}^n}$ thus $|D\bar{u}^\nu_\mu| \le |D\bar{u}_\mu|$. Let us consider the test function $v^\nu := \bar{v}^\nu_\mu := E_\mu\bar{u}^\nu_\mu$, where $E_\mu = \exp\{t\bar{U}_\mu\}$, $C_2 := \sup E_\mu$ and $t := 2C_1^2\|\varrho_2\|^2_{L^\infty(\Omega)}/\alpha^2$. The derivatives of the function $\bar{v}^\nu_\mu$ are

$$D_i\bar{v}^\nu_\mu = E_\mu D_i\bar{u}^\nu_\mu + E_\mu t\bar{u}^\nu_\mu D_i\bar{U}_\mu\,.$$

Now we are going to prove two Lemmas.

Lemma 2.1. *The solutions $\bar{u}_\mu$ are bounded with respect to the norm of $H^1_0(\Omega,\mathbf{R}^M)$ which will follow from the inequality*

$$(2.6)\quad \|\bar{u}_\mu\|^2_{H^1_0(\Omega,\mathbf{R}^M)} \le \text{const}\left[\|\bar{u}_\mu\|^{2\theta}_{L^2(\Omega,\mathbf{R}^M)} + \int_\Omega(\varrho_1)^2dx + \int_\Omega(\varrho_2)^2dx\right]$$

where θ is some number satisfying $0<\theta<1$.

PROOF. Using the test function defined above, and summing over ν the ν–th equation (2.5), we obtain:

$$\sum_{\nu=1}^{M}\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x,\bar{u}_\mu)D_j\bar{u}^\nu_\mu D_iv^\nu_\mu + \sum_{\nu=1}^{M}\int_\Omega a^\nu_0\bar{u}^\nu_\mu v^\nu_\mu dx =$$
$$= -\sum_{\nu=1}^{M}\int_\Omega f^\nu(x,\bar{u}_\mu,D\bar{u}_\mu)v^\nu_\mu dx\,.$$

Consequently

$$(2.6')\qquad \begin{aligned}&\sum_{\nu=1}^{M}\sum_{i,j=1}^{n}\int_\Omega E_\mu a_{ij}(x,\bar{u}_\mu)D_j\bar{u}^\nu_\mu D_i\bar{u}^\nu_\mu dx+\\ &+\frac{t}{2}\sum_{i,j=1}^{n}\int_\Omega E_\mu a_{ij}(x,\bar{u}_\mu)D_i\bar{U}_\mu D_j\bar{U}_\mu dx+\end{aligned}$$

$$+\sum_{\nu=1}^{M}\int_\Omega E_\mu a_0^\nu(x)(\bar u_\mu^\nu)^2 dx = -\sum_{\nu=1}^{M}\int_\Omega E_\mu f^{0\nu}(x,\bar u_\mu, D\bar u_\mu)\bar u_\mu^\nu dx -$$
$$-\sum_{\nu=1}^{M}\int_\Omega E_\mu Q(x,\bar u_\mu, D\bar u_\mu) D\bar u_\mu^\nu \bar u_\mu^\nu dx.$$

By (A_1), the minoration of the left hand side (l.h.s) will be:

$$\alpha\int_\Omega E_\mu |D\bar u_\mu|^2 dx + \alpha\frac{t}{2}\int_\Omega E_\mu |D\bar U_\mu|^2 dx + \alpha_0\int_\Omega E_\mu \bar U_\mu dx. \tag{2.7}$$

On the right hand side (r.h.s.) of (2.6'), we majorise the terms $f^{0\nu}$ and Q by making use of assumptions (A_3), (A_4), Hölder's and Young's inequalities. We consider the second term:

$$-\sum_{\nu=1}^{M}\int_\Omega E_\mu Q(x,\bar u_\mu, D\bar u_\mu) D\bar u_\mu^\nu u_\mu^\nu = -\frac{1}{2}\int_\Omega E_\mu Q(x,\bar u_\mu, D\bar u_\mu) D\bar U_\mu \le$$
$$\le \frac{C_1}{2}\int_\Omega E_\mu \varrho_2(x)|D\bar U_\mu| dx + \frac{C_1}{2}\int_\Omega E_\mu \varrho_2(x)|D\bar u_\mu|\,|D\bar U_\mu| dx,$$

or

$$\begin{aligned}&-\sum_{\nu=1}^{M}\int_\Omega E_\mu Q(x,\bar u_\mu, D\bar u_\mu) D\bar u_\mu^\nu \bar u_\mu^\nu dx \le \frac{\tilde\alpha}{2}\sup E_\mu\int_\Omega(\varrho_2)^2 dx +\\ &+\frac{1}{2\tilde\alpha}\frac{(C_1)^2}{4}\int_\Omega E_\mu|D\bar U_\mu|^2 dx + \frac{\tilde\alpha}{2}\|\varrho_2\|^2_{L^\infty(\Omega)}\int_\Omega E_\mu |D\bar u_\mu|^2 dx +\\ &+\frac{1}{2\tilde\alpha}\frac{(C_1)^2}{4}\int_\Omega E_\mu|D\bar U_\mu|^2 dx;\quad \tilde\alpha := \alpha/(2\|\varrho_2\|^2_{L^\infty(\Omega)}).\end{aligned} \tag{2.7'}$$

The second and fourth terms on the right of (2.7') can be absorbed by the second term of (2.7) and the third term will be absorbed by the first term of (2.7), since $E_\mu > 0$. Let us consider the first term on the right of (2.6'): by (A_3)

$$\begin{aligned}&-\sum_{\nu-1}^{M}\int_\Omega f^{0\nu}(x,\bar u_\mu, D\bar u_\mu)E_\mu \bar u_\mu^\nu dx \le M\int_\Omega E_\mu\varrho_1(x)|\bar u_\mu| dx +\\ &+ C_1 M\int_\Omega E_\mu k_1(x)|D\bar u_\mu|\,|\bar u_\mu| dx\,.\end{aligned} \tag{2.8}$$

By Young's inequality one has

$$M\int_\Omega E_\mu \varrho_1(x)|\bar u_\mu|dx \le \frac{\alpha_0}{2}\int_\Omega E_\mu|\bar u_\mu|^2dx+$$
$$+\frac{1}{2\alpha_0}\sup E_\mu\int_\Omega M^2(\varrho_1)^2dx = \frac{1}{2\alpha_0}C_2\int_\Omega M^2(\varrho_1)^2dx+$$
$$+\frac{\alpha_0}{2}\int_\Omega E_\mu \bar U_\mu dx\,.$$

Here the second term can be absorbed by the last term of (2.7). Next, the following term on the right of (2.8) is estimated:

$$C_1M\int_\Omega E_\mu k_1(x)|D\bar u_\mu||\bar u_\mu|dx \le C_1M(\sup E_\mu)\int_\Omega k_1|D\bar u_\mu||\bar u_\mu|dx \le$$
$$\le C_1MC_2\int_\Omega k_1|D\bar u_\mu||\bar u_\mu|^\theta|\bar u_\mu|^{1-\theta}dx\,,$$

where $1/p+1/q=1/2,\ p>2$ and $2<q<+\infty,\ \theta:=2/q,\ 0<\theta<1$. By Hölder's inequality we have

$$\int_\Omega k_1|D\bar u_\mu|\,|\bar u_\mu|^\theta|\bar u_\mu|^{1-\theta}dx = \int_\Omega |D\bar u_\mu|k_1|\bar u_\mu|^\theta|\bar u_\mu|^{1-\theta}dx \le$$
$$\le\left\{\int_\Omega|D\bar u_\mu|^2\right\}^{\frac12}\left\{\int_\Omega|k_1|^2|\bar u_\mu|^{2\theta}|\bar u_\mu|^{2(1-\theta)}dx\right\}^{\frac12}\le$$
$$\le\left\{\int_\Omega|D\bar u_\mu|^2\right\}^{\frac12}\left[\left\{\int_\Omega|k_1|^p\right\}^{\frac2p}\left\{\int_\Omega|\bar u_\mu|^{\theta q}\right\}^{\frac2q}\right]^{\frac12}\cdot\|\bar u_\mu\|^{1-\theta}_{L^\infty(\Omega,\mathbf{R}^M)}=$$
$$=\|D\bar u_\mu\|_{L^2(\Omega,\mathbf{R}^{Mn})}\cdot\|k_1\|_{L^p(\Omega)}\|\bar u_\mu\|^\theta_{L^2(\Omega,\mathbf{R}^M)}\cdot\|\bar u_\mu\|^{1-\theta}_{L^\infty(\Omega,\mathbf{R}^M)}\,;$$

and so by Young's inequality

$$\text{(2.8')}\qquad C_1M\int_\Omega E_\mu k_1|D\bar u_\mu|\,|\bar u_\mu|dx\le\frac{\alpha}{4}\|E_\mu D\bar u_\mu\|^2_{L^2(\Omega,\mathbf{R}^{Mn})}+$$
$$+\frac{1}{\alpha}C_1^2M^2\sup E_\mu\|k_1\|^2_{L^p(\Omega)}\|\bar u_\mu|^{2\theta}_{L^2(\Omega,\mathbf{R}^M)}\|\bar u_\mu|^{2(1-\theta)}_{L^\infty(\Omega,\mathbf{R}^M)}\,.$$

The first term on the right of (2.8') will be absorbed by the first term of (2.7). Therefore, from (2.7) and the estimations of the terms on the r.h.s. of (2.6') we obtain

$$\|\bar{u}_\mu\|^2_{H^1_0(\Omega,\mathbf{R}^M)} \leq C_3 \left[\|\bar{u}_\mu\|^{2\theta}_{L^2(\Omega,\mathbf{R}^M)} + \int_\Omega (\varrho_1)^2 dx + \int_\Omega (\varrho_2)^2 dx \right],$$

where $C_3 > 0$ is a constant. Since $0 < 2\theta < 2$, one may choose a constant C_4 depending only upon $\|\varrho_1\|_{L^2(\Omega)}$, $\|\varrho_2\|_{L^2(\Omega)}$, $\|k_1\|_{L^p(\Omega)}$, $\|\varrho_1\|_{L^\infty(\Omega)}$ and $\|\varrho_2\|_{L^\infty(\Omega)}$ and $M, \alpha, \tilde{\alpha}, \alpha_0$ such that

$$\|\bar{u}_\mu\|^2_{H^1_0(\Omega,\mathbf{R}^M)} \leq C_4, \quad \forall \mu \in \mathbf{N}. \tag{2.9}$$

which completes the proof of Lemma 2.1.

From the estimate (2.9) it follows that we can extract a subsequence (still denoted by $\bar{u}_\mu$) such that

$$\bar{u}_\mu \to u \quad \text{in} \quad H^1_0(\Omega, \mathbf{R}^M) \quad \text{weakly.}$$

By Rellich's theorem, for every ω compact in Ω, $\bar{u}_\mu$ tends to u strongly in $L^2(\omega)$ and then by the use of the diagonal process, we can extract a subsequence (still denoted by $\bar{u}_\mu$) such that $\bar{u}_\mu \to u$ a.e. in Ω and thus by (2.4) $u \in L^\infty(\Omega, \mathbf{R}^M)$.

Lemma 2.2. *For any ω compactly included in Ω we have:*

$$z^\nu_\mu := \bar{u}^\nu_\mu - u^\nu \to 0 \quad \textit{strongly in} \quad H^1(\omega).$$

PROOF. It is obvious that $z^\nu_\mu \in H^1_0(\Omega) \cap L^\infty(\Omega)$. By using notations $|z_\mu|^2 := \sum_\nu (z^\nu_\mu)^2$, $|Dz_\mu|^2_{\mathbf{R}^{Mn}} := \sum_\nu |Dz^\nu_\mu|^2$, $\bar{z}_\mu := \sum_\nu (z^\nu_\mu)^2$, we have $D_j \bar{Z}_\mu = \sum_{\nu=1}^n 2z^\nu_\mu D_j z^\nu_\mu$.

We choose

$$\bar{t} = 2C_1^2 \|\varrho_2\|^2_{L^\infty(\Omega)}/\alpha^2, \quad \bar{E}_\mu = \exp\{\bar{t}\bar{Z}_\mu\}.$$

Consider the test function $v^\nu = \bar{v}^\nu_\mu := \bar{E}_\mu z^\nu_\mu \theta^2$, where $\theta \in D(\Omega), 0 \leq \theta \leq 1$, $\theta = 1$ on ω. Thus the equation (2.5) can be translated to z^ν_μ in its principal part; multiplying (2.5) by $\bar{v}_\mu^\nu$ and summing over ν, we obtain

$$
\begin{aligned}
&\sum_\nu \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,\bar{u}_\mu) D_j z_\mu^\nu D_i v^\nu dx + \sum_\nu \int_\Omega a_0^\nu(x) z_\mu^\nu v^\nu dx = \\
&= -\sum_\nu \int_\Omega f^{0\nu}(x,\bar{u}_\mu, D\bar{u}_\mu) v^\nu dx - \\
&- \sum_\nu \int_\Omega Q(x,\bar{u}_\mu, D\bar{u}_\mu) D\bar{u}_\mu^\nu v^\nu dx + \\
&- \sum_\nu \sum_{i,j=1}^{n} \int_\Omega \langle D_i[a_{ij}(x,\bar{u}_\mu) D_j u^\nu], v^\nu\rangle - \\
&- \int_\Omega a_0^\nu(x) u^\nu v^\nu dx ,
\end{aligned} \tag{2.10}
$$

for all fixed $v^\nu \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

The derivative of the test function:

$$
D_i \bar{v}_\mu^\nu = D_i z_\mu^\nu \bar{E}_\mu \theta^2 + 2(D_i\theta)\theta \bar{E}_\mu z_\mu^\nu + \bar{t} D_i \bar{Z}_\mu \bar{E}_\mu z_\mu^\nu \theta^2 . \tag{2.11}
$$

The equation (2.10) can be written in the following form:

$$
\begin{aligned}
&\sum_\nu \sum_{i,j=1}^{n} \int_\Omega \bar{E}_\mu a_{ij}(x,\bar{u}_\mu) D_j z_\mu^\nu D_i z_\mu^\nu \theta^2 dx + \\
&\quad + \frac{\bar{t}}{2} \sum_{i,j=1}^{n} \int_\Omega \bar{E}_\mu a_{ij}(x,\bar{u}_\mu) D_i \bar{Z}_\mu D_j \bar{Z}_\mu \theta^2 dx + \\
&\quad + \sum_\nu \int_\Omega \bar{E}_\mu a_0^\nu(x) (z_\mu^\nu)^2 \theta^2 dx = \\
&= -\sum_\nu \int_\Omega \bar{E}_\mu Q(x,\bar{u}_\mu, D\bar{u}_\mu) D z_\mu^\nu z_\mu^\nu \theta^2 dx - \\
&\quad - \sum_\nu \int_\Omega \bar{E}_\mu Q(x,\bar{u}_\mu, D\bar{u}_\mu) D u^\nu z_\mu^\nu \theta^2 dx -
\end{aligned} \tag{2.12}
$$

$$-\sum_{\nu}\int_{\Omega}\bar{E}_{\mu}a_0^{\nu}(x)u^{\nu}z_{\mu}^{\nu}\theta^2dx-$$
$$-\sum_{\nu}\int_{\Omega}\bar{E}_{\mu}f^{0\nu}(x,\bar{u}_{\mu},D\bar{u}_{\mu})z_{\mu}^{\nu}\theta^2dx-$$
$$-\sum_{\nu}\sum_{i,j=1}^{n}\int_{\Omega}\bar{E}_{\mu}a_{ij}(x,\bar{u}_{\mu})D_ju^{\nu}D_iz_{\mu}^{\nu}\theta^2dx-$$
$$-\bar{t}\sum_{\nu}\sum_{i,j=1}^{n}\int_{\Omega}\bar{E}_{\mu}a_{ij}(x,\bar{u}_{\mu})D_ju^{\nu}D_i\bar{Z}_{\mu}z_{\mu}^{\nu}\theta^2dx-$$
$$-2\sum_{\nu}\sum_{i,j=1}^{n}\int_{\Omega}\bar{E}_{\mu}a_{ij}(x,\bar{u}_{\mu})D_ju^{\nu}(D_i\theta)\theta z_{\mu}^{\nu}dx-$$
$$-\sum_{i,j=1}^{n}\int_{\Omega}\bar{E}_{\mu}a_{ij}(x,\bar{u}_{\mu})D_j\bar{Z}_{\mu}(D_i\theta)\theta dx.$$

The l.h.s. of (2.12) is greater than or equal to (by coerciveness)

$$\alpha\int_{\Omega}\bar{E}_{\mu}|Dz_{\mu}|^2\theta^2dx+\alpha\frac{\bar{t}}{2}\int_{\Omega}\bar{E}_{\mu}|DZ_{\mu}|^2\theta^2dx+\alpha_0\int_{\Omega}\bar{E}_{\mu}\bar{Z}_{\mu}\theta^2dx. \tag{2.13}$$

On the r.h.s. of (2.12) first we consider the term

$$\begin{aligned}&-\sum_{\nu}\int_{\Omega}\bar{E}_{\mu}Q(x,\bar{u}_{\mu},D\bar{u}_{\mu})Dz_{\mu}^{\nu}z_{\mu}^{\nu}\theta^2dx=\\ &=-\frac{1}{2}\int_{\Omega}\bar{E}_{\mu}Q(x,\bar{u}_{\mu},D\bar{u}_{\mu})D\bar{Z}_{\mu}\theta^2dx\leq\\ &\leq\frac{C_1}{2}\int_{\Omega}\bar{E}_{\mu}\varrho_2(x)|D\bar{Z}_{\mu}|\theta^2dx+\\ &\quad+\frac{C_1}{2}\int_{\Omega}\bar{E}_{\mu}\varrho_2(x)|D\bar{Z}_{\mu}|\,|D\bar{u}_{\mu}|\theta^2dx.\end{aligned} \tag{2.12'}$$

(where $\bar{Z}_{\mu} := \sum_{\nu}(z_{\mu}^{\nu})^2$, $D_j\bar{Z}_{\mu} = \sum_{\nu}2z_{\mu}^{\nu}D_jz_{\mu}^{\nu}$, $|Dz_{\mu}|^2_{\mathbf{R}^{Mn}} = \sum_{\nu}|Dz_{\mu}^{\nu}|^2$, $|Dz_{\mu}^{\nu}|\leq|Dz_{\mu}|$ and $|D\bar{u}_{\mu}^{\nu}|\leq|D\bar{u}_{\mu}|$).

The term $\frac{C_1}{2}\int\limits_\Omega \bar{E}_\mu \varrho_2(x)|D\bar{Z}_\mu|\theta^2 dx \to 0$, since θ is bounded in $L^\infty(\Omega)$ and $|D\bar{Z}_\mu| \to 0$ weakly in $L^2(\Omega)$, as $z_\mu^\nu \in L^\infty(\Omega)$, $|Dz_\mu^\nu| \leq |Dz_\mu|$ and $Dz_\mu \to 0$ weakly in $L^2(\Omega)$, $\varrho_2 \in L^2(\Omega) \cap L^\infty(\Omega)$ and $\bar{E}_\mu$ is bounded in $L^\infty(\Omega)$. Also, the next term on the right of (2.12') can be estimated in the following manner:

$$\begin{aligned}
&\frac{C_1}{2}\int\limits_\Omega \bar{E}_\mu \varrho_2(x)|D\bar{Z}_\mu|\,|D\bar{u}_\mu|\theta^2 dx \leq \\
&\leq \frac{C_1}{2}\int\limits_\Omega \bar{E}_\mu \varrho_2(x)|D\bar{Z}_\mu|\,|Du|\theta^2 dx + \\
&+\frac{C_1}{2}\int\limits_\Omega \bar{E}_\mu \varrho_2(x)|D\bar{z}_\mu|\,|D\bar{Z}_\mu|\theta^2 dx\,.
\end{aligned}$$

Here, the first term will tend to zero since $|Du|$ is a fixed L^2-function and $|D\bar{Z}_\mu| \to 0$ weakly in $L^2(\Omega)$, and for the second term by the use of Young's inequality we obtain

$$\begin{aligned}
\frac{C_1}{2}\int\limits_\Omega \bar{E}_\mu \varrho_2(x)|Dz_\mu|\,|D\bar{Z}_\mu|\theta^2 dx &\leq \frac{\alpha^*}{2}\int\limits_\Omega \bar{E}_\mu(\varrho_2)^2|Dz_\mu|^2\theta^2 dx + \\
&+\frac{1}{2\alpha^*}\frac{(C_1)^2}{4}\int\limits_\Omega \bar{E}_\mu|D\bar{Z}_\mu|^2\theta^2 dx\,.
\end{aligned}$$

The first of the above two terms can be written as follows:

$$\frac{\alpha^*}{2}\int\limits_\Omega \bar{E}_\mu(\varrho_2)^2|Dz_\mu|^2\theta^2 dx = \frac{\alpha^*}{2}\|\varrho_2\|^2_{L^\infty(\Omega)}\cdot\|\bar{E}_\mu Dz_\mu\theta\|^2_{L^2(\Omega,\mathbf{R}^{Mn})}$$

where $\alpha^* := \frac{\alpha}{2\|\varrho_2\|^2_{L^\infty(\Omega)}}$. This term can be subtracted from the first term of (2.13) and the second one will be absorbed by the second term of (2.13).

In a similar manner it can be shown that all the terms on the right hand side of (2.12) will tend to zero. Consequently, the left hand side of (2.12) will also tend to zero. Next step is to prove the passage to the limit.

PROOF of Theorem 2.1. The limit of (2.5) as $\mu \to \infty$ is to be taken for every v^ν belonging to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and having compact support.

We have

$$|f^{0\nu}(x,\bar u_\mu,D\bar u_\mu)v^\nu| \le [\,\varrho_1(x)+C_1|k_1(x)|\,|D\bar u_\mu|\,]\;|v^\nu| \le \le \varrho_1(x)|v^\nu| + C_1|k_1|\,|D\bar u_\mu|\,|v^\nu|\,. \tag{2.14}$$

Then the r.h.s. of (2.14) converges strongly in $L^1(\Omega)$. Since

$$f^{0\nu}(x,\bar u_\mu,D\bar u_\mu)v^\nu \to f^{0\nu}(x,u,Du)v^\nu \quad \text{a.e. in } \Omega,$$
$$f^{0\nu}(x,\bar u_\mu,D\bar u_\mu)v^\nu \to f^{0\nu}(x,u,Du_\mu)v^\nu \quad \text{in the } L^1\text{-norm}$$

by Vitali's theorem. Similarly, it can be shown that

$$(Q(x,\bar u_\mu,D\bar u_\mu)D\bar u^\nu_\mu)v^\nu \to (Q(x,u,Du)Du^\nu)v^\nu \text{ in the } L^1-\text{norm}. \tag{2.15}$$

Further, for the terms a_{ij}, a^ν_0 passage to the limit can be easily obtained for any v^ν which has compact support and we obtain that equality (2.1) holds for each $v \in H^1_0(\Omega,\mathbf{R}^M)\cap L^\infty(\Omega,\mathbf{R}^M)$ having compact support.

Finally, we show that (2.1) is valid for each $v \in H^1_0(\Omega,\mathbf{R}^M)\cap$ $\cap L^\infty(\Omega,\mathbf{R}^M)$. For $v^\nu \in H^1_0(\Omega)\cap L^\infty(\Omega)$ there is a sequence of functions $\varphi^\nu_\mu \in \mathcal{D}(\Omega)$ such that

$$\|\varphi^\nu_\mu\|_{L^\infty(\Omega)} \le 2\|v^\nu\|_{L^\infty(\Omega)}\,,$$

φ^ν_μ converges strongly to v^ν in $H^1_0(\Omega)$. We have

$$\int_\Omega |f^{0\nu}(x,u,Du)(\varphi^\nu_\mu - v^\nu)|dx \le \int_\Omega \varrho_1|\varphi^\nu_\mu - v^\nu|dx + + C_1\|k_1\|_{L\infty(\Omega)}\int_\Omega |Du^\nu|\,|\varphi^\nu_\mu - v^\nu|dx\,, \tag{2.16}$$

$$\int_\Omega |Q(x,u,Du)Du^\nu(\varphi^\nu_\mu - v^\nu)dx| \le \le C_1\int_\Omega \varrho_2|Du^\nu|\,|\varphi^\nu_\mu - v^\nu|dx + C_1\int_\Omega |Du^\nu|^2|\varphi^\nu_\mu - v^\nu|dx\,, \tag{2.17}$$

and so by the Cauchy–Schwarz inequality and the Lebesgue Convergence Theorem in (2.16), (2.17) it is possible to have limit for $\mu\to\infty$. Since (2.1) holds for $v^\nu := \varphi^\nu_\mu$ thus passing to the limit shows that the cuqality (2.1) holds for each $v^\nu \in H^1_0(\Omega)\cap L^\infty(\Omega)$.

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