Publ. Math. Debrecen
39 / 1–2 (1991), 91–93

Steinhaus Theorem in Summability for P–adic Fields

By V. K. SRINIVASAN (El Paso)

Dedicated to the memory of Professor C. T. Rajagopal

Abstract. A classical theorem of STEINHAUS [11] in summability theory asserts that if C is the complex field and $A = (a_{nk})$ is a regular infinite matrix with coefficients in C, then A cannot sum all sequences of 0's and 1's. The purpose of this note is to give a short proof of the above theorem for p–adic fields Q_p and for local fields.

Introduction

After MONNA [3] gave a complete characterization of regular infinite matrices with coefficients in a complete non–archimedean field K, the investigations in summability theory started developing over complete non–archimedean fields. The works cited in [3], [4], [5], [6], [7], [8], [9], [10] represent the so called non–archimedean summability theory. In his thesis [4] NATARAJAN gave a detailed investigation of Steinhaus' Theorem from different points of view. The purpose of this paper is to give a short and direct proof of the above theorem for p–adic fields Q_p and extend the result to cover local fields.

Throughout let K denote a complete non–archimedean valued field. Let $A = (a_{nk})$ $n = 1, 2, \ldots;$ $k = 1, 2, \ldots$ be an infinite matrix with coefficients in K. Given a sequence $x = \{x_n\}$ in K, the sequence $y = \{y_n\}$ is defined by the procedure

$$y_n = \sum_{k=1}^{\infty} a_{nk} x_k \tag{1}$$

The matrix A is said to be regular if whenever $x = \{x_n\}$ converges to some L in K, the transformed sequence $y = \{y_n\}$ also converges to the very same L. In [3] the following conditions are given as necessary and sufficient for A to be regular

1) $\sup\limits_{n,k} |a_{nk}| < \infty$.
2) $\lim(a_{nk}) = 0$ as $n \to \infty$ for all $k = 1, 2 \ldots$.
3) $\mathrm{Lt}\, (\sum\limits_{k=1}^{\infty} a_{nk}) = 1$ as $n \to \infty$.

The following **Theorem of Osgood** is needed in the sequel. A proof of the above theorem is discussed in STROMBERG [12] in page 120.

Let X be an arbitrary topological space and Y an arbitrary metric space with metric ϱ. Suppose $\{F_n\}$ is a sequence of continuous functions from X to Y, converging to the limit function $F : X \to Y$ pointwise. Then there exists a set $E \subseteq X$ of first category such that F is continuous at each $p \in X - E$. In particular if X is a complete metric space then F is continuous at every point of a dense subset of X.

In the sequel Q_p will denote the p–adic field and $\triangle$ denote the collection of all sequences of 0's and 1's from Q_p. For $x = \{x_n\}$ and $y = \{y_n\}$ in $\triangle$ we define a metric by setting

$$d(x, y) = \operatorname{Sup}_n |p^n(x_n - y_n)| \tag{2}$$

Then $(\triangle, d)$ is a metric space and is complete. (Completion of $\triangle$ is really essential for the investigation later on as $\triangle$ is not empty). We also note that for $x, y \in \triangle$

$$d(x, y) \leq \sum_{n=1}^{\infty} |p^n(x_n - y_n)| .$$

Theorem 1. *A regular infinite matrix method $A = (a_{nk})$ over Q_p cannot sum all sequences of 0's and 1's in Q_p.*

PROOF. As A is regular, for each n the functional

$$F_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k \tag{3}$$

is well defined for all $x = \{x_n\} \in \triangle$ and further it is a continuous function with values in Q_p, in view of the fact

$$|F_n(x) - F_n(y)| \leq \sum_{k=i+1}^{\infty} |a_{nk}| < \varepsilon \tag{4}$$

if $d(x, y) < |p^i|$ for sufficiently large i. Suppose A sums all sequences in $\triangle$ then

$$F(x) = \lim F_n(x) \text{ as } n \to \infty \tag{5}$$

is defined for all $x \in \triangle$. We further note that any open ball around any point of $\triangle$ contains two points z and w where z is eventually all 1's and w is eventually all 0's. The regularity of A implies that

$$|F(z) - F(w)| = 1 \tag{6}$$

Hence the above argument shows that F is discontinuous at each point of $\triangle$. However by Theorem of Osgood, F must be continuous atleast at one point of $\triangle$. The above contradiction shows the validity of the claim made in the Theorem.

Remark 1. In the above Theorem 1, the only thing that was used relating to the p-adic field was p^n. It is a p-adic null sequence. By a local field we will mean a totally disconnected, non-trivial complete locally compact field K. Replacing $\{p^n\}$ by any other suitable null sequence the same result is seen to be valid for any local field, as well as for any arbitrary non-archimedean complete, non-trivial valued field K.

References

[1] J. DUGUNDJI, Topology, *Allyn and Bacon*, 1970.

[2] K. MAHLER, p-adic numbers and their functions, Cambridge tracts in Mathematics 76, *Cambridge University Press* (1980).

[3] A. MONNA, Sur le théorème de Banach Steinhaus, *Indegationes Mathematicae* **25** (1963), 121–131.

[4] P. N. NATARAJAN, Sequence spaces and matrix transformations over valued fields, Ph. d. Thesis, *University of Madras,* 1980.

[5] P. N. NATARAJAN, The Steinhaus Theorem for Toeplitz`matrices in non-archimedean fields, *Comment. Math. Prace Mat.* **20** (1978), 417–422.

[6] P. N. NATARAJAN, Multiplication of series with terms in a non-archimedean field, *Simon Stevin* **52** (1978), 157–160.

[7] P. N. NATARAJAN and M. S. RANGACHARI, Matrix transformations between sequence spaces over non-archimedean fields, *Rev. Roum. Math. Pures et Appl.* **24** (1979), 615–618.

[8] M. S. RANGACHARI and V. K. SRINIVASAN, Matrix transformations in non-archimedean fields, *Indegationes Mathematicae* (1964), 422–429.

[9] D. SOMASUNDARAM, Some properties of T-matrices over non-archimedean fields, *Publ. Mat. Debrecen* **21** (1974), 171–177.

[10] V. K. SRINIVASAN, On certain summation processes in the p-adic field, *Indegationes Mathematicae* **27** (1965), 319–325.

[11] H. STEINHAUS, Some remarks on the generalization of limit, *Prace Mat. Fiz.* **22** (1911), 121–134.

[12] K. R. STROMBERG, Introduction to Classical Real Analysis, *Wadsworth Inc.* 1981.

V. K. SRINIVASAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TEXAS AT EL PASO, EL PASO
TEXAS 79968, U.S.A.

(Received Oktober 6, 1988)