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$f(3,1)$-Finsler structures and their integrability

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Dedicated to Professor Dr. Lajos Tamássy on the occasion of his 65 th birtday

In [11] K. YANO has unified the notions of almost complex structure and almost contact structure by considering on an n-dimensional manifold M a tensor field f of type (1,1) such that $f^3 + f = 0$ and the rank r of f is everywhere a constant even numbers. This structure has been called an f-structure (or an $f(3,1)$-structure). The theory of integrability of this structure [4], [9], is given in [8] using f-connections.

The main purpose of the present paper is to introduce the notion of f(3,1)-Finsler structure on M and to study this structure with the method used by R. MIRON [7].

Terminology and notations belongs to M. MATSUMOTO [5] and R. MIRON [6].

1. Preliminaries

Let M be a C^∞-differentiable manifold, paracompact with n dimensions, let $T(M) = (TM, p, M)$ be its tangent bundle, and let N be a non-linear connection on $T(M)$. We denote by (x^i, y^i) $(i, j, k, \dots, = 1, 2, \dots, n)$ the canonical coordinates on TM. Then $\delta_i = \partial_i - N_i^k \dot{\partial}_k$ $(\partial_i = \frac{\partial}{\partial x^i}$, $\dot{\partial}_k = \frac{\partial}{\partial y^k}$, $\delta_i = \frac{\delta}{\delta x^i})$ is a local basis of horizontal distribution N, and $\dot{\partial}_i$ is a local basis of the vertical distribution $(TM)^v$. The dual basis is $(dx^i, \delta y^i)$ where $\delta y^i = dy^i + N_k^i dx^k$. We have

$$(1.1)\qquad [\delta_j, \delta_k] = R^i_{jk}\dot{\partial}_i, \quad [\delta_j, \dot{\partial}_k] = (\dot{\partial}_k N_j^i)\dot{\partial}_i, \quad [\dot{\partial}_j, \dot{\partial}_k] = 0,$$

$$(1.2)\qquad \delta_k N_j^i - \delta_j N_k^i = R^i_{jk}, \quad \dot{\partial}_k N_j^i - \dot{\partial}_j N_k^i = t^i_{jk},$$

where R^i_{jk} and t^i_{jk} are the curvature and torsion fields of N.

A Finsler connection on M is a triad $F\Gamma = (N, F, C)$, where N is a non-linear connection on M, and F respectively C are the h- and v-connection coefficients, given by

$$\begin{aligned} \nabla_{\delta_k}\delta_j = F^i_{jk}\delta_i, \quad \nabla_{\delta_k}\dot{\partial}_j = F^i_{jk}\dot{\partial}_i, \\ \nabla_{\dot{\partial}_k}\delta_j = C^i_{jk}\delta_i, \quad \nabla_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i. \end{aligned} \tag{1.3}$$

With $|$ and $\Big|$ we denote the h- and v-covariant derivatives with respect to $F\Gamma$.

The torsion Finsler tensor fields of $F\Gamma$ will be denoted by: T^i_{jk}, N^i_{jk}, C^i_{jk}, P^i_{jk}, S^i_{jk}, and the curvature Finsler tensor fields of $F\Gamma$ will denoted by: $R_j{}^i{}_{kh}$, $P_j{}^i{}_{kh}$, $S_j{}^i{}_{kh}$.

2. $f(3,1)$-Finsler structures and $f(3,1)$-Finsler connections

Let M be an n-dimensional differentiable manifold of class C^∞, and let $x = (x^i)$ and $y = (y^i)$ denote a point of M and a supporting element respectively.

Definition 2.1. A Finsler tensor field $f(x,y) \neq 0$ of type (1,1) and of class C^∞ is called an $f(3,1)$-*Finsler structure of index* K, if it satisfies

$$f^3 + f = 0, \quad \text{rank } \|f(x,y)\| = n - K = 2p, \tag{2.1}$$

where K, p are integers and $0 \leq K < n$.

Proposition 2.1. *For any $f(3,1)$-Finsler structure of index K, the operators $h(x,y)$, $v(x,y)$ given by*

$$h = -f^2, \quad v = f^2 + I, \tag{2.2}$$

I denoting the identity operator, are complementary projection operators applied to the tangent bundle of M.

Now we denote by $\mathcal{H}$ and $\mathcal{V}$ the complementary distributions corresponding to the projection operators h and v respectively, and we have: $\dim \mathcal{H}_{(x,y)} = n - K$, $\dim \mathcal{V}_{(x,y)} = K$.

Proposition 2.2. *An f satisfying the relation (2.1), acts on $\mathcal{H}$ as an almost complex Finsler operator and on $\mathcal{V}$ as a null operator.*

In fact, we have

$$\begin{cases} fh = hf = f, & fv = vf = 0 \\ f^2 h = -h, & f^2 v = 0. \end{cases} \tag{2.3}$$

Remark 2.1. If the rank of $f(x,y)$ is n, then $h = I$ and $v = 0$ and $f(x,y)$ satisfies: $f^2 = -I$. Consequently the $f(3,1)$-Finsler structure of *minimum index* (null index) is an almost complex Finsler structure (cf. with G. B. Rizza [10], I. Ichijyo [3], R. Miron [7], etc.). In this case the dimension n must be even.

Remark 2.2. If the $f(3,1)$-Finsler structure is of *index* 1, then $\mathcal{H}$ is $(n-1)$-dimensional, and $\mathcal{V}$ is one-dimensional. Consequently if we denote the local components of $f(x,y)$, $h(x,y)$ and $v(x,y)$ by $f^i_j(x,y)$, $h^i_j(x,y)$ and $v^i_j(x,y)$ $(i,j,\dots=1,2,\dots,n)$ respectively, then $v^i_j(x,y)$ must have the form: $v^i_j=\eta_j\xi^i$, where $\eta(x,y)$ and $\xi(x,y)$ are covariant and contravariant Finsler vector fields respectively. From the relations (2.2) and (2.3) we have

$$f^i_r f^r_j=-\delta^i_j+\eta_j\xi^i,\ f^i_j\xi^j=0,\ \eta_i f^i_j=0,\ \eta_i\xi^i=1.$$

Thus an $f(3,1)$-Finsler structure of index 1 is equivalent to an almost contact Finsler structure [2].

Definition 2.2. We shall call the Finsler tensor fields $\underset{1}{Q}{}^{rs}_{ij}$ and $\underset{2}{Q}{}^{rs}_{ij}$ given by

$$(2.4)\qquad \begin{cases} \underset{1}{Q}{}^{rs}_{ij}=\frac{1}{2}(\delta^r_i\delta^s_j-\delta^r_i v^s_j-v^r_i\delta^s_j-f^r_i f^s_j+3v^r_i v^s_j),\\ \underset{2}{Q}{}^{rs}_{ij}=\frac{1}{2}(\delta^r_i\delta^s_j+\delta^r_i v^s_j+v^r_i\delta^s_j+f^r_i f^s_j-3v^r_i v^s_j),\end{cases}$$

the Obata operators of the $f(3,1)$-Finsler structure.

These operators have the symmetry $\underset{\alpha}{Q}{}^{rs}_{ij}=\underset{\alpha}{Q}{}^{sr}_{ij}$ $(\alpha=1,2)$, and act on a Finsler tensor field K of type (1,2) as $(\underset{\alpha}{Q}K)^i_{jk}=\underset{\alpha}{Q}{}^{ri}_{js}K^s_{rk}$ $(\alpha=1,2)$. Since $(\underset{\alpha}{Q}\underset{\beta}{Q}K)^i_{jk}=\underset{\alpha}{Q}{}^{ri}_{js}\underset{\beta}{Q}{}^{ts}_{rp}K^p_{tk}$, the product $\underset{\alpha}{Q}\underset{\beta}{Q}$ is defined by $(\underset{\alpha}{Q}\underset{\beta}{Q})^{ti}_{jp}=\underset{\alpha}{Q}{}^{ri}_{js}\underset{\beta}{Q}{}^{ts}_{rp}$ $(\alpha,\beta=1,2)$.

Proposition 2.3. *$\underset{1}{Q},\underset{2}{Q}$ are the supplementary projectors on the module τ^1_2 of the tensor fields of type (1,2):*

$$(2.5)\qquad \underset{1}{Q}+\underset{2}{Q}=I,\quad \underset{\alpha}{Q}^2=\underset{\alpha}{Q},\ \underset{1}{Q}\cdot\underset{2}{Q}=\underset{2}{Q}\cdot\underset{1}{Q}=0\quad(\alpha=1,2).$$

The proof is elementary.

Proposition 2.4. *$\underset{2}{Q}X=0$ has solutions, and its general solution is given by $X=\underset{1}{Q}Y$, where $Y\in\tau^2_1$ is arbitrary.*

An important problem concerning an $f(3,1)$-Finsler structure on M is to determine the existence of arbitrary Finsler connections with respect to which $f^i_j(x,y)$ is covariantly constant.

Definition 2.3. Let $f^i_j(x,y)$ be an $f(3,1)$-Finsler structure of index K. A Finsler connection $F\Gamma$ is called an $f(3,1)$-*Finsler connection*, or *compatible* with f^i_j, (2.1), if:

$$f^i_{j|k} = 0, \quad f^i_j\big|_k = 0. \tag{2.6}$$

Proposition 2.5. *With respect to a Finsler connection $F\Gamma$ compatible with and $f(3,1)$-Finsler structure $f^i_j(x,y)$, the tensor fields h^i_j and v^i_j covariantly constant:*

$$h^i_{j|k} = 0, \quad v^i_{j|k} = 0, \quad h^i_j\big|_k = 0, \quad v^i_j\big|_k = 0. \tag{2.7}$$

Theorem 2.1. *(i) The Obata tensor fields $\underset{1}{Q}{}^{rs}_{ij}$ and $\underset{2}{Q}{}^{rs}_{ij}$ are covariantly constant with respect to any $f(3,1)$-Finsler connection $F\Gamma$.*

(ii) The Finsler tensor fields $\underset{2}{Q}{}^{ir}_{sj} R_r{}^s{}_{kl}$, $\underset{2}{Q}{}^{ir}_{sj} P_r{}^s{}_{kl}$, $\underset{2}{Q}{}^{ir}_{sj} S_r{}^s{}_{kl}$ and their h- and v-covariant derivatives of every order vanish for every $F\Gamma$ with the property (2.6).

PROOF. The statement (i) results immediately from (2.4), (2.6) and (2.7). Applying the Ricci identities to f^i_j and taking into account (i) we get the property (ii).

Theorem 2.2. *If on the differentiable manifold M there exists a Finsler connection $F\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$, then there exist $f(3,1)$-Finsler connections with respect to the $f(3,1)$-Finsler structure (2.1). One of these is:*

$$\begin{cases} \hat{N}^i_j = \overset{\circ}{N}{}^i_j, \quad \hat{F}^i_{jk} = \overset{\circ}{F}{}^i_{jk} - \frac{1}{2}\{f^i_h f^h_j{}_{\dagger k} + v^i_j{}_{\dagger k} - 3v^i_h v^h_j{}_{\dagger k}\}, \\ \qquad\qquad\qquad \hat{C}^i_{jk} = \overset{\circ}{C}{}^i_{jk} - \frac{1}{2}\{f^i_h f^h_j\big\rceil_k + v^i_j\big\rceil_k - 3v^i_h v^h_j\big\rceil_k\}. \end{cases} \tag{2.8}$$

where $_\dagger$ and $\rceil$ denote the h- and v-covariant derivatives with respect to $F\overset{\circ}{\Gamma}$.

PROOF. A straightforward calculation shows that the Finsler connection given by (2.8) satisfies equations (2.6).

We shall determine all $f(3,1)$-Finsler connections by a well known method based on Proposition 2.4. Let $F\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ be a fixed Finsler connection on M. Then any Finsler connection $F\Gamma = (N, F, C)$ on M can be expressed in the form, [6]:

$$N^i_j = \overset{\circ}{N}{}^i_j - A^i_j, \;\; F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{jr} A^r_k - B^i_{jk}, \;\; C^i_{jk} = \overset{\circ}{C}{}^i_{jk} - D^i_{jk}, \tag{2.9}$$

where A^i_j, B^i_{jk}, D^i_{jk} are arbitrary Finsler tensor fields.

We consider $\overset{\circ}{F\Gamma} = F\hat{\Gamma}$ in (2.9), where $F\hat{\Gamma} = (\hat{N}, \hat{F}, \hat{C})$ is given by (2.8). In order that $F\Gamma$ is an $f(3,1)$-Finsler connection, that is, (2.6) holds for $F\Gamma$, it is necessary and sufficient that $A^i_j, B^i_{jk}, D^i_{jk}$ satisfy

$$B^i_{jk} + (f^i_s f^h_j - v^i_s v^h_s) B^s_{hk} = 0, \;\; D^i_{jk} + (f^i_s f^h_j - v^i_s v^h_j) D^s_{hk} = 0,$$

which is equivalent to $\underset{2}{Q}B = 0$, $\underset{2}{Q}D = 0$. From Proposition 2.4, however, the last system has solutions in B^i_{jk}, D^i_{jk} for any Finsler tensor field $A^i_j = X^i_j$. Hence

Theorem 2.3. *Let $\overset{\circ}{F\Gamma}$ be a fixed Finsler connection. The set of all $f(3,1)$-Finsler connections $F\Gamma$ with respect to the $f(3,1)$-Finsler structure (2.1) is given by:*

$$\text{(2.10)} \qquad \begin{cases} N^i_j = \overset{\circ}{N}{}^i_j - X^i_j \\ F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{jr} X^r_k - \frac{1}{2}\{ f^i_h (f^h_j \overset{\circ}{|}_k + f^h_j \overset{\circ}{|}_m X^m_k) + v^i_j \overset{\circ}{|}_k + \\ \qquad + v^i_j \overset{\circ}{|}_m X^m_k - 3 v^i_h (v^h_j \overset{\circ}{|}_k + v^h_j \overset{\circ}{|}_m X^m_k)\} + \underset{1}{Q}{}^{ih}_{sj} Y^s_{hk}, \\ C^i_{jk} = \overset{\circ}{C}{}^i_{jk} - \frac{1}{2}\{ f^i_h f^h_j \overset{\circ}{|}_k + v^i_j \overset{\circ}{|}_k - 3 v^i_h v^h_j \overset{\circ}{|}_k \} + \underset{1}{Q}{}^{ih}_{sj} Z^s_{hk}, \end{cases}$$

where X^i_j, Y^i_{jk}, Z^i_{jk} are arbitrary Finsler tensor fields.

Remark 2.1. The $f(3,1)$-Finsler connection $F\hat{\Gamma} = (\hat{N}, \hat{F}, \hat{C})$ given by (2.8) is obtained from (2.10) for $X^i_j = Y^i_{jk} = Z^i_{jk} = 0$.

Corollary 2.1. *If $\overset{\circ}{F\Gamma}$ is a fixed $f(3,1)$-Finsler connection, then the set of all $f(3,1)$-Finsler connections $F\Gamma$ is given by:*

$$\text{(2.11)} \qquad \begin{aligned} N^i_j &= \overset{\circ}{N}{}^i_j - X^i_j, \\ F^i_{jk} &= \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{jr} X^r_k + \underset{1}{Q}{}^{ih}_{sj} Y^s_{hk}, \\ C^i_{jk} &= \overset{\circ}{C}{}^i_{jk} - \underset{1}{Q}{}^{ih}_{sj} Z^s_{hk}, \end{aligned}$$

where X^i_j, Y^i_{jk}, Z^i_{jk} are arbitrary Finsler tensor fields.

We denote by $F\Gamma(N)$ the Finsler connections having the same non-linear connection N.

Theorem 2.4. *The set of all $f(3,1)$-Finsler connections $F\Gamma(\overset{\circ}{N})$ is given by*

$$(2.12)\qquad \begin{cases} F^i_{jk} = \overset{\circ}{F}{}^i_{jk} - \frac{1}{2}\{f^i_h f^h_j{}_{\overset{\circ}{|}k} + v^i_j{}_{\overset{\circ}{|}k} - 3v^i_h v^h_j{}_{\overset{\circ}{|}k}\} + \underset{1}{Q}{}^{ih}_{sj} Y^s_{hk}, \\ C^i_{jk} = C^i_{jk} - \frac{1}{2}\{f^i_h f^h_j \overset{\circ}{|}_k + v^i_j \overset{\circ}{|}_k - 3v^i_h v^h_j \overset{\circ}{|}_k\} + Q^{ih}_{sj} Z^s_{hk}, \end{cases}$$

where $F\overset{\circ}{\Gamma}$ is a fixed Finsler connection, and Y^i_{jk}, Z^i_{jk} are arbitrary Finsler tensor fields.

3. The group of transformations of $f(3,1)$-Finsler connections

Let us consider the transformations $F\Gamma(N) \to F\bar{\Gamma}(N)$ of $f(3,1)$-Finsler connections that conserve the non-linear connection N. According to Theorem 2.4 they are given by:

$$(3.1)\qquad \bar{N}^i_j = N^i_j, \quad \bar{F}^i_{jk} = F^i_{jk} + \underset{1}{Q}{}^{ih}_{sj} Y^s_{hk}, \quad \bar{C}^i_{jk} = C^i_{jk} + \underset{1}{Q}{}^{ih}_{sj} Z^s_{hk}.$$

Theorem 3.1. *The set of all transformations (3.1) together with the mapping product is an Abelian group G_f isomorphic with the additive group of pairs of Finsler tensor fields $(\underset{1}{Q}{}^{ih}_{sj} Y^s_{hk}, \underset{1}{Q}{}^{ih}_{sj} Z^s_{hk})$.*

By a straightforward calculation we can prove:

Theorem 3.2. *The following Finsler tensor fields are invariants by the action of the group G_f:*

$$(3.2)\qquad R^i_{jk}, \quad t^i_{jk}$$

$$(3.3)\qquad \begin{aligned} \overset{1}{N}{}^i_{jk} &= \underset{2}{Q}{}^{ri}_{pq} \underset{1}{Q}{}^{ps}_{jk} T^q_{rs}, & \tilde{C}^i_{jk} &= \underset{2}{Q}{}^{ri}_{pq} \underset{1}{Q}{}^{ps}_{jk} C^q_{rs}, \\ \tilde{P}^i_{jk} &= \underset{2}{Q}{}^{ri}_{pq} \underset{1}{Q}{}^{ps}_{jk} P^q_{rs}, & \overset{2}{N}{}^i_{jk} &= \underset{2}{Q}{}^{ri}_{pq} \underset{1}{Q}{}^{ps}_{jk} S^q_{rs}, \end{aligned}$$

$$(3.4)\qquad \begin{cases} \overset{1}{T}{}^i_{jk} = h^i_m T^m_{jk} - f^r_j f^s_k T^i_{rs} + (f^r_j T^m_{rk} + f^s_k T^m_{js}) f^i_m, \\ \overset{1}{R}{}^i_{jk} = h^i_m R^m_{jk} - f^r_j f^s_k R^i_{rs} + (f^r_j R^m_{rk} + f^s_k R^m_{js}) f^i_m, \\ \overset{1}{C}{}^i_{jk} = h^i_m C^m_{jk} - f^r_j f^s_k C^i_{rs} + (f^r_j C^m_{rk} + f^s_k C^m_{js}) f^i_m, \\ \overset{1}{P}{}^i_{jk} = h^i_m P^m_{jk} - f^r_j f^s_k P^i_{rs} + (f^r_j P^m_{rk} + f^s_k P^m_{js}) f^i_m, \\ \overset{1}{S}{}^i_{jk} = h^i_m S^m_{jk} - f^r_j f^s_k S^i_{rs} + (f^r_j S^m_{rk} + f^s_k S^m_{js}) f^i_m, \end{cases}$$

$$
(3.5)\qquad \begin{cases}
\overset{2}{R}{}^i_{jk} = h^i_m R^m_{jk} - f^r_j f^s_k R^i_{rs} - (f^r_j R^m_{rk} + f^s_k R^m_{js}) f^i_m,\\
\overset{2}{C}{}^i_{jk} = h^i_m C^m_{jk} - f^r_j f^s_k C^i_{rs} + (f^r_j C^m_{rk} - f^s_k C^m_{js}) f^i_m,\\
\overset{2}{P}{}^i_{jk} = h^i_m P^m_{jk} + f^r_j f^s_k P^i_{rs} - (f^r_j P^m_{rk} - f^s_k P^m_{js}) f^i_m,
\end{cases}
$$

$$
(3.6)\qquad \begin{cases}
\overset{3}{T}{}^i_{jk} = h^i_m T^m_{jk} - (f^r_j P^m_{kr} - f^r_k P^m_{jr}) f^i_m\\
\overset{3}{R}{}^i_{jk} = h^i_m R^m_{jk} - f^r_j f^s_k S^i_{rs} - (f^r_j C^m_{kr} - f^s_k C^m_{js}) f^i_m,\\
\overset{3}{C}{}^i_{jk} = h^i_m C^m_{jk} + f^r_j f^s_k C^i_{sr} + (f^r_j S^m_{rk} + f^s_k R^m_{js}) f^i_m,\\
\overset{3}{P}{}^i_{jk} = h^i_m P^m_{jk} + f^r_j f^s_k P^i_{sr} + f^s_k T^m_{js} f^i_m\\
\overset{3}{S}{}^i_{jk} = h^i_m S^m_{jk} - f^r_j f^s_k R^i_{rs} + (f^r_j C^m_{rk} - f^s_k C^m_{sj}) f^i_m,\\
\tilde{T}^i_{jk} = f^r_j f^s_k T^i_{rs} - (f^r_j P^m_{rk} - f^s_k P^m_{sj}) f^i_m.
\end{cases}
$$

Theorem 3.3. *The Finsler tensor fields* $\overset{1}{N}{}^i_{jk}$, $\overset{2}{N}{}^i_{jk}$, $\overset{1}{T}{}^i_{jk}$, $\overset{1}{S}{}^i_{jk}$ *vanish if and only if there exists on M a h- and v-semi-symmetric $f(3,1)$-Finsler connection $F\Gamma(N)$.*

Remark 3.1. If the $f(3,1)$-Finsler structure is of index null, that is, if it is an almost complex Finsler structure ($h^i_j = \delta^i_j$, $v^i_j = 0$), then we have $\overset{1}{N} = \overset{1}{T} = \overset{*}{N}$, $\overset{1}{R} = \overset{*}{R}$, $\tilde{C} = \overset{1}{C} = \overset{*}{C}$, $\tilde{P} = \overset{1}{P} = \overset{*}{P}$, $\overset{2}{N} = \overset{1}{S} = \overset{**}{N}$, $\overset{2}{R} = \overset{**}{R}$, $\overset{2}{C} = \overset{**}{C}$, $\overset{2}{P} = \overset{**}{P}$, and (3.6) can be constructed with $\overset{3}{R}{}^i_{jk} = \overset{**}{S}{}^i_{jk}$, and $f^i_m f^r_j f^s_k \tilde{T}^m_{jk} = \overset{**}{T}{}^i_{jk}$, only (cf. with R. Miron [7], Theorem 3.2, where there are $\overset{*}{N}, \overset{*}{R}, \dots, \overset{**}{T}$).

4. $\tilde{F}(3,1)$-structures on the tangent bundle

Let N be a fixed non-linear connection on TM.

An $\tilde{F}(3,1)$-structure of index k' on TM is given by a tensor field $\tilde{F} \in \tau^1_1(TM)$ with property:

$$
(4.1)\qquad \begin{aligned}
&\tilde{F}^3 + \tilde{F} = 0 \quad \operatorname{rank} \|\tilde{F}(x,y)\| = 2n - k' = 2p',\\
&0 \le k' < 2n,\ \forall (x,y) \in TM
\end{aligned}
$$

In the adapted basis $X_A = \{\delta_i, \dot{\partial}_i\}$, $A = \overline{1,2n}$, $i = \overline{1,n}$, $\tilde{F}$ can be represented by:

$$\begin{aligned}\tilde{F} =& \overset{1}{F}{}^i_j \delta_i \otimes dx^j + \overset{2}{F}{}^i_j \delta_i \otimes \delta y^j + \\ &+ \overset{3}{F}{}^i_j \dot{\partial}_i \otimes dx^j + \overset{4}{F}{}^i_j \dot{\partial}_i \otimes \delta y^j,\end{aligned} \tag{4.2}$$

where $\overset{\alpha}{F}{}^i_j$ $(\alpha = 1,2,3,4)$ are Finsler tensor on M. Then we have

$$\tilde{F}(\delta_j) = \overset{1}{F}{}^i_j \delta_i + \overset{3}{F}{}^i_j \dot{\partial}_i, \quad \tilde{F}(\dot{\partial}_j) = \overset{2}{F}{}^i_j \delta_i + \overset{4}{F}{}^i_j \dot{\partial}_i, \tag{4.3}$$

and the condition (4.1) is equivalent with

$$\begin{cases} (\overset{1}{F}{}^i_h \overset{1}{F}{}^h_s + \overset{2}{F}{}^i_h \overset{3}{F}{}^h_s)\overset{1}{F}{}^s_j + (\overset{1}{F}{}^i_h \overset{2}{F}{}^h_s + \overset{2}{F}{}^i_h \overset{4}{F}{}^h_s)\overset{3}{F}{}^s_j + \overset{1}{F}{}^i_j = 0, \\ (\overset{1}{F}{}^i_h \overset{1}{F}{}^h_s + \overset{2}{F}{}^i_h \overset{3}{F}{}^h_s)\overset{2}{F}{}^s_j + (\overset{1}{F}{}^i_h \overset{2}{F}{}^h_s + \overset{2}{F}{}^i_h \overset{4}{F}{}^h_s)\overset{4}{F}{}^s_j + \overset{2}{F}{}^i_j = 0, \\ (\overset{3}{F}{}^i_h \overset{1}{F}{}^h_s + \overset{4}{F}{}^i_h \overset{3}{F}{}^h_s)\overset{1}{F}{}^s_j + (\overset{3}{F}{}^i_h \overset{2}{F}{}^h_s + \overset{4}{F}{}^i_h \overset{4}{F}{}^h_s)\overset{3}{F}{}^s_j + \overset{3}{F}{}^i_j = 0, \\ (\overset{3}{F}{}^i_h \overset{1}{F}{}^h_s + \overset{4}{F}{}^i_h \overset{3}{F}{}^h_s)\overset{2}{F}{}^s_j + (\overset{3}{F}{}^i_h \overset{2}{F}{}^h_s + \overset{4}{F}{}^i_h \overset{4}{F}{}^h_s)\overset{4}{F}{}^s_j + \overset{4}{F}{}^i_j = 0, \end{cases} \tag{4.4}$$

Also, we suppose that the components of $\tilde{F}$ fulfil the conditions

$$H = -\tilde{F}^2, \quad V = \tilde{F}^2 + I, \tag{4.5}$$

so as to be orthogonal and supplementary projectors.

The Nijenhuis tensor of $\tilde{F}$ is given by

$$\begin{aligned}\tilde{N}(X,Y) =& H[X,Y] + \tilde{F}[\tilde{F}X,Y] + \tilde{F}[X,\tilde{F}Y] - \\ &- [\tilde{F}X,\tilde{F}Y], \quad \forall X,Y \in \Xi(TM).\end{aligned} \tag{4.6}$$

The integrability condition of the $\tilde{F}(3,1)$-structure $\tilde{F}$ is $\tilde{N}(X,Y) = 0$ $\forall X,Y \in \Xi(TM)$, [4], [9]. It is sufficient to calculate $\tilde{N}(\delta_j,\delta_k)$, $\tilde{N}(\delta_j,\dot{\partial}_k)$ and $\tilde{N}(\dot{\partial}_j,\dot{\partial}_k)$, and we can determine $\tilde{N}(X,Y)$.

If $f^i_j(x,y)$ is an $f(3,1)$-Finsler structure of index K on M, then on TM, in the presence of a non-linear connection, we have some important special cases:

$$\begin{cases} \overset{1}{\tilde{F}} = f^i_j \delta_i \otimes dx^j + f^i_j \dot{\partial}_i \otimes \delta y^j, \\ \overset{2}{\tilde{F}} = f^i_j \delta_i \otimes dx^j - f^i_j \dot{\partial}_i \otimes \delta y^j, \\ \overset{3}{\tilde{F}} = f^i_j \delta_i \otimes \delta y^j + f^i_j \dot{\partial}_i \otimes dx^j. \end{cases} \tag{4.7}$$

The tensor fields $\overset{\alpha}{\tilde{F}}$ $(\alpha = 1,2,3)$ given by (4.6) are $\tilde{F}(3,1)$-structures of special type on TM. Indeed, conditions (2.2) and (2.3) being fulfilled for f^i_j, we get

$$(4.8)\qquad \begin{cases} \overset{\alpha}{\tilde{F}}{}^3 + \overset{\alpha}{\tilde{F}} = 0 \quad \operatorname{rank} \|\overset{\alpha}{\tilde{F}}(x,y)\|_{\mathcal{H}TM} = \\ \qquad\qquad = \operatorname{rank} \|\overset{\alpha}{\tilde{F}}(x,y)\|_{\mathcal{V}TM} = n-k \\ H = -\overset{\alpha}{\tilde{F}}{}^2 = h^i_j \delta_j \otimes dx^j + h^i_j \dot{\partial}_i \otimes \delta y^j \\ V = \overset{\alpha}{\tilde{F}}{}^2 + I = v^i_j \delta_i \otimes dx^j + v^i_j \dot{\partial}_i \otimes \delta y^j, \quad \forall\, \alpha = 1,2,3, \end{cases}$$

where $h^i_j = -f^i_h f^h_j$, $v^i_j = f^i_h f^h_j + \delta^i_j$.

To the formulae (4.7) we give $\tilde{F}(3,1)$-structures on TM by the lift of an $f(3,1)$-Finsler structure from M to the total space TM of the tangent bundle $T(M)$.

5. The integrability of $f(3,1)$-Finsler structures

Let N be a non-linear connection of $T(M)$. Then an $f(3,1)$-Finsler structure on the base manifold M can be lifted to an $\tilde{F}(3,1)$-structure on $T(M)$ in three manners (4.7). The values of the Finsler components of $\tilde{F}$ from (4.2) are given in the following table:

$\tilde{F}$	$\overset{1}{F}{}^i_j$	$\overset{2}{F}{}^i_j$	$\overset{3}{F}{}^i_j$	$\overset{4}{F}{}^i_j$
$\overset{1}{\tilde{F}}$	f^i_j	0	0	f^i_j
$\overset{2}{\tilde{F}}$	f^i_j	0	0	$-f^i_j$
$\overset{3}{\tilde{F}}$	0	f^i_j	f^i_j	0

We remark the following relations

$$\begin{aligned} \overset{1}{\tilde{F}}(\delta_j) &= f^i_j \delta_i, & \overset{1}{\tilde{F}}(\dot{\partial}_j) &= f^i_j \dot{\partial}_i, \\ \overset{2}{\tilde{F}}(\delta_j) &= f^i_j \delta_i, & \overset{2}{\tilde{F}}(\dot{\partial}_j) &= -f^i_j \dot{\partial}_i, \\ \overset{3}{\tilde{F}}(\delta_j) &= f^i_j \dot{\partial}_i, & \overset{3}{\tilde{F}}(\dot{\partial}_j) &= f^i_j \delta_i. \end{aligned}$$

Definition 5.1. An $f(3,1)$-Finsler structure of index K on a differentiable manifold M is called integrable of type I, II or III with respect to the non-linear connection N, if the corresponding lifted $\overset{1}{\tilde{F}}(3,1)$-, $\overset{2}{\tilde{F}}(3,1)$-, or $\overset{3}{\tilde{F}}(3,1)$-structure is integrable.

We characterize these cases of integrability using only the invariants of the group G_f.

Theorem 5.1. *The $f(3,1)$-Finsler structure, (2.1) is integrable of type I, II or III if and only if the invariants of the group G_f have the values given in the following table*

Type of integrability	Characterization by invariants
I	$\overset{1}{T}{}^i_{jk}=0;\ \overset{1}{R}{}^i_{jk}=0;\ \overset{1}{C}{}^i_{jk}=0;\ \overset{1}{P}{}^i_{jk}=0,\ \overset{1}{S}{}^i_{jk}=0$
II	$\overset{1}{T}{}^i_{jk}=0;\ \overset{2}{R}{}^i_{jk}=0;\ \overset{2}{C}{}^i_{jk}=0;\ \overset{2}{P}{}^i_{jk}=0,\ \overset{1}{S}{}^i_{jk}=0$
III	$\overset{3}{T}{}^i_{jk}=0;\ \overset{3}{R}{}^i_{jk}=0;\ \overset{3}{C}{}^i_{jk}=0;\ \overset{3}{P}{}^i_{jk}=0,$ $\overset{3}{S}{}^i_{jk}=0;\ \tilde{T}^i_{jk}=0$

PROOF. The $f(3,1)$-Finsler structure is integrable of type I if and only if $\tilde{N}(X,Y)=0$ for $\overset{1}{\tilde{F}}$. But $\tilde{N}(X,Y)=0$, $\forall X,Y\in\Xi(TM)$ is equivalent to

$$\tilde{N}(\delta_j,\delta_k)=0,\quad \tilde{N}(\delta_j,\dot{\partial}_k)=0,\quad \tilde{N}(\dot{\partial}_j,\dot{\partial}_k)=0,$$

which are equivalent to $\overset{1}{T}{}^i_{jk}=0$, $\overset{1}{R}{}^i_{jk}=0$, $\overset{1}{C}{}^i_{jk}=0$, $\overset{1}{P}{}^i_{jk}=0$, $\overset{1}{S}{}^i_{jk}=0$. The proofs of II and III follow the same pattern.

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