Publ. Math. Debrecen
39 / 1–2 (1991), 107–111

Nonlinear connection in cotangent bundle

By GH. ATANASIU (Braşov) and F.C. KLEPP (Timişoara)

Dedicated to Professor Lajos Tamássy on the occasion of his 65th birthday

1. Introduction

In the present paper we give a dual theory corresponding to the theory of nonlinear connections in Lagrange spaces, given by the first author in [1].

We study the possibility of determination of a nonlinear connection N on the cotangent bundle $T^*(M) = (T^*M, \pi^*, M)$ of a differentiable manifold M, starting from a linear connection $\overset{*}{\nabla}{}^v$ on the vertical subbundle VT^*M of T^*M. We also study the possibility of the determination of a distinguished connection $F^*\Gamma = (N_{jk}, F_j{}^i{}_k, C_i{}^{jk})$ on T^*M, whose restriction to VT^*M coincides with $\overset{*}{\nabla}{}^v$.

The terminology and notations follow those in MIRON's papers [4], [5], [6], with some modifications given in ATANASIU–HASHIGUCHI's paper [2]. For convenience sake, in §2 we shall sketch the material from MIRON's theory necessary for our discussions.

The authors wish to express here their sincere gratitude to Professor Dr. R. MIRON for his invaluable suggestions and encouragement.

2. Distinguished geometrical objects on the cotangent bundle

Let M be an n–dimensional C^∞ -manifold and $\pi^* : T^*M \to M$ its cotangent bundle. Since a point of T^*M is a covector (x,p) at a point x of the base manifold M, a coordinate system $x = (x^i)$ in M induces a canonical coordinate system $(x,p) = (x^i, p_i)$ in T^*M by $p = p_i(dx^i)$.

A transformation of the canonical coordinate system in T^*M is given by

$$(2.1)\qquad \widetilde{x}^i = \widetilde{x}^i(x^1, \dots, x^n), \quad \widetilde{p}_i = \frac{\partial x^r}{\partial \widetilde{x}^i} p_r; \quad \det\left(\frac{\partial \widetilde{x}^i}{\partial x^r}\right) \neq 0$$

The kernel of the differential of the maping $\pi^* : T^*M \to M$ gives us the vertical subbundle VT^*M of the tangent bundle TT^*M. If $V_{(x,p)}$ is the fiber of VT^*M at the point $(x,p) \in T^*M$, then we have the vertical distribution $V : (x,p) \in T^*M \to V_{(x,p)} \subset T_{(x,p)}T^*M$.

A *nonlinear connection* N on T^*M is a distribution of class C^∞ given by $(x,p) \in T^*M \to N_{(x,p)} \subset T_{(x,p)}T^*M$ such that $T_{(x,p)}T^*M = N_{(x,p)} \oplus V_{(x,p)}$. N is spanned by

$$\delta_i = \partial_i + N_{ri}(x,p)\dot{\partial}^r \,, \tag{2.2}$$

where $\partial_i = \partial/\partial x^i$, $\dot{\partial} = \partial/\partial p_i$. $\{\delta_i, \dot{\partial}_i\}$ is a local basis adapted to the distributions N and V, and its dual basis is $\{dx^i, \delta p_i\}$, where

$$\delta p_i = dp_i - N_{ir}dx^r \,. \tag{2.3}$$

The functions $N_{jk}(x,p)$ are called the *coefficients* of the nonlinear connection N and these are uniquely determined.

With respect to (2.1), we have

$$\begin{aligned} \widetilde{\delta}_i &= \frac{\partial x^r}{\partial \widetilde{x}^i}\delta_r \,, \quad \widetilde{\dot{\partial}}^i = \frac{\partial \widetilde{x}^i}{\partial x^r}\dot{\partial}^r \,, \\ d\widetilde{x}^i &= \frac{\partial \widetilde{x}^i}{\partial x^r}dx^r \,, \quad \delta\widetilde{p}_i = \frac{\partial x^r}{\partial \widetilde{x}^i}\delta_{p_r} \,, \end{aligned} \tag{2.4}$$

and the transformation formula of N_{jk} with respect to (2.1) is

$$\widetilde{N}_{jk}(\widetilde{x},\widetilde{p}) = \frac{\partial x^s}{\partial \widetilde{x}^j}\frac{\partial x^t}{\partial \widetilde{x}^k}N_{st}(x,p) + p_r\frac{\partial^2 x^r}{\partial \widetilde{x}^j \partial \widetilde{x}^k} \,. \tag{2.5}$$

A *distinguished tensor field*, a *d–tensor field* for short, of type (r,s) on M is defined by its components, satisfying the classical law of transformation with respect to (2.1).

A *d–connection* on M is a triad $F^*\Gamma = (N,F,C)$, where $N = (N_{jk})$ is a nonlinear connection, $F = (F_j{}^i{}_k)$ is a special d–object and $C = (C_i{}^{jk})$ is a d–tensor field of type (2.1). The transformation formulal of N_{jk}, $F_j{}^i{}_k$, $C_i{}^{jk}$ with respect to (2.1) are (2.5) and

$$\widetilde{F}_j{}^i{}_k = \frac{\partial \widetilde{x}^i}{\partial x^r}\frac{\partial x^s}{\partial \widetilde{x}^j}\frac{\partial x^t}{\partial \widetilde{x}^k}F_s{}^r{}_t + \frac{\partial \widetilde{x}^i}{\partial x^r}\frac{\partial^2 x^r}{\partial \widetilde{x}^j \partial \widetilde{x}^k} \,, \tag{2.6}$$

$$\widetilde{C}_i{}^{jk} = \frac{\partial x^r}{\partial \widetilde{x}^i}\frac{\partial \widetilde{x}^j}{\partial x^s}\frac{\partial \widetilde{x}^k}{\partial x^t}C_r{}^{st} \,. \tag{2.7}$$

Given a d–connection, the h– and v–*covariant derivatives* are defined for a d–tensor field, e.g. for K^i_j, by

(2.8) $K^i_{j|k} = \delta_k K^i_j + K^r_j F_r{}^i{}_k - K^i_r F_j{}^r{}_k,\ K^i_j|^k = \dot\partial^k K^i_j + K^r_j C_r{}^{jk} - K^i_r C_j{}^{rk}.$

For example the, h– and v–covariant derivatives of the d–covector p_i are:

$$p_{i|h} = N_{ih} - F_i{}^o{}_h\,,\quad p_i|^h = \delta^h_i - C_i{}^{oh}\,, \tag{2.9}$$

where the index "o" means the contraction with p_i.

We denote

$$D_{ih} = N_{ih} - F_i{}^o{}_h\,,\quad d_i{}^h = \delta^h_i - C_i{}^{oh}\,. \tag{2.10}$$

The d–tensor field D_{ih} is called the h–*deflection* tensor field and $d_i{}^h$ is called the v–*deflection* tensor field of the d–connection $F^*\Gamma$ on T^*M.

3. Nonlinear connections and d–connections on T^*M derived from linear connections given only on the vertical subbundle VT^*M

Let us denote by $\overset{*}{\Gamma}_j{}^i{}_k$, $C_i{}^{jk}$ the coefficients of the linear connection $\overset{*}{\nabla}{}^v$ obtained by restricting the d–connection $F^*\Gamma$ on the vertical subbundle VT^*M:

$$\overset{*}{\nabla}{}^v_{\partial_k}\dot\partial^i = -\overset{*}{\Gamma}_j{}^i{}_k\dot\partial^j\,,\quad \overset{*}{\nabla}{}^v_{\dot\partial k}\dot\partial^j = -C_i{}^{jk}\dot\partial^i\,. \tag{3.1}$$

Effecting a change of the canonical coordinates $(x,p)\to(\widetilde{x},\widetilde{p})$ on T^*M we can deduce by a simple calculation the transformation law of $\overset{*}{\Gamma}_j{}^i{}_k$

$$\widetilde{\overset{*}{\Gamma}}_j{}^i{}_k = \frac{\partial\widetilde{x}^i}{\partial x^r}\frac{\partial x^s}{\partial\widetilde{x}^j}\frac{\partial x^t}{\partial\widetilde{x}^k}\overset{*}{\Gamma}_s{}^r{}_t + \left(\delta^r_j\frac{\partial\widetilde{x}^i}{\partial x^m} - \widetilde{C}_j{}^{ir}p_m\right)\frac{\partial^2 x^m}{\partial\widetilde{x}^r\partial\widetilde{x}^k}\,, \tag{3.2}$$

and $C_i{}^{jk}$ has the transformation law given by (2.7).

There arises the following problem: given a linear connection $\overset{*}{\nabla}{}^v\left(\overset{*}{\Gamma}_j{}^i{}_k, C_i{}^{jk}\right)$ on the vertical subbundle VT^*M, under which conditions does $\overset{*}{\nabla}{}^v$ determine

(i) a nonlinear connection on T^*M

(ii) a d–connection on T^*M whose restriction to VT^*M coincides with $\overset{*}{\nabla}{}^v$.

To solve this problem we first give the following definition.

Definition 3.1. A linear connection $\overset{*}{\nabla}{}^v$ on the vertical bundle is called *strongly regulated* if the v–deflection tensor field $d_i{}^k = \delta_i{}^k - C_i{}^{ok}$ is non-degenerate, i.e. rank $\|d_i{}^k\| = n$.

We show that for a strongly regulated linear connection $\overset{*}{\nabla}{}^v$ the above problem has a natural solution. – In fact, we multiply (3.2) with $\widetilde{p}_i$, and we get

$$\widetilde{\overset{*}{\Gamma}}{}_j{}^o{}_k = \frac{\partial x^s}{\partial \widetilde{x}^j}\frac{\partial x^t}{\partial \widetilde{x}^k}\overset{*}{\Gamma}{}_s{}^o{}_t + \widetilde{d}_j{}^r p_m \frac{\partial^2 x^m}{\partial \widetilde{x}^r \partial \widetilde{x}^k}. \tag{3.3}$$

Since $\overset{*}{\nabla}{}^v$ is strongly regulated, the inverse of the matrix $(d_j{}^r)$ exists, and it will be denoted by $(d_j{}^r)^{-1} = (b^r{}_j)$. Then by mutiplying (3.3) with $\widetilde{b}^j{}_i$ we obtain

$$\widetilde{b}^j{}_i \widetilde{\overset{*}{\Gamma}}{}_j{}^o{}_k = \frac{\partial x^r}{\partial \widetilde{x}^i}\frac{\partial x^s}{\partial \widetilde{x}^k} b^t{}_r \overset{*}{\Gamma}{}_t{}^o{}_s + p_m \frac{\partial^2 x^m}{\partial \widetilde{x}^i \partial \widetilde{x}^k}. \tag{3.4}$$

We define

$$N_{ik} = b^r{}_i \overset{*}{\Gamma}{}_r{}^o{}_k, \tag{3.5}$$

and from (3.4) we get

$$\widetilde{N}_{ik} = \frac{\partial x^r}{\partial \widetilde{x}^i}\frac{\partial x^s}{\partial \widetilde{x}^k} N_{rs} + p_m \frac{\partial^2 x^m}{\partial \widetilde{x}^i \partial \widetilde{x}^k} \tag{3.6}$$

that is, the transformation law of the nonlinear connection on T^*M.

In this way have obtained

Theorem 3.1. *Let be given a strongly regulated linear connection* $\overset{*}{\nabla}{}^v$ *on the vertical subbundle* VT^*M *of the tangent bundle* $\pi^* : T^*M \to M$, *with coefficients* $\left(\overset{*}{\Gamma}{}_j{}^i{}_k, C_i{}^{jk}\right)$, *then:*

(a) there exists a global nonlinear connection N *defined on* T^*M, *and determined only by* $\overset{*}{\nabla}{}^v$. *The coefficients* N_{jk} *of* N *are given by the formulal (3.5), where* $(b^i{}_j) = (d_j{}^i)^{-1}$*;*

(b) the triad $(N_{jk}, F_j{}^i{}_k, C_i{}^{jk})$ *with*

$$F_j{}^i{}_k = \overset{*}{\Gamma}{}_j{}^i{}_k + C_j{}^{ir} N_{kr} \tag{3.7}$$

is a d–connection, which depends only on the vertical connection $\overset{*}{\nabla}{}^v$ *and whose restriction at* VT^*M *is* $\overset{*}{\nabla}{}^v$.

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GHEORGHE ATANASIU
CATEDRA DE MATEMATICĂ
UNIVERSITATEA DIN BRAŞOV
2200–BRAŞOV, ROMANIA

FRANCISC C. KLEPP
CATEDRA DE MATEMATICI
INST. POLITEHNIC "TRAIAN VUIA"
1900–TIMIŞOARA, ROMANIA

(Received July 21, 1988)