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On the geometry of generalized metric spaces I. Connections and identities

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Dedicated to the memory of Professor Arthur Moór

§0. Introduction

The investigation of generalized metric space began at the same time as that of Finsler spaces. While the latter had been studied in the famous works of E. CARTAN, L. BERWALD and H. RUND, there were many persons who were interested in the geometry of generalized metric spaces, such as M.S. KNEBELMAN, A. MOÓR and others. Recently, from the view point of theoretical physics, R. MIRON'S school defined a generalized Lagrange space (for instance, [1], [3], [8], [9]) which is considered a generalized metric space without the homogeneity condition. In the present paper, the mathematical discussions are based on the theories due to Prof. M. MATSUMOTO [7] and the author [6].

Let M be an n–dimensional manifold of class C^∞ with local coordinates (x^i) and $T(M) \mid \pi : T \to M$ its tangent vector bundle with local coordinates (x^i, y^i). Let us denote by M_T the manifold of non–vanishing tangent vectors: $M_T := T(M) - \{0\}$. A Finsler space is a pair $(M_T, F(x,y))$, and a Lagrange space is a pair $(T(M), L(x,y))$. A generalized metric space is called a generalized Finsler space (M_T, g_{ij}), or a generalized Lagrange space $(T(M), g_{ij})$ depending on the fact, whether the tensor $g_{ij}(x,y)$ is positively homogeneous (p–homogeneous) of degree zero in y, or not. The generalized metric space mentioned in the present paper is a generalized Finsler space, and the tensor g_{ij} is called the metric tensor.

Homogeneity condition follows from the well known

Lemma 0.1. (Zermelo). *The arc length s of a curve $x^i = x^i(t)$ in M given by*

$$s = \int_{t_0}^{t} \sqrt{g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j}\,dt, \quad \dot{x}^i := dx^i/dt\,,$$

is invariant under any parameter change $\bar{t}=\bar{t}(t)$, $d\bar{t}/dt>0$, if and only if the metric tensor $g_{ij}(x,y)$ is p–homogeneous of degree zero.

The required conditions for the metric tensor $g_{ij}(x,y)$ are as follows:

(a) class C^∞ in (x^i,y^i), (b) positive definite, (c) positively homogeneous of degree zero in y^i and (d) $g^*_{ij}:=\dot{\partial}_i\dot{\partial}_jF^2/2$ is non–degenerate, where $\dot{\partial}_j:=\partial/\partial y^j$ and $F^2:=g_{ij}y^iy^j$.

From the above conditions, we see that a pair $(M_T,F(x,y))$ is a Finsler space and a function $F(x,y)$ is a called a Finsler metric associated with the generalized metric g_{ij}.

In the geometry of M_T a non–linear connection N^i_k plays an important role. We shall define N^i_k in the following way. Let us calculate below

$$2G^i:=g^{*ih}(y^j\partial_j\dot{\partial}_hF^2-\partial_hF^2)/2,\quad \partial_j:=\partial/\partial x^j,\quad g^{*ih}g^*_{hj}:=\delta^i_j\,.$$

A non–linear connection N^i_k is required to satisfy $2G^i=N^i_ky^k$, hence we have

$$(0.1)\qquad N^i_k=G^i_k-P^i{}_k;\quad G^i_k:=\dot{\partial}_kG^i,\quad P^i{}_k:=(y^j\dot{\partial}_kN^i_j-N^i_k)/2\,.$$

It is easily shown that the G^i_k are transformed as non–linear connection parameters and therefore $P^i{}_k$ is an arbitrary tensor. It is p–homogeneous of degree one. This requirement is based on the fact that in the theory of (M_T,g_{ij}) a geodesic in M is expressed by $\delta y^i/ds=0$, $y^i:=dx^i/ds$.

§1 is devoted to a short discussion of the geometry of the bundle M_T. In §2 and §3 metrical connection $\mathrm{C}\Gamma(N)$, h–metrical connection $\mathrm{R}\Gamma(N)$ and non–metrical connection $\mathrm{B}\Gamma(G)$ are introduced, and many identities of their torsion and curvature tensors are given. In §4 we shall consider the special cases of generalized metric spaces and give some results.

In the usual manner, we raise or lower indices by means of g_{ij}.

§1. Preliminaries on geometry of the bundle M_T

The content of this section refers to an idea of S. KOBAYASHI and K. NOMIZU [2]. To make clear the discussion, the content is arranged in sections: A. ~ J., in which some proofs are omitted. Most of the equations will be expressed without indices and written in the matrix product form. The last of these sections refers to Lie derivation.

A. In a generalized metric space we define the non–linear connection N by (0.1). The tangent space $T_y(M_T)$ at $y=(x^i,y^i)$ in M_T is decomposed into two distributions $H_y(M_T)$ and $V_y(M_T)$ in the following way. If we introduce two 1–forms θ, θ' in $T_y^*(M_T)$ by

$$\theta := dx, \quad \theta' := dy + N dx \quad ; \quad \theta^i, \theta^{(i)} \,,$$

we can define

$$(1.1) \qquad \begin{aligned} H_y(M_T) &:= \{X \in T_y(M_T) \,|\, \theta'(X) = 0\}\,, \quad \text{(horizontal space)}\,; \\ V_y(M_T) &:= \{X \in T_y(M_T) \,|\, \theta(X) = 0\}\,, \quad \text{(vertical space)}\,, \end{aligned}$$

such that $T_y(M_T) = H_y(M_T) \oplus V_y(M_T)$. The dual bases (d, d') of (θ, θ') are given by

$$d := \partial x - N \partial y, \quad d' := \partial y; \quad d_i, \quad d_{(i)}$$

so that the distributions $H_y(M_T)$ and $V_y(M_T)$ are spanned by $(d_i)_y$ and $(d_{(i)})_y$, respectively.

B. Because the linear spaces H_y, V_y and $T_p(M)$, $p = \pi(y) \in M$ are linearly isomorphic to a single vector space V of dimension n, we can define a horizontal (vertical) lift $l_N(l^v)$ as follows:

$$(1.2) \qquad \begin{aligned} & l_N : T_p(M) \to H_y(M_T) \,|\, l_N \cdot (\partial x)_p = d_y = (\partial x)_y - N(\partial y)_y\,; \\ & l^v : T_p(M) \to V_y(M_T) \,|\, l^v \cdot (\partial x)_p = d'_y = (\partial y)_y\,. \end{aligned}$$

Let $L(M) = (L, \tau, M, GL(n))$ be the frame bundle of $M, z_n = (x^i, z_a^i) \in L(M)$, such that n vectors $z_a := z_a^i(\partial/\partial x^i)_p$ form the basis of $T_p(M)$ at p in M. On the other hand, the frame bundle $L(M_T) = (\bar{L}, \bar{\tau}, M_T, GL(2n))$ of M_T consists of a set of $2n$ basis vectors of $T_y(M_T)$. Let us denote by P_N the image of the product lifts $\bar{l}_N := l_N \times l^v : L(M) \to L(M_T)$. On taking account of (1.2), we find for $u = (B, B') \in P_N$

$$B = (B_a) = l_N \cdot z_n = (z_a^i d_i), \qquad B' = (B_{(a)}) = l^v \cdot z_n = (z_a^i d_{(i)})\,.$$

Here we shall define the action $g \in GL(n)$ to $u \in P_N$ as follows:

$$u\bar{g} = (Bg, B'g) = (B, B') \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \quad \bar{g} := \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \in D(GL(n))\,,$$

where the double group $D(GL(n))$ [4] defined by $D(GL(n)) := \{\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \mid g \in GL(n)\}$ is a sub–group of $GL(2n)$. By virtue of the non–linear connection N, we can state that the frame bundle $L(M_T)$ is reduced to its sub–frame bundle $B_D = (P_N, \bar{\tau}, M_T, D(GL(n)))$. The image bundle $P_N(M_T) := (P_N, \bar{\tau}, M_T, GL(n))$ is essentially regarded as $B_D(M_T)$.

C. A vector field X_y in $T_y(M_T)$ is uniquely expressed as

$$X = \xi^i(x,y)d_i + \xi^{(i)}(x,y)d_{(i)}\,,$$

which implies that any vector X in the $\mathcal{F}(M_T)$–module $\mathcal{X}(M_T)$ is decomposed into h– and v– parts: $X = X^h + X^v$. The same applies to (r,s)–type tensor field K on M_T, that is K has 2^{r+s} parts. For example, a $(1,2)$–type tensor K has 8 parts; one of the 8 parts is denoted by $K^h_{hv} := K^i_{j(k)}(x,y)d_i \otimes \theta^j \otimes \theta^{(k)}$, where the indices i, j and (k) indicate the specified components of the horizontal or the vertical part of K. To simplify the calculations, we shall introduce an alternative definition of tensor fields on M_T, i.e. of a tensor function (or a tensor–valued p–form) on P_N in the following way. Let e_a, $e_{(a)}$ be a natural basis of the product spaces $\bar{V} := V \times V\,;\, V \times \{0\}$, $\{0\} \times V$, and e^b_a the basis of $V^1_1 = V \otimes V^*$. On the other hand, the basis of $\bar{V}^1_1$ is composed of $e^b_a, e^{(b)}_a, e^b_{(a)}$ and $e^{(b)}_{(a)}$. A tensor function $\bar{K}_u$ is defined by an isomorphism

$$(1.3) \qquad I_u : \mathcal{I}^r_s(M_T) \to \bar{V}^r_s\,,$$

or we can say that a tensor function $\bar{K}_u = I_u \cdot K_y$ of (r,s)–type at y in M_T is a $\bar{V}^r_s$–valued function (each part of specified component) of $u = (x^i, y^i, z^i_a)$ in P_N. E.g. for $K_y = (K^h_{hv})_y$, we get

$$(1.3)' \qquad \bar{K} : u \to \bar{K}_u \in \bar{V}^r_s \mid \bar{K}_u = z^a_i K^i_{j(k)}(x,y) z^j_b z^k_c e^{b(c)}_a = I_u \cdot K_y\,.$$

D. Now let us consider the tangent space $T_u(P_N)$ at u in P_N and define the connection $\tilde{\omega}^a_b$ on P_N (accordingly the connection ω^i_j on M_T). This is carried out by the following decomposition of $T_u(P_N)$: $T_u(P_N) = \Gamma_u(P_n) \oplus G_u(P_N)$, where

$$\Gamma_u(P_N) := \{\bar{X} \in T_u(P_N) \,|\, \tilde{\omega}(\bar{X}) = 0\}\,,$$
$$G_u(P_N) := \{\bar{X} \in T_u(P_N) \,|\, \tilde{\Theta}(\bar{X}) = 0\}\,;$$

$$(1.4) \quad \begin{aligned} &(a) \quad \tilde{\omega} = z^{-1}(dz + \omega z), \quad \tilde{\omega}^a_b := z^a_i(dz^i_b + \omega^i_j z^j_b), \\ &\qquad\;\; \omega^i_j := F_j{}^i{}_k \theta^k + C_j{}^i{}_k \theta^{(k)}\,; \\ &(b) \quad \tilde{\Theta} = (\tilde{\theta}, \tilde{\theta}'), \quad \tilde{\theta} := z^{-1}\theta, \quad \tilde{\Theta}' := z^{-1}\theta', \quad z^{-1} = (z^a_i). \end{aligned}$$

Remark. Fundamentally we should consider the decomposition of $T_u(B_D)$ instead of $T_u(P_N)$. In this case the connection $\tilde{\omega}$ must be replaced by $\begin{pmatrix} 0 & \tilde{\omega} \\ \tilde{\omega} & 0 \end{pmatrix}$ and the transition from $T_u(P_N)$ to $T_u(B_D)$ is obvious.

The $\bar{V}$–valued 1–form $\tilde{\Theta} = \tilde{\theta}^a e_a + \tilde{\theta}^{(a)} e_{(a)}$ and the Lie algebra–valued (or V_1^1–valued) 1–form $\tilde{\omega} := \tilde{\omega}_b^a L_a^b$ are called a basic form and a fundamental form on $T_u^*(P_N)$ respectively. It is easily seen that the dual bases of $\tilde{\theta}, \tilde{\theta}'$ and $\tilde{\omega}$ are given as follows:

$$(1.5)\qquad \begin{aligned} &(a)\quad H_a := z_a^i(d_i - F_j{}^k{}_i L_k^j),\ V_a := z_a^i(d_{(i)} - C_j{}^k{}_i L_k^j),\\ &(b)\quad L_a^b := z_a^i(\partial/\partial z_b^i);\qquad L_k^j := z_b^j(\partial/\partial z_b^k) = z_b^j L_a^b z_k^a\,, \end{aligned}$$

where L_a^b is the basis of left invariant vectors on $T(GL(n))$ (Lie algebra of $GL(n)$;[2], p. 38) and the bases H_a and V_a span the linear space $\Gamma_u(P_N) = H_u(P_N) \oplus V_u(P_N)$.

E. If the tensor K_u of the (r,s)–type module $(T(P_N))_s^r$ is contained in $(\Gamma_u(P_N))_s^r$, then the K_u is called tensorial. Here we shall show a geometrical example of a non–tensorial vector $\bar{X}_u$ on $T_u(P_N)$. When a vector field $X = \xi^i(x)(\partial/\partial x^i)$ on a domain U of the underlying space M generates a curve $x_t : t \to x_t^i = \phi^i(t,x)$, we find $dx_t^i/dt = \xi^i(x_t)$. The curve x_t in M is extended to a curve $u_t = (x_t, y_t, z_t)$ in P_N such that u_t should be the solution of

$$(1.6)\quad x_t^i = \phi^i(t,x),\ y_t^i = \xi^i{}_{,j}(x_t)y^j,\ (z_a^i)_t = \xi^i{}_{,j}(x_t)z_a^j\,;\ \xi^i{}_{,j} := \partial_j\xi^i(x_t)\,.$$

The curve u_t induces a vector field $\bar{X}_u$ $(t=0)$ along the curve:

$$(1.7)\qquad \bar{X}_u := \xi^i(x)(\partial/\partial x^i)_u + \xi^i{}_{,j}(x)y^j(\partial/\partial y^i)_u + \xi^i{}_{,j}(x)z_a^j(\partial/\partial z_a^i)_u\,,$$

which, using (1.2) and (1.5)(b), is rewritten as

$$(1.7)'\qquad \bar{X}_u := \xi^i d_i + \xi^{(i)} d_{(i)} + \xi^i{}_{,j} L_i^j\,;\quad \xi^{(i)} := \xi^i{}_{,j}y^j + N_j^i\xi^j\,.$$

Moreover, noting (1.5)(a), we obtain

$$(1.7)''\qquad \bar{X}_u = z_i^a\xi^i H_a + z_i^a\xi^{(i)} V_a + \xi^a{}_b L_a^b\,,$$

where

$$\begin{aligned} &\tilde{\theta}^a(\bar{X}_u) = z_i^a\xi^i,\quad \tilde{\theta}^{(a)}(\bar{X}_u) = z_i^a\xi^{(i)}\,,\\ &\tilde{\omega}_b^a(\bar{X}_u) = \xi^a{}_b = z_i^a\xi^i{}_j z_b^j,\quad \xi^i{}_j := \xi^i{}_{,j} + F_j{}^i{}_k\xi^k + C_j{}^i{}_k\xi^{(k)}, \end{aligned}$$

in which, if the connection parameters satisfy $F_j{}^i{}_k = F_k{}^i{}_j$ and $C_j{}^i{}_k = C_k{}^i{}_j$, we find $\xi^i{}_j = \xi^i{}_{/j} + C_j{}^i{}_k\xi^k{}_{/0}$ (see $C\Gamma(N)$ in §2). Thus we find that the vector $\bar{X}_u$ is not tensorial.

F. Covariant differentiation D of a tensor function on P_N (accordingly covariant differentiation δ on M_T) is defined by $D := \bar{h}^* \cdot d$ in the usual way. The projection $\bar{h}^*$ means $\bar{h}^* : T^*_u(P_N) \to (\Gamma_u(M_T))^*$ such that

$$(1.8) \qquad (a) \quad \bar{h}^* \cdot \tilde{\theta} = \tilde{\theta}, \quad \bar{h}^* \cdot \tilde{\theta}' = \tilde{\theta}', \quad (b) \quad \bar{h}^* \cdot \tilde{\omega} = 0 \,.$$

Since the connection form ω on M_T is tensorial, we see $\bar{h}^* \cdot \omega = \omega$. We shall show

Lemma 1.1. *The following facts hold:*

$$(1.9) \qquad Dz = -\omega z, \quad Dz^{-1} = z^{-1}\omega; \quad Dz^i_a = -\omega^i_j z^j_a, \quad Dz^a_j = z^a_i \omega^i_j \,.$$

PROOF. From the definition (1.4)(a), we have $z\tilde{\omega} = dz + \omega z$, which leads us to

$$\bar{h}^* \cdot (z\tilde{\omega}) = z\bar{h}^* \cdot \tilde{\omega} = 0 = \bar{h}^* \cdot (dz + \omega z) = Dz + \omega z$$

from (1.8)(b). Hence we obtain (1.9).

Remark. This Lemma tells us that for the covariant differentiation D the matrices z^i_a and z^a_j behave like a "moving frame" as in the theory of E. CARTAN.

G. **Lemma 1.2.** *The operations of the basis L^b_a of the Lie algebra $gl(n)$ on the elements z and z^{-1} of $GL(n)$ are*

$$(1.10) \qquad \begin{aligned} L^b_a \cdot z^i_c &= \delta^b_c z^i_a, \quad L^j_k \cdot z^i_c = \delta^i_k z^j_c; \\ L^b_a \cdot z^c_j &= -\delta^c_a z^b_j, \quad L^j_k \cdot z^c_i = -\delta^j_i z^c_k \,. \end{aligned}$$

Let $L(\alpha)$ be a Lie algebra–valued p–form on M_T given by $L(\alpha) := \alpha^i_j(x,y)L^j_i$, where α is a (1,1)–type p–form (tensor or not). In connection with Lemma 1.2, we shall introduce the following

Definition. The operation of the (1,1)–type p–form α on M_T to a tensor field K_y (or tensor–valued q–form) is defined by

$$(1.11) \qquad L(\alpha) \cdot \bar{K}_u = -I_u \cdot (\alpha \cdot K) \,.$$

Really, using (1.10) for an example K^h_{hv} we find

$$\begin{aligned} L(\alpha) \cdot \bar{K}_u &= \alpha^n_m L^m_n \cdot (z^a_i K^i_{j(k)} z^j_b z^k_c) e^{b(c)}_a = \\ &= - z^a_i (\alpha^i_m K^m_{j(k)} - \alpha^m_j K^i_{m(k)} - \alpha^m_k K^i_{j(m)}) z^j_b z^k_c e^{b(c)}_a = \\ &= - I_u \cdot (\alpha \cdot K)_y \,, \end{aligned}$$

where we have put

$$(1.12) \qquad (\alpha \cdot K)^i_{j(k)} := \alpha^i_m K^m_{j(k)} - \alpha^m_j K^i_{m(k)} - \alpha^m_k K^i_{j(m)} \,.$$

Remark. The above definition implies that the relation $L(\alpha) \cdot I_u \cdot = -I_u \cdot \alpha \cdot$ holds for any tensor K_y.

H. Here we shall be concerned with covariant differentiations of tensor fields. Using Lemma 1.1 and (1.12), we have

Theorem 1.3. *The relation* $DI_u\cdot = I_u \cdot \delta$ *holds for any tensor* K_y, *namely*

$$D\bar{K}_u = I_u \cdot \delta K_y\,;$$

(1.13) (*a*) $\delta K := dK + \omega \cdot K$ *for tensor fields on* M_T,

(*b*) $[\delta K] := [dK] + [\omega \cdot K]$ *for tensor–valued* q*–forms on* M_T.

PROOF. For an example $K_y = K^h_{hv}$ we find

$$\begin{aligned} D\bar{K}_u &= D(z^a_i K^i_{j(k)} z^j_b z^k_c) e^{b(c)}_a = \\ &= z^a_i (dK^i_{j(k)} + \omega^i_m K^m_{j(k)} - \omega^m_j K^i_{m(k)} - \omega^m_k K^i_{j(m)}) z^j_b z^k_c e^{b(c)}_a = \\ &= z^a_i \{dK^i_{j(k)} + (\omega \cdot K)^i_{j(k)}\} z^j_b z^k_c e^{b(c)}_a = \\ &= z^a_i (\delta K^i_{j(k)}) z^j_b z^k_c e^{b(c)}_a = I_u \cdot \delta K_y, \end{aligned}$$

where we have put $\delta K^i_{j(k)} := dK^i_{j(k)} + (\omega \cdot K)^i_{j(k)}$. The bracket [ ] means

$$[dK](X,Y) := X(K(Y)) - Y(K(X)) - K([X,Y]) \quad \text{for 1–form } K,$$
$$[\omega K](X,Y) := \omega(X)K(Y) - \omega(Y)K(X) \quad \text{for} \quad X, Y \in T_y(M_T).$$

On M_T the inner product of $\iota_X \cdot$ by $X = \xi^i d_i + \xi^{(i)} d_{(i)}$ with the bases vectors θ^i, $\theta^{(i)}$ gives $\iota_X \cdot \theta^i = \theta^i(X) = \xi^i$ and $\iota_X \cdot \theta^{(i)} = \theta^{(i)}(X) = \xi^{(i)}$. We shall define $\nabla_X := \iota_X \cdot \delta$, so that

$$\nabla_X K^i_{j(k)} = \delta K^i_{j(k)}(X) = K^i_{j(k)/m} \xi^m + K^i_{j(k)/(m)} \xi^{(m)},$$

where we have put

$$\begin{aligned} K^i_{j(k)/m} :=& \delta K^i_{j(k)}(d_m) = d_m K^i_{j(k)} + F_h{}^i{}_m K^h_{j(k)} \\ & - F_j{}^h{}_m K^i_{h(k)} - F_k{}^h{}_m K^i_{j(h)}, \\ K^i_{j(k)/(m)} :=& \delta K^i_{j(k)}(d_{(m)}) = \dot{\partial}_m K^i_{j(k)} + C_h{}^i{}_m K^h_{j(k)} \\ & - C_j{}^h{}_m K^i_{h(k)} - C_k{}^h{}_m K^i_{j(h)}. \end{aligned}$$

By virtue of (1.5) and Lemma 1.2, let us define the operations $H_a\cdot$ and $V_a\cdot$ to a tensor function as

$$(1.14) \qquad \begin{aligned} H_d \cdot \bar{K}_u &:= D\bar{K}_u(B_d) = z^a_i K^i_{j(k)/m} z^j_b z^k_c z^m_d e^{b(c)}_a, \\ V_d \cdot \bar{K}_u &:= D\bar{K}_u(B_{(d)}) = z^a_i K^i_{j(k)/(m)} z^j_b z^k_c z^m_d e^{b(c)}_a. \end{aligned}$$

I. For torsion tensors $\Omega_1 = (\Omega^i)$, $\Omega_0 = (\Omega^{(i)})$ and curvature tensor $\Omega = (\Omega^i_j)$, we have

Theorem 1.4. *The following facts hold:*

$$
(1.15)\qquad
\begin{aligned}
&(a)\quad D\tilde{\theta} = I_u \cdot \Omega_1 \quad ; \quad \Omega_1 := [\delta\ \theta]\,,\\
&(b)\quad D\tilde{\theta}' = I_u \cdot \Omega_0 \quad ; \quad \Omega_0 := [\delta\ \theta']\,,\\
&(c)\quad D\tilde{\omega} = I_u \cdot \Omega \quad ; \quad \Omega := [d\ \omega] + [\omega\ \omega]\,.
\end{aligned}
$$

PROOF. For (a) and (b), using (1.4)(b) and Theorem 1.3, we see that

$$D\tilde{\theta} = DI_u \cdot \theta = I_u \cdot [\delta\ \theta] = I_u \cdot \Omega_1\,.$$

For (c), using (1.4)(a), we have $dz = z\tilde{\omega} - \omega z$, and hence we find

$$
\begin{aligned}
\bar{h}^* \cdot ddz = 0 &= \bar{h}^* \cdot ([dz\ \tilde{\omega}] + z[d\ \tilde{\omega}] - [d\ \omega]z + [\omega\ dz])\\
&= [Dz, \bar{h}^* \cdot \tilde{\omega}] + zD\tilde{\omega} - [d\ \omega]z + [\omega\ Dz]\\
&= zD\tilde{\omega} - ([d\ \omega] + [\omega\ \omega])z = 0\,,
\end{aligned}
$$

which gives $D\tilde{\omega} = z^{-1}\Omega z = I_u \cdot \Omega$ and $\Omega = [d\ \omega] + [\omega\ \omega]$.

For the Bianchi and Ricci identities, we can easily show

Theorem 1.5. *The following facts hold:*
(a) $DD\tilde{\omega} = I_u \cdot [\delta\ \Omega] = 0 \quad : \quad [\delta\ \Omega] = 0$ *(Bianchi),*
(b) *the relation* $DDI_u\cdot = I_u \cdot \Omega\cdot = -L(\Omega)\cdot$ *holds for any tensor* K_y, *namely, we have* $DD\bar{K}_u = I_u \cdot [\delta\ \delta K] = I_u \cdot (\Omega \cdot K)\ :\ [\delta\ \delta K] = \Omega \cdot K$ *(Ricci).*

Really, for an example $K^i_{j(k)}$, we have

$$(1.16)\qquad [\delta\ \delta K^i_{j(k)}] = (\Omega \cdot K)^i_{j(k)} = \Omega^i_m K^m_{j(k)} - \Omega^m_j K^i_{m(k)} - \Omega^m_k K^i_{j(m)}\,.$$

J. Last we shall refer to Lie derivation with respect to the vector X induced by a curve x_t in a domain of the underlying space M. The curve x_t is lifted to the curve u_t in P_N. Let us define on P_N the Lie operation as

$$(1.17)\qquad \bar{L}_{\bar{X}} := \iota_{\bar{X}} \cdot d + d \cdot \iota_{\bar{X}}\,,$$

where $\iota_{\bar{X}}$ means the inner product by $\bar{X}_u$ in (1.7). L_X on M_T is defined similarly. We have

Theorem 1.6. *The following facts hold:*
(a) the relation $\bar{L}_{\bar{X}} I_u\cdot = I_u \cdot L_X$ *holds for any tensor* K_y, *namely we have*

$$(1.18)\quad \bar{L}_{\bar{X}} \bar{K}_u = I_u \cdot L_X K_y\ :\ L_X K := \nabla_X K - \xi \cdot K \text{ for any 0-form } K\,,$$

(b) for the connection $\tilde{\omega}$ *and* ω, *we have*

$$\bar{L}_{\bar{X}}\tilde{\omega} = I_u \cdot L_X\omega \quad : \quad L_X\omega := \delta\xi + \iota_X \cdot \Omega \quad \textit{for 1–form} \quad \omega .$$

Really, we have

$$(a)' \quad L_X K^i_{j(k)} = K^i_{j(k)/l}\xi^l + K^i_{j(k)/(l)}\xi^l_{/0} - (\xi \cdot K)^i_{j(k)} ,$$

$$(b)' \quad L_X\omega^i_j = \delta\xi^i{}_j + \iota_X \cdot \Omega^i_j ,$$

where $\xi^i{}_j := z^i_a\tilde{\omega}^a_b(\bar{X})z^b_j = \xi^i{}_{/j} + C_j{}^i{}_k\xi^k{}_{/0}$, for $C\Gamma(N)$ in §2.

PROOF. Because $\iota_{\bar{X}} \cdot \bar{K}_u = 0$ for any 0–form K_u, we obtain $\bar{L}_{\bar{X}}\bar{K}_u = \iota_{\bar{X}} \cdot d\bar{K}_u = \bar{X} \cdot \bar{K}_u$, where

$$(1.7)''' \quad \bar{X}\cdot := \xi^a H_a \cdot + \xi^{(a)}V_a \cdot + \xi^i{}_j L^j_i = \iota_{\bar{X}} \cdot D + L(\xi)\cdot = I_u \cdot (\nabla_X - \xi\cdot) ,$$

and hence we obtain (1.18)(a), using (1.11) and (1.14). To prove (b), we shall give the following

Lemma. *The following relations hold:*

$$(1) \qquad dz = z\tilde{\omega} - \omega z, \qquad dz^{-1} = -\tilde{\omega}z^{-1} + z^{-1}\omega ,$$

$$(2) \qquad \iota_{\bar{X}} \cdot \tilde{\omega} = \tilde{\omega}(\bar{X}) = I_u \cdot \xi = z^{-1}\xi z ,$$

$$(3) \qquad \delta\xi = d\xi + \omega\xi - \xi\omega .$$

$$(4) \qquad [d\ \tilde{\omega}] = -[\tilde{\omega}\ \tilde{\omega}] + I_u \cdot \Omega .$$

PROOF of (4). Noting $\tilde{\omega} = z^{-1}(dz + \omega z)$, we see

$$\begin{aligned}[d\ \tilde{\omega}] &= [dz^{-1}(dz + \omega z)] + z^{-1}([d\ \omega]z - [\omega\ dz]) = \\ &= [dz^{-1}z\tilde{\omega}] + z^{-1}([d\ \omega]z - [\omega(z\tilde{\omega} - \omega z)]) = \\ &= -z^{-1}[dz\ \tilde{\omega}] + z^{-1}(\Omega z - [\omega\ z\tilde{\omega}]) = \\ &= -z^{-1}[(z\tilde{\omega} - \omega z)\tilde{\omega}] + I_u \cdot \Omega - z^{-1}[\omega\ z\tilde{\omega}] = \\ &= -[\tilde{\omega}\ \tilde{\omega}] + I_u \cdot \Omega.\end{aligned}$$

Using the above lemma, we find that

$$(a) \qquad \iota_{\bar{X}} \cdot d\tilde{\omega} = \iota_{\bar{X}} \cdot (I_u \cdot \Omega - [\tilde{\omega}\ \tilde{\omega}]) = I_u \cdot \iota_{\bar{X}} \cdot \Omega - \iota_{\bar{X}} \cdot [\tilde{\omega}\ \tilde{\omega}];$$

$$\begin{aligned}(b) \quad d(\iota_{\bar{X}} \cdot \tilde{\omega}) &= d(z^{-1}\xi z) = dz^{-1}\xi z + z^{-1}d\xi z + z^{-1}\xi dz \\ &= (-\tilde{\omega}z^{-1} + z^{-1}\omega)\xi z + z^{-1}(\delta\xi - \omega\xi + \xi\omega)z + \\ &\quad + z^{1}\xi(z\tilde{\omega} - \omega z) \\ &= -\tilde{\omega}(I_u \cdot \xi) + I_u \cdot \delta\xi + (I_u \cdot \xi)\tilde{\omega} \\ &= I_u \cdot \delta\xi + \tilde{\omega}(\bar{X})\tilde{\omega} - \tilde{\omega}\tilde{\omega}(\bar{X}) = I_u \cdot \delta\xi + \iota_{\bar{X}} \cdot [\tilde{\omega}\ \tilde{\omega}] .\end{aligned}$$

Thus we obtain $\bar{L}_{\bar{X}}\tilde{\omega} = (\iota_{\bar{X}} \cdot d + d \cdot \iota_{\bar{X}})\tilde{\omega} = I_u \cdot (\delta\xi + \iota_{\bar{X}} \cdot \Omega)$.

Remark. For the case of the connection $C\Gamma(N)$ (see in §2), we have

$$\Omega^i_j := -\frac{1}{2}R_j{}^i{}_{kl}[\theta^k\theta^l] - P_j{}^i{}_{kl}[\theta^k\theta^{(l)}] - \frac{1}{2}S_j{}^i{}_{kl}[\theta^{(k)}\theta^{(l)}]\,, \tag{1.19}$$

and then we obtain

$$L_X\omega^i_j = A_j{}^i{}_k\theta^k + B_j{}^i{}_k\theta^{(k)}\,, \tag{1.20}$$

where

$$\begin{aligned} &(a) \quad A_j{}^i{}_k := \xi^i{}_{j/k} + R_j{}^i{}_{kl}\xi^l + P_j{}^i{}_{kl}\xi^l{}_{/0}\,, \\ &(b) \quad B_j{}^i{}_k := \xi^i{}_{j/(k)} - P_j{}^i{}_{lk}\xi^l + S_j{}^i{}_{kl}\xi^l{}_{/0}\,. \end{aligned} \tag{1.21}$$

§2. Metrical connection $C\Gamma(N)$ and h–metrical connection $R\Gamma(N)$

In this section all tensors are expressed with their components and the two 1–forms θ^i and $\theta^{(i)}$ from §1 are replaced by dx^i and δy^i. This implies that $d\theta^i = ddx^i = 0$, that is the considered coordinate system (x^i) is holonomic (c.f. [6]) and $\theta^{(i)} = \delta y^i$ depends on the choice of our assumptions of connection (cf. (2.3)(b)). Most of the tensors are in accordance with those in a Finsler space.

The metrical connection $C\Gamma(N) = (F_j{}^i{}_k, C_j{}^i{}_k)$: $\omega^i_j := F_j{}^i{}_k dx^k + C_j{}^i{}_k\delta y^k$ satisfies

$$(a)\, N^i_k := F_j{}^i{}_k y^j, \quad (b)\, F_j{}^i{}_k = F_k{}^i{}_j, \quad (c)\, C_j{}^i{}_k = C_k{}^i{}_j \tag{2.1}$$

such that $\delta g_{ij} = dg_{ij} - g_{hi}\omega^h_j - g_{hj}\omega^h_i = g_{ij/k}dx^k + g_{ij/(k)}\delta y^k = 0$:

$$\begin{aligned} &(a) \quad g_{ij/k} = d_k g_{ij} - F_i{}^h{}_k g_{hj} - F_j{}^h{}_k g_{hi} = 0\,; \\ &(b) \quad g_{ij/(k)} = g_{ij(k)} - C_i{}^h{}_k g_{hj} - C_j{}^h{}_k g_{hi} = 0, \quad g_{ij(k)} := \dot\partial_k g_{ij}\,. \end{aligned} \tag{2.2}$$

The property of homogeneity leads us to

Proposition 2.1. *In a generalized metric space with connection* $C\Gamma(N)$, *we have*

$$(a)\ C_o{}^i{}_j = C_j{}^i{}_o = 0, \quad (b)\ \delta y^i = dy^i + \omega^i_j y^j = dy^i + N^i_j dx^j\,, \tag{2.3}$$

where the index o means contraction by y.

PROOF. Because the non–linear connection $N(x,y)$ is p–homogeneous of degree one, we see that $N(x,\lambda y) = \lambda N(x,y)$ for an arbitrary $\lambda > 0$ and then we get

$$\delta(\lambda y) = \lambda(\delta y) + (d\lambda)y\,. \tag{2.4}$$

On the other hand, for an arbitrary vector $v(x,y)$ which is p–homogeneous of degree zero, we have $\delta v(x,\lambda y)=\delta v(x,y)$, that is

$$\delta v^i(x,\lambda y)=dv^i(x,\lambda y)+\omega^i_j(x,\lambda y)v^j(x,\lambda y)=\delta v^i(x,y),$$

which yields that $\omega(x,y)$ is p–homogeneous of degree zero, and then we get

$$\omega^i_j(x,\lambda y)=F_j{}^i{}_k(x,\lambda y)dx^k+C_j{}^i{}_k(x,\lambda y)\delta(\lambda y^k)=\omega^i_j(x,y)\,.$$

The above equation and (2.4) imply

$$(a)\ \ F_j{}^i{}_k(x,\lambda y)=F_j{}^i{}_k(x,y),\quad (b)\ \ C_j{}^i{}_k(x,\lambda y)\lambda=C_j{}^i{}_k(x,y),$$
$$(c)\ \ d\lambda C_j{}^i{}_o=0\,.$$

Thus we have $C_o{}^i{}_j=0$ for an arbitrary λ, and then we obtain our assertion.

Proposition 2.2. *In a generalized metric space with connection* $C\Gamma(N)$ *we have*

$$y^i{}_{/k}=0,\ \ y_{j/k}=0,\ \ y^i{}_{/(k)}=\delta^i_k\,. \tag{2.5}$$

Proposition 2.3. *In a generalized metric space with connection* $C\Gamma(N)$ *we have*

$$g_{ij(k)}y^iy^j=0\,. \tag{2.6}$$

PROOF. From (2.2)(b) and (2.3)(a), we find

$$g_{ij(k)}y^iy^j=(C_i{}^h{}_kg_{hj}+C_j{}^h{}_kg_{hi})y^iy^j=0\,.$$

Remark. Condition (2.6) is called "Miron's condition" in a generalized Lagrange space (cf. [8](4.6)).

Using (2.2), we obtain

$$\begin{aligned}&(a)\quad F_j{}^i{}_k=\frac{1}{2}g^{ih}(d_kg_{hj}+d_jg_{hk}-d_hg_{jk})\,,\\ &(b)\quad C_j{}^i{}_k=\frac{1}{2}g^{ih}(g_{hj(k)}+g_{hk(j)}-g_{jk(h)})\,.\end{aligned}\tag{2.7}$$

On noting (2.6), easy calculations give us

$$\begin{aligned}&(a)\quad y_k:=\dot\partial_k(\frac{1}{2}F^2)=\frac{1}{2}g_{ij(k)}y^iy^j+g_{ik}y^i=g_{ik}y^i\,,\\ &(b)\quad g^*_{jk}=\dot\partial_k(y_j)=\dot\partial_k(g_{jh}y^h)=g_{jk}+C_{jk},\\ &\qquad\quad C_{jk}:=g_{hj(k)}y^h=C_{kj}\,,\end{aligned}\tag{2.8}$$

where the difference tensor C_{jk} plays an important role in the sequel and satisfies

$$C_{jk}=C_j{}^o{}_k=\frac{1}{2}(g_{hj(k)}+g_{hk(j)}-g_{jk(h)})y^h\,, \tag{2.9}$$

because of $g_{ij(h)}y^h=0$.

Theorem 2.4. *If the tensor C_{jk} vanishes, then the generalized metric space is a Finsler space. If the tensor $C_j{}^i{}_k$ vanishes, then the generalized metric space is a Riemannian space.*

PROOF. The relation (2.8)(b) implies the first assertion, that is $g^*_{jk} = g_{jk}$. The second assertion follows from (2.2)(b), that is

$$g_{ij(k)} = g_{ih}C_j{}^h{}_k + g_{jh}C_i{}^h{}_k = C_{jik} + C_{ijk}\,. \tag{2.10}$$

Here we shall be concerned with torsion tensors of the connection $C\Gamma(N)$. On considering (1.15)(a), (b) and (2.1)(b), (c), we obtain

$$\begin{aligned} &\Omega^i := [\delta\ dx^i] = [\omega^i_j\ dx^j] = -C_j{}^i{}_k[dx^j\ \delta y^k]\,;\\ &\Omega^{(i)} := [\delta\ \delta y^i] = \Omega^i_j y^j = \Omega^i_0 \qquad (\text{see } (1.16))\,. \end{aligned} \tag{2.11}$$

From (1.19), we have

$$\Omega^{(i)} := -\frac{1}{2}R^i{}_{kl}[dx^k\ dx^l] - P^i{}_{kl}[dx^k\ \delta y^l]\,,$$

where $R^i{}_{jk} := R_o{}^i{}_{jk}$, $P^i{}_{jk} := P_o{}^i{}_{jk}$ (see (2.12)(a), (b)).

From the Ricci identities (1.16) for a vector v^i we have

$$\begin{aligned} &(a)\quad v^i{}_{/j/k} - v^i{}_{/k/j} = R_h{}^i{}_{jk}v^h - R^h{}_{jk}v^i{}_{/(h)}\,,\\ &(b)\quad v^i{}_{/j/(k)} - v^i{}_{/(k)/j} = P_h{}^i{}_{jk}v^h - C_j{}^h{}_k v^i{}_{/h} - P^h{}_{jk}v^i{}_{/(h)}\,,\\ &(c)\quad v^i{}_{/(j)/(k)} - v^i{}_{/(k)/(j)} = S_h{}^i{}_{jk}v^h\,, \end{aligned}$$

where we have put

$$\begin{aligned} (a)\quad & R_h{}^i{}_{jk} := K_h{}^i{}_{jk} + C_h{}^i{}_m R^m{}_{jk}, \quad R^m{}_{jk} := d_k N^m_j - j|k\,,\\ & K_h{}^i{}_{jk} := d_k F_h{}^i{}_j + F_h{}^m{}_j F_m{}^i{}_k - j|k\,;\\ (b)\quad & P_h{}^i{}_{jk} := F_h{}^i{}_{jk} - C_h{}^i{}_{k/j} + C_h{}^i{}_m P^m{}_{jk}\,,\\ & F_h{}^i{}_{jk} := \dot\partial_k F_h{}^i{}_j, \quad P^i{}_{jk} := \dot\partial_k N^i_j - F_k{}^i{}_j\,;\\ (c)\quad & S_h{}^i{}_{jk} := C_h{}^i{}_{j(k)} + C_h{}^m{}_j C_m{}^i{}_k - j|k\,, \end{aligned} \tag{2.12}$$

where $j|k$ means the interchange of the indices j, k in the foregoing terms and the index (k) means partial differentiation by y^k. Using the Ricci identities for a supporting element y and a metric tensor g, we have

Proposition 2.5. *In a generalized metric space the following relations hold:*

$$\begin{aligned} &(a)\quad S_o{}^i{}_{jk} = 0, \quad S_h{}^o{}_{jk} = C_{hj(k)} - g_{mk}C_h{}^m{}_j - j|k = 0\,,\\ &(b)\quad R_{hijk} = -R_{ihjk}, \quad P_{hijk} = -P_{ihjk}, \quad S_{hijk} = -S_{ihjk}\,. \end{aligned} \tag{2.13}$$

Remark. The latter of (2.13)(a) shows that if the relation $C_{hj(k)} = C_{hk(j)}$ holds, then the space is a Finsler space.

Proposition 2.6. *In a generalized metric space we have* $P^i{}_{ok} = 2P^i{}_k$.

PROOF. We see from (0.1), (2.1)(a), (b) and from the last equation of (2.12)(b) that

$$P^i{}_{ok} = y^j(\dot{\partial}_k N^i_j - F_k{}^i{}_j) = \dot{\partial}_k(N^i_j y^j) - N^i_k - N^i_k = 2(G^i_k - N^i_k) = 2P^i{}_k .$$

Proposition 2.7. *In a generalized metric space the following relations hold:*

$$(2.14)\quad \begin{aligned} &(a)\quad R^o{}_{jk} = 0,\ P^o{}_{jk} = 0, \qquad (b)\quad P^i{}_{jo} = 0,\ P_h{}^i{}_{jo} = 0,\\ &(c)\quad P^i{}_o = 0,\ P^o{}_k = 0 . \end{aligned}$$

Now we refer to an h-metrical connection $\mathrm{R}\Gamma(N) = (F_j{}^i{}_k)$: $\omega^i_j := F_j{}^i{}_k dx^k$, which corresponds to the Rund connection RF(G) in a Finsler space.

The Ricci Identities with respect to $\mathrm{R}\Gamma(N)$ are written as

$$(2.15)\quad \begin{aligned} &(a)\quad v^i{}_{/j/k} - v^i{}_{/k/j} = K_h{}^i{}_{jk} v^h - R^h{}_{jk} v^i{}_{(h)} ,\\ &(b)\quad v^i{}_{/j(k)} - v^i{}_{(k)/j} = F_h{}^i{}_{jk} v^h - P^h{}_{jk} v^i{}_{(h)} . \end{aligned}$$

Using the Ricci identities for a supporting element y and a metric tensor g, we have

Proposition 2.8. *In a generalized metric space the following identities hold:*

$$(2.16)\quad \begin{aligned} &(a)\quad R^i{}_{jk} = K_o{}^i{}_{jk}, \quad P^i{}_{jk} = F_o{}^i{}_{jk} = F_j{}^i{}_{ok} ,\\ &(b)\quad K_{hijk} + K_{ihjk} = -g_{hi(m)} R^m{}_{jk} ,\\ &\qquad F_{hijk} + F_{ihjk} = g_{hi(k)/j} - g_{hi(m)} P^m{}_{jk} ,\\ &(c)\quad K_h{}^o{}_{jk} = -(g_{hm} + C_{hm}) R^m{}_{jk} ,\\ &\qquad F_h{}^o{}_{jk} = C_{hk/j} - (g_{hm} + C_{hm}) P^m{}_{jk} ,\\ &(d)\quad C_{hk/j} - C_{jk/h} = (g_{hm} + C_{hm}) P^m{}_{jk} -\\ &\qquad\qquad - (g_{jm} + C_{jm}) P^m{}_{hk} ,\\ &(e)\quad g_{hi(k)/o} = g_{im} P^m{}_{hk} + g_{hm} P^m{}_{ik} + 2P^m{}_k g_{hi(m)} ,\\ &(f)\quad C_{jk/o} = 2(g_{jm} + C_{jm}) P^m{}_k =\\ &\qquad = (g_{jm} + C_{jm}) P^m{}_k + (g_{km} + C_{km}) P^m{}_j . \end{aligned}$$

(f) of (2.16) implies

Proposition 2.9. *In a generalized metric space the tensor* $P^i{}_k$ *vanishes if and only if the tensor* $C_{jk/o}$ *vanishes.*

By virtue of Christoffel's process and (2.1)(b) the latter of (2.16)(b) gives

Proposition 2.10. *In a generalized metric space the following identity holds:*

$$2F_{hijk} = g_{hi(k)/j} + g_{ij(k)/h} - g_{jh(k)/i} - g_{hi(m)}P^m{}_{jk} - g_{ij(m)}P^m{}_{hk} + g_{jh(m)}P^m{}_{ik}\,. \tag{2.17}$$

Next we shall introduce the Bianchi identities of the space. Comparing coefficients of $[dx^j\ dx^k\ dx^l]$, $[dx^j\ dx^k\ \delta y^l]$, $[dx^j\ \delta y^k\ \delta y^l]$ and $[\delta y^j\ \delta y^k\ \delta y^l]$, the Ricci identities for $dx^i : [\delta\ \delta\ dx^i] = [\Omega^i_j\ dx^j]$ give for $C\Gamma(N)$

Proposition 2.11. *In a generalized metric space the following identities hold:*

$$\begin{aligned}
&(a) \quad R_j{}^i{}_{kl} - C_j{}^i{}_m R^m{}_{kl} + j|k|l = 0,\\
&\qquad (g_{jm} + C_{jm})R^m{}_{kl} + j|k|l = 0\,,\\
&(b) \quad P_j{}^i{}_{kl} + C_j{}^i{}_{l/k} - C_j{}^i{}_m P^m{}_{kl} - j|k = 0,\\
&\qquad P_k{}^i{}_{ol} = P^i{}_{kl} - C_k{}^i{}_{l/o} + 2C_k{}^i{}_m P^m{}_l\,,\\
&(c) \quad S_j{}^i{}_{kl} = C_j{}^i{}_{k/(l)} - C_j{}^m{}_k C_m{}^i{}_l - k|l,\\
&\qquad C_{jk/(l)} + (g_{km} + C_{km})C_j{}^m{}_l - k|l = 0\,,
\end{aligned} \tag{2.18}$$

where $j|k|l$ means the cycle change of indices j, k, l in the foregoing terms.

The Ricci identities for $\delta y^i : [\delta\ \delta\ \delta y^i] = [\delta\ \Omega^i_0] = [\Omega^i_j\ \delta y^j]$ give

Proposition 2.12. *In a generalized metric space the following identities hold:*

$$\begin{aligned}
&(a) \quad R^i{}_{jk/l} - P^i{}_{lm}R^m{}_{jk} + j|k|l = 0,\\
&(b) \quad R^i{}_{jk/(l)} - R_l{}^i{}_{jk} = P^i{}_{jl/k} + P^m{}_{jl}P^i{}_{km} - R^i{}_{jm}C_k{}^m{}_l - j|k\,,\\
&(c) \quad P^i{}_{jk/(l)} - P_l{}^i{}_{jk} - P^i{}_{ml}C_j{}^m{}_k - k|l = 0\,,\\
&(d) \quad S_j{}^i{}_{kl} + j|k|l = 0\,.
\end{aligned} \tag{2.19}$$

The Bianchi identities: $[\delta\ \Omega^i_j] = 0$ for the $C\Gamma(N)$ give

Proposition 2.13. *In a generalized metric space the following identities hold:*

$$\begin{aligned}
&(a) \quad R_h{}^i{}_{jk/l} + P_h{}^i{}_{lm}R^m{}_{jk} + j|k|l = 0,\\
&(b) \quad R_h{}^i{}_{jk/(l)} + S_h{}^i{}_{lm}R^m{}_{jk} = P_h{}^i{}_{jl/k} - P_h{}^i{}_{jm}P^m{}_{kl} -\\
&\qquad\qquad - R_h{}^i{}_{jm}C_k{}^m{}_l - j|k\,,\\
&(c) \quad -S_h{}^i{}_{kl/j} = P_h{}^i{}_{jk/(l)} + P_h{}^i{}_{mk}C_j{}^m{}_l + S_h{}^i{}_{lm}P^m{}_{jk} - k|l\,,\\
&(d) \quad S_h{}^i{}_{jk/(l)} + j|k|l = 0\,.
\end{aligned} \tag{2.20}$$

We can easily see that contracting (2.20) by y^h, we have (2.19).

For the case of an h–metrical connection $R\Gamma(N)$, the identities in the preceding Propositions reduce to a more simple form.

Proposition 2.14. *In a generalized metric space the following identities hold:*

$$(2.21)\quad \begin{aligned} &(a)\quad K_j{}^i{}_{kl} + j|k|l = 0\,,\\ &(b)\quad K_h{}^i{}_{jk/l} + F_h{}^i{}_{lm}R^m{}_{jk} + j|k|l = 0\,,\\ &(c)\quad K_h{}^i{}_{jk(l)} = F_h{}^i{}_{jl/k} + F_h{}^i{}_{km}P^m{}_{jl} - j|k\,,\\ &(d)\quad R^i{}_{jk(l)} = K_l{}^i{}_{jk} + (P^i{}_{jl/k} + P^m{}_{jl}P^i{}_{km} - j|k)\,. \end{aligned}$$

The above identities contain several more detailed equations, such as contraction of indices or indicatrized decomposition [5].

§3. Non–metrical connection $\mathrm{B}\Gamma(G)$

In this section we refer to the non–metrical connection $\mathrm{B}\Gamma(G) = (G_j{}^i{}_k)$: $\omega^i_j = G_j{}^i{}_k dx^k$, $G_j{}^i{}_k := G^i_{k(j)}$ which corresponds to the Berwald connection $\mathrm{BF}(G)$ in a Finsler space.

Covariant derivative of a vector v^i by x^k will be denoted by

$$v^i_{//k} := \bar{d}_k v^i + G_j{}^i{}_k v^j, \quad \bar{d}_k := \partial_k - G^h_k\dot{\partial}_h = d_k - P^h{}_k\dot{\partial}_h\,,$$

which is rewritten as

$$(3.1)\qquad v^i{}_{//k} := v^i{}_{/k} + D_j{}^i{}_k v^j - P^h{}_k v^i{}_{(h)}, \quad D_j{}^i{}_k := G_j{}^i{}_k - F_j{}^i{}_k\,.$$

Hence we get

$$(3.1)'\qquad g_{ij//k} = -g_{ih}D_j{}^h{}_k - g_{jh}D_i{}^h{}_k - P^h{}_k g_{ij(h)}\,.$$

Proposition 3.1. *The difference tensor $D_j{}^i{}_k$ in (3.1) satisfies the following relations:*

$$(3.2)\quad \begin{aligned} &(a)\quad D_j{}^i{}_k = P^i{}_{jk} + P^i{}_{j(k)},\\ &(b)\quad D_j{}^o{}_k = -(g_{jm} + C_{jm})P^m{}_k,\\ &(c)\quad D_o{}^i{}_k = P^i_{k\cdot}. \end{aligned}$$

Proof. (a) We see from the definitions (2.12)(b) and (3.1) that

$$D_j{}^i{}_k = D_k{}^i{}_j = \dot{\partial}_k(N^i_j + P^i{}_j) - F_k{}^i{}_j = \dot{\partial}_k N^i_j - F_k{}^i{}_j + P^i{}_{j(k)} = P^i{}_{jk} + P^i{}_{j(k)}.$$

(b) We see from (a) and (2.8) that

$$\begin{aligned} D_j{}^o{}_k &= P^o{}_{jk} + y_m P^m{}_{j(k)} = (y_m P^m{}_j)_{(k)} - y_{m(k)}P^m{}_j = \\ &= -(g_{mk} + C_{mk})P^m{}_j = -(g_{mj} + C_{mj})P^m{}_k\,. \end{aligned}$$

(c) is evident.

Using (3.1)', we get

Proposition 3.2. *The difference tensor $D_j{}^i{}_k$ satisfies the following relation:*

$$(3.3)\qquad -2D_j{}^i{}_k = g^{im}\Big(g_{mj//k} + g_{mk//j} - g_{jk//m} + P^h{}_j g_{mk(h)} + P^h{}_k g_{mj(h)} - P^h{}_m g_{jk(h)}\Big).$$

The Ricci identities for $B\Gamma(G)$ gives us

$$(3.4)\qquad \begin{aligned} &(a)\quad v^i{}_{//j//k} - v^i{}_{//k//j} = H_h{}^i{}_{jk}v^h - H^h{}_{jk}v^i{}_{(h)},\\ &(b)\quad v^i{}_{//j(k)} - v^i{}_{(k)//j} = G_h{}^i{}_{jk}v^h, \end{aligned}$$

where we have put

$$(3.5)\qquad \begin{aligned} &(a)\quad H_h{}^i{}_{jk} := \bar{d}_k G_h{}^i{}_j + G_h{}^m{}_j G_m{}^i{}_k - j|k,\\ &(b)\quad H^i{}_{jk} := \bar{d}_k G^i_j - j|k, \quad (c)\quad G_h{}^i{}_{jk} := G_h{}^i{}_{j(k)}. \end{aligned}$$

Direct calculations lead to the following

Proposition 3.3. *In a generalized metric space we have*

$$(3.6)\qquad \begin{aligned} &(a)\quad y^i{}_{//k} = 0, \quad y_{j//k} = 0,\\ &(b)\quad H^i{}_{jk(h)} = H_h{}^i{}_{jk}, \ H^i{}_{k(j)} - j|k = 3H^i{}_{jk}, \ H^i{}_k := H^i{}_{ok}. \end{aligned}$$

PROOF. The second equation of (a) follows from (3.1)' and (3.2)(c).

Proposition 3.4. *The following three conditions are equivalent:*

$$(a)\ H_h{}^i{}_{jk} = 0, \quad (b)\ H^i{}_{jk} = 0, \quad (c)\ H^i{}_k = 0.$$

Using the Ricci identities for y and a metric g, we have

Proposition 3.5. *In a generalized metric space we have*

$$(3.7)\qquad \begin{aligned} &(a)\quad H_o{}^i{}_{jk} = H^i{}_{jk}, \quad H_h{}^o{}_{jk} = -(g_{hm} + C_{hm})H^m{}_{jk},\\ &(b)\quad G_o{}^i{}_{jk} = G_h{}^i{}_{jo} = 0, \quad G_b{}^o{}_{jk} = g_{hj//k} + C_{hj//k},\\ &(c)\quad H_{hijk} + H_{ihjk} = -g_{hi//j//k} + g_{hi//k//j} - g_{hi(m)}H^m{}_{jk},\\ &(d)\quad G_{hijk} + G_{ihjk} = -g_{hi//j(k)} + g_{hi(k)//j}. \end{aligned}$$

Direct calculations lead to

Proposition 3.6. *In a generalized metric space we obtain*

$$(3.8)\qquad \begin{aligned} &H_h{}^i{}_{jk} = K_h{}^i{}_{jk} + E_h{}^i{}_{jk},\\ &E_h{}^i{}_{jk} := D_h{}^i{}_{j/k} + D_h{}^m{}_j D_m{}^i{}_k - P^m{}_k G_h{}^i{}_{jm} - j|k\,,\end{aligned}$$

and a tensor $E_h{}^i{}_{jk}$ (called a stretch tensor) satisfies

$$(3.9)\qquad \begin{aligned} &(a)\quad E^i{}_{jk} := E_o{}^i{}_{jk} = H^i{}_{jk} - R^i{}_{jk} = P^i{}_{j/k} + P^m{}_j D_m{}^i{}_k - j|k\,,\\ &(b)\quad E_h{}^o{}_{jk} = -(g_{hm} + C_{hm})E^m{}_{jk},\\ &\qquad E^o{}_{jk} = 0, \quad E^i{}_{ok} = -(P^i{}_{k/o} + P^i{}_m P^m{}_k)\,,\\ &(c)\quad E^i{}_{jk(h)} = E_h{}^i{}_{jk} - (P^i{}_{jh/k} + P^m{}_{jh} P^i{}_{km} - j|k)\,.\end{aligned}$$

For the Bianchi identities of the $\mathrm{B}\Gamma(G)$ we have

Proposition 3.7. *In a generalized metric space the following identities hold:*

$$(3.10)\qquad \begin{aligned} &(a)\quad H_j{}^i{}_{kl} + j|k|l = 0, \quad E_j{}^i{}_{kl} + j|k|l = 0\,,\\ &(b)\quad H^i{}_{jk//l} + j|k|l = H^i{}_{jk/l} + H^m{}_{jk} D_m{}^i{}_l - P^m{}_l H_m{}^i{}_{jk} +\\ &\qquad\qquad + j|k|l = 0\,,\\ &(c)\quad H_h{}^i{}_{jk(l)} = G_h{}^i{}_{jl//k} - j|k\,,\\ &(d)\quad H_h{}^i{}_{jk//l} + G_h{}^i{}_{lm} H^m{}_{jk} + j|k|l = 0\,,\\ &\qquad H_h{}^i{}_{jk/l} + H_h{}^m{}_{jk} D_m{}^i{}_l - D_h{}^m{}_l H_m{}^i{}_{jk} - P^m{}_l H_h{}^i{}_{jk(m)} +\\ &\qquad + G_h{}^i{}_{lm} H^m{}_{jk} + j|k|l = 0\,.\end{aligned}$$

Now we shall use the angular metric tensor defined by $h_{ij} := g_{ij} - l_j l_j$, $l^i := y^i/F$. The tensor $h^i_j := \delta^i_j - l^i l_j$ helps us to the notion of projection on the indicatrix defined by $g_{ij}(x, X)X^i X^j = 1$ in a generalized metric space. We can prove the following

Proposition 3.8. *In a generalized metric space we have*

$$(3.11)\qquad \begin{aligned} &(a)\quad F l_{i(k)} = h_{ik} + C_{ik}, \quad F l^i{}_{(k)} = h^i_k\,,\\ &(b)\quad F h^i_{j(k)} = -h^i_k l_j - l^i(h_{jk} + C_{jk})\,,\\ &\qquad F h_{ij(k)} = F g_{ij(k)} - l_i(h_{jk} + C_{jk}) - l_j(h_{ik} + C_{ik})\,,\\ &(c)\quad h_{ij/k} = 0, \quad F h_{ij/(k)} = -l_i h_{jk} - l_j h_{ik}\,.\end{aligned}$$

Last we shall show

Proposition 3.9. *In a generalized metric space we have*

$$(3.12)\qquad \begin{aligned} &(a)\quad 2(R_{hijk} - R_{jkhi}) = T_{hijk} + (C_{him} - C_{ihm})R^m{}_{jk} -\\ &\qquad\qquad - (C_{jkm} - C_{kjm})R^m{}_{hi}\,,\\ &(b)\quad 2(K_{hijk} - K_{jkhi}) = T_{hijk} - g_{hi(m)} R^m{}_{jk} + g_{jk(m)} R^m{}_{hi}\,,\\ &(c)\quad 2(H_{hijk} - H_{jkhi}) = T_{hijk} - g_{hi(m)} R^m{}_{jk} + g_{jk(m)} R^m{}_{hi} +\\ &\qquad\qquad + 2(E_{hijk} - E_{jkhi})\,,\end{aligned}$$

where we have put

$$T_{hijk} := g_{hj(m)}R^m{}_{ik} + g_{ik(m)}R^m{}_{hj} - j|k\,. \tag{3.13}$$

PROOF. Let us recall the following identities:

$$(3.14)\quad \begin{aligned} &(a)\quad R_{hijk} = K_{hijk} + C_{him}R^m{}_{jk} && (2.12)(a)\,,\\ &(b)\quad K_{hijk} + K_{ihjk} + g_{hi(m)}R^m{}_{jk} = 0 && (2.16)(b)\,,\\ &(c)\quad K_{hijk} - K_{ihjk} = 2R_{hijk} - (C_{him} - C_{ihm})R^m{}_{jk}, &&\\ &(d)\quad K_{hijk} + h|j|k = 0 && (2.21)(a)\,. \end{aligned}$$

On making use of $+i|j|k$ of (3.14)(b) and using (3.14)(d) we get

$$K_{hijk}+K_{hjki}+K_{hkij}+g_{hi(m)}R^m{}_{jk}+g_{hj(m)}R^m{}_{ki}+g_{hk(m)}R^m{}_{ij} = 0\,. \tag{3.15}$$

On making use of $-h|i$ of (3.15) and using (3.14)(c) we get

$$2R_{hijk} - (C_{him} - C_{ihm})R^m{}_{jk} + K_{hjki} + K_{hkij} + K_{ijhk} + K_{ikjh} - T_{hijk} = 0\,,$$

which can be rewritten as

$$\begin{aligned} 2R_{hijk} =& K_{hjik} + K_{hkji} + K_{ijkh} + K_{ikhj} +\\ &+ (C_{him} - C_{ihm})R^m{}_{jk} + T_{hijk}\,. \end{aligned} \tag{3.16}$$

Interchangement of the pair of indices (h,j) and (i,k) in (3.16) gives us

$$\begin{aligned} 2R_{jkhi} =& K_{jhki} + K_{jihk} + K_{khij} + K_{kijh} +\\ &+ (C_{jkm} - C_{kjm})R^m{}_{hi} + T_{jkhi}\,. \end{aligned} \tag{3.16'}$$

Using (3.14)(b) and (3.13), we obtain

$$\begin{aligned} 2(R_{hijk} - R_{jkhi}) =& -(g_{hj(m)}R^m{}_{ik} + g_{hk(m)}R^m{}_{ji} + g_{ij(m)}R^m{}_{kh} +\\ &+ g_{ik(m)}R^m{}_{hj}) + (C_{him} - C_{ihm})R^m{}_{jk} -\\ &- (C_{jkm} - C_{kjm})R^m{}_{hi} + T_{hijk} - T_{jkhi}\\ =& T_{hijk} + (C_{him} - C_{ihm})R^m{}_{jk} - (C_{jkm} - C_{kjm})R^m{}_{hi}\,. \end{aligned}$$

(b) and (c) of (3.12) easily follow from (3.12)(a), (3.14)(a) and (3.8).

§4. Special cases of generalized metric space and theorems

In this section we shall consider the cases in which a generalized metric space satisfies some conditions. To clear the discussion, the material is arranged under titles: A. $\sim$ C.

A. Let us consider the case when the arbitrary tensor $P^i{}_j$ vanishes. This means that the generalized metric space has a unique connection $C\Gamma(G) = (F_j{}^i{}_k, C_j{}^i{}_k)$ called a native metrical connection. The following Proposition is evident.

Proposition 4.1. *In a generalized metric space with connection* $C\Gamma(G)$ *we have*

$$(4.1)\quad \begin{aligned} &(a) \quad D_j{}^i{}_k = P^i{}_{jk}\,, \\ &(b) \quad C_{ij/o} = 0\,, \\ &(c) \quad R^i{}_{jk} = H^i{}_{jk}\,, \\ &(d) \quad P_h{}^i{}_{ok} = P^i{}_{hk} - C_h{}^i{}_{k/o}\,, \\ &(e) \quad E_h{}^i{}_{jk} = P^i{}_{hj/k} + P^m{}_{hj} P^i{}_{mk} - j|k, \\ &\qquad E^i{}_{jk} = E_h{}^o{}_{jk} = 0,\ E_h{}^i{}_{jo} = P^i{}_{hj/o}\,, \\ &(f) \quad R^i{}_{jk(h)} = H^i{}_{jk(h)} = K_h{}^i{}_{jk} + E_h{}^i{}_{jk}\,. \end{aligned}$$

Remark. (d) of (4.1) does not appear in a Finsler space.

Proposition 2.9, (3.2)(c) and (2.18)(b) show the following

Theorem 4.2. *In a generalized metric space the connection* $C\Gamma(N)$ *reduces to* $C\Gamma(G)$*, if one of the following conditions holds:*

$$(a)\ C_{ij/o} = 0, \quad (b)\ D_j{}^i{}_k = 0, \quad (c)\ P_h{}^i{}_{ok} = 0\,.$$

B. *Definition.* If the connection parameters $F_j{}^i{}_k(x,y)$ are independent of y^i, that is $F_j{}^i{}_{kl} = 0$, then the space is called a g–Berwald space (an affinely connected space).

Proposition 4.3. *In a* g*–Berwald space the curvature tensor* $G_h{}^i{}_{jk}$ *vanishes.*

Proof. We have $P^i{}_{kl} = F_o{}^i{}_{kl} = 0$, $2P^i{}_l = P^i{}_{ol} = 0$, hence $G_j{}^i{}_k - F_j{}^i{}_k = D_j{}^i{}_k = P^i{}_{jk} + P^i{}_{j(k)} = 0$ holds, and then we find $G_h{}^i{}_{jk} = F_h{}^i{}_{jk} = 0$.

Remark. Conditions $G_h{}^i{}_{jk} = 0$ and $P^i{}_{jk} = 0$ yield $F_h{}^i{}_{jk} = 0$.

Lemma 4.4. *In a generalized metric space the conditions* $g_{ij(k)/l} = 0$ *and* $C_j{}^i{}_{k/l} = 0$ *are equivalent.*

PROOF. From the definition (2.7)(b) of $C_j{}^i{}_k$, we see that $g_{ij(k)/l} = 0$ yields $C_j{}^i{}_{k/l} = 0$. On the other hand, (2.10) shows that $C_{jik/l} = 0$ yields $g_{ij(k)/l} = 0$.

Theorem 4.5. *A g–Berwald space is characterized by* $C_j{}^i{}_{k/l} = 0$.

PROOF. In a g–Berwald space $F_h{}^i{}_{kl} = 0$, and then $P^i{}_{kl} = F_o{}^i{}_{kl} = 0$. In this case (2.17) reduces to

$$g_{hi(k)/j} + g_{ij(k)/h} - g_{jh(k)/i} = 0\,.$$

Making use of $+i|j$ in the above equation, we have $2g_{ij(k)/h} = 0$, which means $C_{jik/h} = 0$. Conversely, if $C_{jik/l} = 0$ and thus $g_{hi(k)/j} = 0$, then (2.17) reduces to

$$(4.2) \qquad 2F_{hijk} = -g_{hi(m)}P^m{}_{jk} - g_{ij(m)}P^m{}_{hk} + g_{hj(m)}P^m{}_{ik}\,.$$

On transvecting (2.16)(e) by y^h, and using (2.16)(f) and (2.14)(c), we find that

$$\begin{aligned} 0 = g_{hi(k)/o}y^h &= (g_{im}P^m{}_{hk} + g_{hm}P^m{}_{ik} + 2P^m{}_k g_{hi(m)})y^h = \\ &= C_{ik/o} = 2(g_{im} + C_{im})P^m{}_k\,, \end{aligned}$$

which implies $P^m{}_k = 0$. Hence wes see $D_j{}^i{}_k = P^i{}_{jk} = P^i{}_{kj}$ from (3.2)(a) and (3.1). (2.16)(e) gives us $P_{ihk} + P_{hik} = 0$. Christoffel process of the last equation and $P_{ijk} = P_{ikj}$ give us $2P_h{}^i{}_k = 0$. Thus from (4.2) we obtain $F_h{}^i{}_{jk} = 0$.

C. *Definition.* If the tensor $P^i{}_{jk}$ vanishes, then the generalized metric space is called a g–Landsberg space.

Theorem 4.6. *A g–Landsberg space is characterized by* $D_j{}^i{}_k = 0$.

PROOF. Because of $D_j{}^i{}_k = P^i{}_{jk} + P^i{}_{j(k)}$ and $P^i{}_{ok} = 2P^i{}_k$, the assertion is evident.

T. SAKAGUCHI recently obtained the following

Theorem 4.7. *In a generalized metric space the following identity holds:*

$$(4.3) \qquad \begin{aligned} 2P_{ikjl} =&(P_{kjl} + P_{ljk})_{/(i)} + P_{ijk/(l)} + \\ &+ (P_{kml} + P_{lmk})C_j{}^m{}_i + P_{imk}C_j{}^m{}_l - i|k, \end{aligned}$$

which yields that the condition $P^i{}_{jk} = 0$ *is equivalent to the condition* $P_h{}^i{}_{jk} = 0$.

PROOF. By means of the Christoffel process with respect to the indices i, k, l of P_{lijk} in (2.19)(c), we have (4.3).

Theorem 4.8. *A g–Landsberg space is characterized by* $C_j{}^i{}_{k/o} = 0$.

PROOF. In a g–Landsberg space we obtain $C_j{}^i{}_{k/o} = 0$ from (4.1)(d) and Theorem 4.7. Conversely, by means of Lemma 4.4, the condition $C_j{}^i{}_{k/o} = 0$ yields $g_{ij(k)/o} = 0$. Hence $g_{ij(k)/o}y^i = C_{jk/o} = 0$ implies $P^i{}_k = 0$ from (2.16)(f). Thus the relation (2.16)(e) reduces to $P_{ihk} + P_{hik} = 0$. The last equation implies $P^i{}_{jk} = 0$.

Theorem 4.9. *In a generalized metric space the following facts hold:*

(a) *the condition* $P^i{}_{jk} = 0$ *implies the condition* $g_{ij//k} = 0$ *(*$B\Gamma(G)$ *is h–metrical).*

(b) *the conditions* $g_{ij//k} = 0$ *and* $P^i{}_k = 0$ *yield* $P^i{}_{jk} = 0$.

PROOF. The proof of (a) is evident from (3.1)'. For (b), the conditions $g_{ij//k} = 0$ and $P^i{}_k = 0$ yield $D_{jik} + D_{ijk} = 0$ from (3.1)'. The last equation implies $D_j{}^i{}_k = 0$.

Remark. It is well known that in a Finsler space the following four conditions are equivalent: (a) $P_h{}^i{}_{jk} = 0$, (b) $P^i{}_{jk} = 0$, (c) $C_j{}^i{}_{k/o} = 0$ and (d) $g_{ij//k} = 0$.

Proposition 4.10. *In a g–Landsberg space we have*

$$(4.3)\quad \begin{aligned} &(a) \quad F_j{}^i{}_k = G_j{}^i{}_k = N^i_{k(j)}, \quad P^i{}_k = 0, \quad C_{ij/o} = 0,\\ &(b) \quad F_h{}^i{}_{jk} = G_h{}^i{}_{jk} = C_h{}^i{}_{k/j}, \quad C_{ij/k} = C_{ik/j},\\ &(c) \quad E_h{}^i{}_{jk} = 0, \quad H_h{}^i{}_{jk} = K_h{}^i{}_{jk},\\ &(d) \quad H_h{}^i{}_{jk/l} + G_h{}^i{}_{lm}H^m{}_{jk} + j|k|l = 0\,. \end{aligned}$$

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