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A Note on Topological Spaces

By MANSOOR AHMAD (Aminabad)

Introduction

The most interesting special class of topological spaces is certainly that of Hausdorff spaces. As is known:

(i) A topological space is a Hausdorff space, if and only if each net in the space converges to at most one point. [1, pp. 67].

(ii) The product space of Hausdorff spaces is a Hausdorff space. [1, pp. 92].

The object of this note is to generalize the two theorems (i) and (ii), in the best possible way, so that the generalized statements hold for a wide class of topological spaces, which includes Hausdorff spaces as the simplest case.

1. Let X bet any topological space which satisfies the condition that, given any two points a, b of X, it is possible to choose neighbourhoods A and B of a and b respectively, so that $A \cap B$ is minimal, in the sense that, if U and V are any neighbourhoods of a and b respectively, then $A \cap B \subseteq U \cap V$. Let S denote the union of all the sets $A \cap B$, chosen in this manner, for all pairs of points a and b of X.

The set S just defined can be called the *neighbourhood kernel* of X or, in view of a further generalization, the 2-kernel of X. One easily sees that $S = 0$, if and only if the space is Hausdorff.

A simple example of a topological space X which does not satisfy this condition is as follows: Let X be any countable set; and let any subset $A \neq \emptyset$ of X be open, if and only if A is the complement of a finite number of points of X. Let the null set be open. Although this topological space does not satisfy the required condition, as a counterexample it has a serious defect: every infinite net in X converges to every point of X, and every finite net in X, converges to no point of X. Consequently, this topology is of little interest, so far as the convergence of nets'is concerned.

On the other hand, interesting examples of topological spaces which satisfy this condition, and which prove that there exists a wide class of topological spaces of this type, are as follows:

(i) Let X be the union of a linear set H and of a set I of disjoint intervals which contain no point of H. Let us choose the open sets in X, so that the union of any collection of intervals of I is open, that every subset of H is open, and that the union of any collection of subsets of H and of intervals of I is open. It is easy to verify that X is a non-Hausdorff topological space and that the set S is the set of all the points which belong to the disjoint intervals I.

(ii) Let R denote the set of all real numbers and let the only open sets in R be the following sets and the union of any collection of these sets: (a) the null set, (b) the set of all irrational points of every interval in R, (c) the sets I and $I_m\,(m = 0, 1, 2, \dots)$, where I denotes the set of all rational points of R, $I_m\,(m = 1, 2, \dots)$ are subsets of I, such that $I_{m+1} \subseteq I_m$, for all m $(m = 1, 2, \dots)$, I_m converges to a subset R_0 of R, as m tends to infinity, and I_0 is the set of all rational points of R_0, I_0 being supposed to contain more than one rational point. It is easily verified that R is a non-Hausdorff topological space, and that S is the set I or I_1, according as I contains or does not contain at least two points which do not belong to I_1.

Returning to the general case we have the following theorem:

Theorem 1. *The set S is the smallest subset of X, which has the property that every net in X, which is not eventually in S, converges to at most one point.*

PROOF. Suppose there is a net which converges to two points. Let a and b denote these points. Then the net is eventually in $A \cap B$, and so it is eventually in S. Therefore, it follows that every net which is not eventually in S, converges to at most one point.

On the other hand, suppose that S' is any proper subset of S. Let $c \in S - S'$. By the definition of S, c belongs to, at least, one set $A \cap B$. The net $\{S_n, n \in D, \geq\}$, $S_n = c$ for all n, converges to both a and b, because by definition $A \cap B$ is the smallest set such that A and B are neighbourhoods of a and b respectively, and so, it follows that, if A' and B' are any other neighbourhoods of a and b respectively, then $A \cap B \subseteq A' \cap B'$, i.e. $c \in A' \cap B'$. So, it follows that S' does not have the property that every net in X, which is not eventually in S', converges to at most one point. This completes the proof of the theorem.

We call a topological space X with neighbourhood kernel S an S-space.

Theorem 1 can also be stated in another form as follows:

A topological space X has neighbourhood kernel S, if and only if S is the smallest subset of X which has the property that every net in X, which is not eventually in S, converges to at most one point.

Theorem 2. *The Cartesian product of an S_1-space and of an S_2-space is an $S_1 \times S_2$-space.*

PROOF. Let (x_1, y_1) and (x_2, y_2) be any two points of the cartesian product of the two spaces. Let $(U \times V)$ and $(R \times S)$ be neighbourhoods of (x_1, y_1) and (x_2, y_2) respectively,where U, R belong to the S_1-space and V, S belong to the S_2-space. By [1, pp. 89], we have

$$(U \times V) \cap (R \times S) = (U \cap R) \times (V \cap S).$$

Let us choose U, V, R, S such that each of $U \cap R$ and $V \cap S$ is minimal. Consequently, $U \cap R$ and $V \cap S$ belong to S_1, and S_2 respectively. We have thus proved that $(U \times V) \cap (R \times S)$ is minimal and that this is a subset of $S_1 \times S_2$. Consequently, the union of all these sets, corresponding to all the pairs of points of the cartesian product of the two spaces, is a subset of $S_1 \times S_2$, but, by the definition of S_1 and S_2, $S_1 \times S_2$ is minimal, because if not, then S_1 and S_2 will not be minimal, and so, it follows that this union is the set $S_1 \times S_2$. Thus the theorem follows.

2. Let X be any topological space, and let $a_1, a_2, \ldots, a_p$ be any p points of X. Let us choose neighbourhoods $A_1, A_2, \ldots, A_p$, in X, of $a_1, a_2, \ldots, a_p$ respectively, such that $\bigcap_{n=1}^{p} A_n$ is minimal, in the sense that, if $B_1, B_2, \ldots, B_p$ are any neighbourhoods of $a_1, a_2, \ldots, a_p$ respectively, then $\bigcap_{n=1}^{p} A_n \subseteq \bigcap_{n=1}^{p} B_n$. Let S_p denote the union of all the sets $\bigcap_{n=1}^{p} A_n$, chosen in this manner*, for all the sets of p points $a_1, a_2, \ldots, a_p$ of X. We have the following

Theorem 3. *The set S_p is the smallest subset of X,which has the property that every net in X, which is not eventually in S_p, converges to at most $(p-1)$ points.*

The proof follows the same lines as that of Theorem 1.

The set S_p just defined is said to be the p-kernel of X, and it follows that the set S_p is the p-kernel of any topological space X, if and only if every net in X, which is not eventually in S_p, converges to at most $(p-1)$ points.

Let us suppose that there is a net which converges to p points. Let $a_1, a_2, \ldots, a_p$ denote these points. Then the net is eventually in $A_1 \cap A_2 \cap \ldots \cap A_p$, and so it is eventually in S_p.

On the other hand, let us suppose that S'_p is any proper subset of S_p. Let $c \in S_p - S'_p$. By the definition of S_p, c belongs to, at least, one set $A_1 \cap A_2 \cap \ldots \cap A_p$. The net $\{S_n, n \in D, \geq\}$, $S_n = c$ for all n, converges to all the points $a_1, a_2, \ldots, a_p$, because, by definition, $A_1 \cap A_2 \cap \ldots \cap A_p$ is the minimal set such that $A_1, A_2, \ldots, A_p$ are neighbourhoods of $a_1, a_2, \ldots, a_p$ respectively, and so, it follows that if $A'_1, A'_2, \ldots, A'_p$ are any other neighbourhoods of $a_1, a_2, \ldots, a_p$ respectively, then

$$A_1 \cap A_2 \cap \ldots \cap A_p \subseteq A'_1 \cap A'_2 \cap \ldots \cap A'_p,$$

i.e. $c \in A'_1 \cap A'_2 \cap \ldots \cap A'_p$.

*See Referee's Remark at the end of the paper

So, it follows that S'_p does not have the property that every net in X, which is not eventually in S'_p, converges to at most $(p-1)$ points.

In the particular case when S_p is the null set, we call X a Hausdorff space of order $(p-1)$, and we say that every p points of X are separated. In this case, Theorem 3 be stated as follows:

Any topological space X is a Hausdorff space of order $(p-1)$, if and only if every net in X, converges to at most $(p-1)$ points.

Theorem 4. *The Cartesian product of an S_p-space and of an S'_p-space is an $S_p \times S'_p$-space.*

PROOF. Let $(x_1, y_2), (x_2, y_2), \dots, (x_p, y_p)$ be any p points of the Cartesian product of the two spaces.
Let $(U_1 \times V_1), (U_2 \times V_2), \dots, (U_p \times V_p)$ be neighbourhoods of $(x_1, y_1), (x_2, y_2), \dots, (x_p, y_p)$ respectively, where $U_1, U_2, \dots, U_p$ belong to the S_p-space and $V_1, V_2, \dots, V_p$ to the S'_p-space.

By induction, it follows easily that the result of [1, pp. 89] which has been used in the proof of Theorem 2, can be put in the following general form:

$$\bigcap_{n=1}^{p} (U_n \times V_n) = (\bigcap_{n=1}^{p} U_n) \times (\bigcap_{n=1}^{p} V_n).$$

Let us choose U_n, V_n $(n = 1, 2, \dots, p)$ such that each of $\bigcap_{n=1}^{p} U_n$ and $\bigcap_{n=1}^{p} V_n$ is the minimal subset. The rest of the proof follows the same lines as that of Theorem 2.

Remarks. In section 1, if the minimal set $A \cap B$ does not exist, it follows that there exist two infinite sequences (A_n) and (B_n) of neighbourhoods of a and b respectively, such that $A_{n+1} \cap B_{n+1} \subseteq A_n \cap B_n$ for all n $(n = 1, 2, \dots)$. In this case, we take the minimal set to be $\bigcap_{n=1} O_n$, instead of $A \cap B$, where $O_n = A_n \cap B_n$. With this change, it is easily verified that the proofs of Theorems 1,2,3 are valid, and so these theorems are true in a more general form.

In section 2, we can choose $a_1, a_2, \dots$, to be a countable set of points of the topological space X, instead of a set of p points; and thus Theorem 3 can be extended to a countable set, instead of a set of $(p-1)$ points.

Remark (of the referee S. GACSÁLYI). The existence of $S = S_2$ implies the existence of S_p for any natural number $p \geq 2$. Let us consider the case $p = 3$:

If a_1, a_2, a_3 are any three pairwise different points, then let A_1, A_2; B_1, B_3; C_2, C_3 be those sets for which the intersections $A_1 \cap A_2$; $B_1 \cap B_3$; $C_2 \cap C_3$ are the contributions to S of the pairs of points a_1, a_2; a_1, a_3 and

a_2, a_3 respectively. Let now be U, V, W be any three neighbourhoods of a_1, a_2 and a_3 respectively. Put

$$D = (A_1 \cap A_2) \cap (B_1 \cap B_3) \cap (C_2 \cap C_3) = (A_1 \cap B_1) \cap (A_2 \cap C_2) \cap (B_3 \cap C_3).$$

Clearly, the inclusions

$$A_1 \cap A_2 \subseteq U \cap V, \quad B_1 \cap B_3 \subseteq U \cap W, \quad C_2 \cap C_3 \subseteq V \cap W$$

together imply that $U \cap V \cap W = (U \cap V) \cap (U \cap W) \cap (V \cap W) \supseteq (A_1 \cap A_2) \cap (B_1 \cap B_3) \cap (C_2 \cap C_3) = D$.
Since the points a_1, a_2, a_3 are arbitrary, S_3 exists, the set D being the conribution to it of the triple a_1, a_2, a_3.

In the general case we can reason along similar lines, the set D being now the intersection of $\binom{p}{2}$ sets, each of them the contribution to S_2 of a pair of points chosen from among $a_1, a_2, \dots, a_p$.

References

[1] J. L. KELLEY, General Topology, *New York*, 1955.

MANSOOR AHMAD
AMINABAD
LUCKNOW, INDIA

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