

Multiplicative functions satisfying a congruence property. III

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Let \mathcal{M} be the set of integer-valued multiplicative functions. For a fixed positive integer M let $K_M \subseteq \mathcal{M}$ denote the set of those functions $f(n)$ for which

$$(1) \quad \Delta_M f(n+p) \equiv \Delta_M f(n) \pmod{p}$$

holds for every positive integer n and for every prime p , where

$$\Delta_M f(n) = f(n+M) - f(n).$$

From a result of [1] it follows that if $f \in K_1$, then $f(n)$ is a power of n with non-negative integer exponent. The aim of this note is to extend this result for K_M . We shall prove the following.

Theorem. *Let M be a positive integer and let $f \in K_M$. Then for every positive integer n either*

$$f(n) = n^a \quad \text{or} \quad f(n+M) = f(n),$$

where a is a positive integer.

We need two lemmas for the proof of our theorem.

Lemma 1. *Let $f \in K_M$. Then for each prime p with $p^e \parallel M$ ($e \geq 0$) and $f(p^e) \neq 0$, we have*

$$(2) \quad f(p^{e+k}) = \left(\frac{f(p^{e+1})}{f(p^e)} \right)^k f(p^e) \quad (k = 1, 2, \dots).$$

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PROOF. For a positive integer m let m^* denote the product of all distinct prime factors of m . First we note that by the definition of K_M it follows that if $f \in K_M$, then

$$(3) \quad \Delta_{tM} f(n+m) \equiv \Delta_{tM} f(n) \pmod{m^*}$$

for every positive integer n, m, t . Thus

$$(4) \quad \begin{aligned} f(n+hmM) - f(n) &= \sum_{i=0}^{h-1} \{f(n+(i+1)mM) - f(n+imM)\} = \\ &= \sum_{i=0}^{h-1} \Delta_{mM} f(n+imM) \equiv \sum_{i=0}^{h-1} \Delta_{mM} f(n) = \\ &= h\Delta_{mM} f(n) \pmod{(mM)^*} \end{aligned}$$

and so, replacing h by hm^* ,

$$(5) \quad f(n+hmm^*M) \equiv f(n) \pmod{m^*}$$

holds for every positive integer n, m and h .

We prove that for an integer $m \geq 2$ and for a positive integer n with $(n, M) = 1$

$$(6) \quad m^* | f(n) \text{ implies } (n, m^*) > 1.$$

Assume indirectly that for an integer $m \geq 2$ and for a positive integer n with $(n, M) = 1$

$$m^* | f(n) \text{ and } (n, m^*) = 1.$$

Then by Dirichlet's theorem there exist positive integers x, y such that

$$(7) \quad (n, x) = 1 \text{ and } nx = 1 + mm^*My.$$

Applying (5), by using (7) we have

$$0 \equiv f(n)f(x) = f(nx) = f(1 + mm^*My) \equiv f(1) = 1 \pmod{m^*},$$

which is a contradiction, since $m \geq 2$. Thus (6) is proved.

Let p be a prime number with $p^e \parallel M$ ($e \geq 0$) and $f(p^e) \neq 0$. We shall prove that for each positive integer $k > 0$

$$(8) \quad f(p^{e+k}) = \frac{f(p^{e+1})}{f(p^e)} f(p^{e+k-1}).$$

Let $q(> p)$ be an arbitrary prime. For each prime q there exist positive integers $u = u(q)$, $v = v(q)$ such that

$$(9) \quad (u, pM) = 1 \quad \text{and} \quad p^{e+k-1}u = p^e + q^2Mv.$$

Since $(u, qM) = 1$, using (6) we have

$$(10) \quad f(u) \not\equiv 0 \pmod{q}.$$

Applying (5), by using (9) and (10) we get

$$\begin{aligned} f(p^{e+k}u)f(p^e) &\equiv f(p^{e+1})f(p^e) \\ &\equiv f(p^{e+1})f(p^{e+k-1}u) \pmod{q} \end{aligned}$$

and so by the multiplicativity of f

$$f(p^{e+k}) \equiv \frac{f(p^{e+1})}{f(p^e)} f(p^{e+k-1}) \pmod{q},$$

which proves (8), since $q(> p)$ is an arbitrary prime, and from (8) the lemma follows.

Lemma 2. *Let $f \in K_M$. Then there is a non-negative integer a such that*

$$(11) \quad f(nM) = n^a f(M)$$

for every positive integer n .

PROOF. First we prove that

$$(12) \quad f(nM + mM) \equiv f(nM) \pmod{m^*}$$

holds for every positive integer n, m .

By (3) and (4), applying (3) with $m = mM$, $n = M$, $t = n - 1$, we can derive

$$(13) \quad \begin{aligned} f(nM + hmM) - f(nM) &\equiv h\Delta_{mM}f(nM) \\ &\equiv h\Delta_{mM}f(M) \pmod{m^*} \end{aligned}$$

for every positive integer n, m, h . Let $m = p$ be a prime for which $p^e \parallel M$ ($e \geq 0$) and $M = p^e M'$. By using (13) we get

$$f(pM + p^2M) - f(pM) = f(p^{e+1}) \{f(M' + pM') - f(M')\} \equiv 0 \pmod{p}.$$

If

$$f(M' + pM') - f(M') \equiv 0 \pmod{p} \quad \text{or} \quad f(p^e) \equiv 0 \pmod{p},$$

then by (13)

$$\begin{aligned} f(nM + hpM) - f(nM) &\equiv h\Delta_{pM}f(M) \\ &= hf(p^e) \{ f(M' + pM') - f(M') \} \equiv 0 \pmod{p}, \end{aligned}$$

from which (12) follows. If

$$f(p^{e+1}) \equiv 0 \pmod{p} \quad \text{and} \quad f(p^e) \not\equiv 0 \pmod{p},$$

then by using Lemma 1, we have

$$f(p^{e+k}) \equiv 0 \pmod{p}$$

for $k = 1, 2, \dots$, and so (13) implies that

$$f(M + pM) - f(M) \equiv f(pM + pM) - f(pM) \equiv 0 \pmod{p}.$$

This with (13) implies that

$$f(nM + hpM) - f(nM) \equiv 0 \pmod{p},$$

for any $n > 0$, from which (12) also follows. Thus (12) is proved.

Let q be a fixed prime, for which $(q, M) = 1$. Then, using (6) (see the proof of Lemma 1),

$$(14) \quad f(q) = \pm q^a,$$

where $a = a(q)$ is a non-negative integer. Let n be a positive integer.

Applying (12) and using Lemma 1 we have

$$f(nq^{2s}M) \equiv f(M) \pmod{(nq^{2s} - 1)^*}$$

and so

$$(15) \quad f(q^{2s})f(nM) \equiv f(M) \pmod{(nq^{2s} - 1)^*}.$$

By (14) and (15) it follows that

$$f(nM) = n^a f(M),$$

because $f(q^k) = f(q)^k$ ($k = 1, 2, \dots$) by Lemma 1, and

$$(nq^{2s} - 1)^* \rightarrow \infty \quad \text{as} \quad s \rightarrow \infty.$$

Thus Lemma 2 is proved.

PROOF. of the theorem. First we assume that $f(M) \neq 0$. Then, by Lemma 2, we have

$$(16) \quad f(n) = n^a \quad \text{if } (n, M) = 1.$$

Let n be an arbitrary positive integer. Then there exist infinitely many positive integers m , for which

$$(17) \quad (n + m, M) = 1 \quad \text{and} \quad m^* \rightarrow \infty.$$

Since $f \in K_M$, for each integer m satisfying (17), by using (3) and (16), we have

$$\begin{aligned} f(n + tM) - f(n) &\equiv f(n + m + tM) - f(n + m) = \\ &= (n + m + tM)^a - (n + m)^a \equiv \\ &\equiv (n + tM)^a - n^a \pmod{m^*} \end{aligned}$$

and so

$$(18) \quad f(n + tM) - f(n) = (n + tM)^a - n^a.$$

Applying (18) in the case $t = n^2$, using (16) we get

$$f(n) \{(1 + nM)^a - 1\} = n^a \{(1 + nM)^a - 1\}.$$

If $a > 0$, then

$$(19) \quad f(n) = n^a$$

for every positive integer n . If $a = 0$, then by (18) it follows that

$$(20) \quad f(n + M) = f(n)$$

for every positive integer n .

Now assume that $f(M) = 0$. In this case, using Lemma 2,

$$(21) \quad f(nM) = n^a f(M) = 0$$

for every positive integer n . Since $f \in K_M$, by using (21) and (3) we have

$$f(n + M) - f(n) \equiv f(ntM + M) - f(ntM) = 0 \pmod{(Mt - 1)^*},$$

which implies (20), because

$$(Mt - 1)^* \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

This completes the proof of our theorem.

References

- [1] B. M. Phong, Multiplicative functions satisfying a congruence property II, *Ann. Univ. Sci. Budapest, Sec. Math.* **33** (1990).

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