Publ. Math. Debrecen
51 / 3-4 (1997), 265–271

A note on the denseness of complete invariant metrics

By CSABA VARGA (Cluj-Napoca) and MARIUS CRAINIC (Cluj-Napoca)

Abstract. Many authors have obtained interesting results concerning the existence and the denseness of complete metrics on metrisable topological spaces and on differentiable manifolds (see [3], [4], [5], [7]).

In this note we extend these results to the invariant case and simplify some proofs and we extend these results to Finsler spaces.

1. Introduction

Let M be a finite dimensional topological manifold and G a compact topological group which acts continuously on M. Then we know that M is metrisable. Therefore we can consider the space of metrics on M that generate the topology τ_M of the space M, i.e. let

$$\mathcal{M} = \{\delta : M \times M \to \mathbb{R} \mid \delta \text{ is a metric and } \tau_\rho = \tau_M\},$$

where τ_ρ is the topology generated by the metric ρ on M. We say that $\delta \in \mathcal{M}$ is a G-invariant metric if $\delta(ax, ay) = \delta(x, y)$ for every $x, y \in M$ and $a \in G$.

We introduce the following notations:

$$\begin{aligned} \mathcal{M}^G &= \{\delta \in \mathcal{M} \mid \delta \text{ is a } G\text{-invariant metric}\} \\ \mathcal{M}_0 &= \{\delta \in \mathcal{M} \mid \delta \text{ is a complete metric}\} \\ \mathcal{M}_0^G &= \mathcal{M}^G \cap \mathcal{M}_0. \end{aligned}$$

Mathematics Subject Classification: 54E50, 54C35, 54H11.
Key words and phrases: complete metrics, complete Riemannian metrics, complete Finslerian metrics, topological groups.

We say that the function $f : M \to \mathbb{R}$ is proper if for every compact $K \subset \mathbb{R}$ the subset $f^{-1}(K) \subset M$ is compact. For the existence of continuous and differentiable proper functions see [3] and [8]. We need the following simple result, which will be very useful in the sequel:

Lemma 1.1. *If $\delta \in \mathcal{M}$ and there exists a continuous proper function $f : M \to \mathbb{R}$ such that $\delta(x, y) \geq |f(x) - f(y)|$ for every $x, y \in M$, then we have $\delta \in \mathcal{M}_0$.*

Proof. Let $(x_n)_{n\geq 1} \subset M$ be a δ-Cauchy sequence, then $(f(x_n))_{n\geq 1}$ is also a Cauchy sequence, hence bounded. In particular there exist $a, b \in \mathbb{R}$ such that $f(x_n) \in [a, b]$ for every $n \geq 1$. It follows that $(x_n)_{n\geq 1} \subset f^{-1}([a, b])$ but $f^{-1}([a, b])$ is a compact set because f is a proper function, hence $(x_n)_{n\geq 1}$ has a convergent subsequence.

Because M has a G-invariant exhaustion, we obtain as in the trivial case $G = \{e\}$ the following result:

Lemma 1.2. *If M is a finite dimensional topological G-manifold, then there exists a continuous proper G-invariant function $p : M \to \mathbb{R}$.*

More generally we have:

Lemma 1.3. *Let K be a compact subset of M and $f_0 : M \to \mathbb{R}$ a continuous G-invariant function. Then there exists a continuous G-invariant proper function $f : M \to \mathbb{R}$ such that $f|_K = f_0|_K$.*

Proof. We can suppose that K is G-invariant (or else, we can replace K by GK). We choose a compact G-invariant set L such that $K \subseteq \overset{\circ}{L}$ and then, for the G-invariant covering $\{\overset{\circ}{L},\ M \backslash K\}$ of M, we find a G-invariant function $\varphi : M \to \mathbb{R}$ such that $\varphi|_K = 1$ and $\varphi|_{M\backslash L} = 0$. We define the function $f = \varphi f_0 + (1 - \varphi)p$ with p as in Lemma 1.2. Because L is a compact set, p is a proper function and $f|_{M\backslash L} = p|_{M\backslash L}$ we get that f is a proper function. Obviously $f|_K = f_0|_K$ and this completes the proof.

Let now M be a C^∞ finite dimensional, connected differentiable manifold and G a compact Lie group which acts differentiably on M. We denote by $T(M)$ the tangent bundle of M. If M is a G-manifold and $\psi : G \times M \to M$ is the action of G on M, then the tangent bundle TM is a G-bundle, with the action $g \cdot X = d_x\psi(g, x) \cdot (X)$, for every $X \in T_x(M)$ and $x \in M$.

Definition 1.1. Let $\|\cdot\| : T(M) \to [0,\infty)$ be a continuous function. We say that $\|\cdot\|$ is a Finsler norm on M if the following conditions are satisfied:

(i) for each $x \in M$, the restriction of $\|\cdot\|$ to T_xM denoted by $\|\cdot\|_x$ is a continuous norm on T_xM.

(ii) for each $x_0 \in M$ and $k > 1$ there is a trivializing neighbourhood U of x_0 such that:

$$\frac{1}{k}\|\cdot\|_x \le \|\cdot\|_{x_0} \le k\|\cdot\|_x \quad \text{for every } x \in U.$$

We say that $\|\cdot\|$ is G-invariant if $\|aX\| = \|X\|$ for every $X \in TM$ and $a \in G$. If $\|\cdot\|$ is a Finsler norm on M, then for each C^1 path $\sigma : [a,b] \to M$ let us define the length of σ by $l(\sigma) = \int_a^b \|\dot\sigma(t)\|dt$. If x, y are two points in M define the distance $\delta_{\|\cdot\|}(x,y) = \inf\{l(\sigma) \mid \sigma$ is a C^1 path joining x and $y\}$. Then $\delta_{\|\cdot\|}$ is a metric on M and $\tau_M = \tau_{\delta_{\|\cdot\|}}$ (see [6]).

We say that $\|\cdot\|$ is a complete Finsler norm if $\delta_{\|\cdot\|}$ is a complete metric on M.

We define:

$$\begin{aligned}
\mathcal{F} \;\; &= \{\|\cdot\| \;\big|\; \|\cdot\| \text{ is a Finsler norm on } M\} \\
\mathcal{F}^G &= \{\|\cdot\| \in \mathcal{F} \;\big|\; \|\cdot\| \text{ is } G\text{-invariant}\} \\
\mathcal{F}_0 \;\; &= \{\|\cdot\| \in \mathcal{F} \;\big|\; \|\cdot\| \text{ is complete}\} \\
\mathcal{F}_0^G &= \mathcal{F}^G \cap \mathcal{F}_0.
\end{aligned}$$

If $g : TM \oplus TM \to \mathbb{R}$ is a Riemannian metric on M then we associate to g the following Finsler norm on M: $h_g : TM \to \mathbb{R}$ defined by $h_g(X) = (g(X,X))^{\frac{1}{2}}$.

We say that g is a complete Riemannian metric on M iff h_g is a complete Finsler norm. Also, we say that g is G-invariant if $g(aX,aY) = g(X,Y)$ for each $(X,Y) \in TM \oplus TM$ and for each $a \in G$. We define $\mathcal{R}$, $\mathcal{R}^G$, $\mathcal{R}_0$ and $\mathcal{R}_0^G$ in a natural way.

2. The case of metrics

Let now M be a finite dimensional topological manifold and G a topological compact group which acts continuously on M. We suppose that $\mathcal{M}$ is endowed with the topology induced by the compact-open topology of $C(M \times M, \mathbb{R})$. We are interested to the inclusion $\mathcal{M}_0^G \subseteq \mathcal{M}^G \subseteq \mathcal{M}$ from topological viewpoint.

Theorem 2.1. *The set* $\mathcal{M}_0^G$ *is nonempty. Moreover, for each* $\delta \in \mathcal{M}^G$ *and each compact set* $K \subseteq M \times M$*, there exists* $\tilde{\delta} \in \mathcal{M}_0^G$ *such that* $\tilde{\delta} \mid_K = \delta \mid_K$.

Proof. First, we prove the last statement. Let $\delta \in \mathcal{M}^G$ and $K \subset M \times M$ a compact set. We choose a compact set $K_1 \subseteq M$ such that $K \subseteq K_1 \times K_1$ and we apply the Lemma 1.3 for $f_0 = 0$ and K_1. Hence we find a proper G-invariant function $f : M \to \mathbb{R}$ with $f \mid_{K_1} = 0$ and we define $\tilde{\delta} : M \times M \to \mathbb{R}$ by $\tilde{\delta}(x, y) = \delta(x, y) + |f(x) - f(y)|$. Obviously $\tilde{\delta}$ is a G-invariant metric on M, and using Lemma 1.1 we get that $\tilde{\delta}$ is a complete metric on M. We have the following implications:

1) $x_n \xrightarrow{\tilde{\delta}} x_0$ implies that $x_n \xrightarrow{\delta} x_0$ because $\tilde{\delta} \geq \delta$.

2) $x_n \xrightarrow{\delta} x_0$ implies that $x_n \xrightarrow{\tau_M} x_0$ and $f(x_n) \mapsto f(x_0)$ (because f is continuous and $\tau_M = \tau_\delta$), hence $x_n \xrightarrow{\tilde{\delta}} x_0$. Then we have $\tau_{\tilde{\delta}} = \tau_\delta = \tau_M$, hence $\tau_{\tilde{\delta}} \in \mathcal{M}_0^G$. Obviously $\tilde{\delta} \mid_K = \delta$. It remains to prove that $\mathcal{M}^G \neq \emptyset$. We know that $\mathcal{M} \neq \emptyset$, hence we choose $d \in \mathcal{M}$ and we define $\delta : M \times M \to \mathbb{R}$ by $\delta(x, y) = \sup_{a \in G} d(ax, ay)$. Because G is a compact group we have

$$\delta(x, y) = \max_{a \in G} d(ax, ay). \tag{1}$$

Also we have $\delta(x, z) = \sup_{a \in G} d(ax, az) \leq \sup_{a \in G} [d(ax, ay) + d(ay, az)] \leq \delta(x, y) + \delta(y, z)$ for each $x, y, z \in M$ and hence δ is a G-invariant metric on M. Because $\delta \geq d$ it remains to prove that $x_n \xrightarrow{d} x_0$ implies $x_n \xrightarrow{\delta} x_0$.

We suppose that this is not true. We find a sequence $(x_n)_{n \geq 1}$ in M, a point $x_0 \in M$ and a positive real number $r > 0$ such that:

$$x_n \xrightarrow{d} x_0 \text{ and } \delta(x_n, x_0) \geq r \text{ for every } n \geq 1. \tag{2}$$

From (1) and from the fact that G is compact we infer that there exists a sequence $(a_n)_{n \geq 1}$ in G, convergent to $a_0 \in G$ and such that: $\delta(x_n, x_0) = d(a_n x_n, a_n x_0)$ for every $n \geq 1$. Now we have $\delta(x_n, x_0) \leq d(a_n x_n, a_0 x_0) + d(a_0 x_0, a_n x_0)$, and because $a_n \xrightarrow{G} a_0$, $x_n \xrightarrow{\tau_M} x_0$, $\tau_M = \tau_d$ and G acts continuously on M we have $\delta(x_n, x_0) \to 0$, which contradicts (2). Now the proof is complete.

Theorem 2.2. *$\mathcal{M}_0^G$ is dense in $\mathcal{M}^G$ and $\mathcal{M}^G$ is closed in $\mathcal{M}$.*

Proof. The first part is clear from Theorem 2.1. For the last part let $\delta \in \mathcal{M} \backslash \mathcal{M}^G$ be a metric and we prove that $\delta \notin \overline{\mathcal{M}^G}$. Because $\delta \notin \mathcal{M}^G$ we find $a \in G$ and $x, y \in M$ such that $\delta(x,y) \neq \delta(ax,ay)$. We choose D_1, D_2 to be two disjoint neighbourhoods in $\mathbb{R}$ of $\delta(x,y)$ and $\delta(ax,ay)$. Then we have $U := \mathcal{M} \cap B(\{(x,y)\}, D_1) \cap B(\{(ax,ay)\}, D_2) \in \mathcal{V}_{\mathcal{M}}(\delta)$ and $U \cap \mathcal{M}^G = \emptyset$ (where $B(K,D) = \{f : M \times M \to \mathbb{R} \mid f$ is continuous and $f(K) \subset D\}$, for a compact subset K of $M \times M$ and an open subset D of $\mathbb{R}$), hence $\delta \notin \overline{\mathcal{M}^G}$.

3. The case of Finsler norms

We suppose that M is a finite dimensional C^∞ differentiable manifold and G a Lie compact group which acts differentiably on M. We suppose that $\mathcal{F}$ is endowed with the topology induced by the compact-open topology of $\mathcal{C}(TM, \mathbb{R})$. We are interested in the inclusion $\mathcal{F}_0^G \subseteq \mathcal{F}^G \subseteq \mathcal{F}$ from a topological viewpoint.

Theorem 3.1. *The set $\mathcal{F}_0^G$ is nonempty. Moreover, for each $h \in \mathcal{F}^G$ and compact $K \subseteq M$, there exists $\tilde{h} \in \mathcal{M}_0^G$ such that $\tilde{h}_x = h_x$ for every $x \in K$.*

Proof. First, we prove the last statement. Let $h \in \mathcal{F}^G$ and $K \subseteq M$ a compact set. We choose an open set U with compact closure such that $K \subseteq U$ and we apply Lemma 1.3 for $f_0 = 0$ and $\overline{U}$.

Hence we find a proper G-invariant function $f : M \to \mathbb{R}$ with $f\mid_{\overline{U}} = 0$. Obviously, we can suppose that $f \in C^\infty$ (see the proof of Lemma 1.3). We define the function $\tilde{h} : TM \to \mathbb{R}$ by $\tilde{h} = h + |df|$ which is a G-invariant Finsler norm on M.

For the completeness of $\tilde{h}$ we observe that if $\sigma : [a,b] \to M$ is a C^1 path joining x and y then we have

$$\int_a^b \tilde{h}(\dot{\sigma}(t))dt \geq \int_a^b |df(\dot{\sigma}(t))|dt \geq \left| \int_a^b (f \circ \sigma)'(t)dt \right| = |f(x) - f(y)|$$

and we can apply Lemma 1.1. For the first statement of the theorem it is sufficient to prove that $\mathcal{F}^G \neq \emptyset$. But we know that there exists $h \in \mathcal{F}$ see [6]. We define $\tilde{h} : TM \to \mathbb{R}$ by $\tilde{h}(X) = \sup_{a \in G} h(aX) = \max_{a \in G} h(aX)$ (or $\tilde{h}(X) = \int_G |aX| da$, where da is a Haar measure on G). Obviously, $\tilde{h} \in \mathcal{F}^G$.

Theorem 3.2. *$\mathcal{F}_0^G$ is dense in $\mathcal{F}^G$ and $\mathcal{F}^G$ is closed in $\mathcal{F}$.*

The proof is similar to Theorem 2.2.

4. The case of Riemannian metrics

We suppose that M is a finite dimensional differentiable manifold of class C^∞ and G is a compact Lie group which acts differentiably on M. We suppose that $\mathcal{R}$ is endowed with the topology induced by the compact-open topology of $C(TM \oplus TM, \mathbb{R})$. We are concerned with the inclusion $\mathcal{R}_0^G \subseteq \mathcal{R}^G \subseteq \mathcal{R}$ from a topological viewpoint.

Theorem 4.1. *The set $\mathcal{R}_0^G$ is nonempty. Moreover, for each $g \in \mathcal{R}^G$ and compact $K \subseteq M$, there exists $\tilde{h} \in \mathcal{R}_0^G$ such that $\tilde{g}_x = g_x$ for every $x \in K$.*

Proof. We follow the line of [3] First, we prove the last statement. Let $g \in \mathcal{R}^G$ and $K \subseteq M$ a compact set. We choose an open set $U \subseteq M$ with compact closure such that $K \subseteq U$ and we apply Lemma 1.3 for $f_0 = 0$ and $\overline{U}$. Hence we get a proper, G-invariant, C^∞-function $f : M \to \mathbb{R}$ such that $f\,|_{\overline{U}}= 0$. We define $\tilde{g} : TM \oplus TM \to \mathbb{R}$ by $\tilde{g} = g + (df) \otimes (df)$ and we see that $\tilde{g}$ is a G-invariant Riemannian metric on M. For the completeness of $\tilde{g}$ we observe that if $\sigma : [a, b] \to M$ is a C^1 path joining x and y then we have the inequalities

$$\int_a^b [\tilde{g}(\dot{\sigma}(t), \dot{\sigma}(t))]^{\frac{1}{2}}\, dt \geq \int_a^b |df(\dot{\sigma}(t))| \geq \left|\int_a^b (f \circ \sigma)'(t)dt\right| = |f(x) - f(y)|$$

and we can apply Lemma 1.1. For the first statement of the theorem it is sufficient to prove then, that $\mathcal{R}^G \neq \emptyset$. We choose a Haar measure da on G and an element $g \in \mathcal{R}$ (we know that $\mathcal{R} \neq \emptyset$) and we define $\tilde{g} : TM \oplus TM \to \mathbb{R}$ by $\tilde{g}(X, Y) = \int_G g(aX, aY)da$. We have $\tilde{g} \in \mathcal{R}^G$ and this completes the proof.

Theorem 4.2. *$\mathcal{R}_0^G$ is dense in $\mathcal{R}^G$ and $\mathcal{R}^G$ is closed in $\mathcal{R}$.*

The proof is similar to that of Theorem 2.2. If G is the trivial group then we get the results of J.A. Morrow and H.D. Fegan, R.S. Millman see [4] respectively [2].

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CSABA VARGA
BABEŞ–BOLYAI UNIVERSITY
FACULTY OF MATHEMATICS
STR. KOGALNICEANU 1
3400 CLUJ–NAPOCA
ROMANIA

MARIUS CRAINIC
BABEŞ–BOLYAI UNIVERSITY
FACULTY OF MATHEMATICS
STR. KOGALNICEANU 1
3400 CLUJ–NAPOCA
ROMANIA

(Received September 23, 1996; revised April 14, 1997)