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On a method of Galambos and Kátai concerning the frequency of deficient numbers

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Abstract. A number n is called deficient if $\sigma(n) < 2n$, where $\sigma(n)$ denotes the sum of divisors of n. In this note it is proved that, for $n \geq n_0$, there is a deficient number between n and $n + (\log n)^2$.

With the method of GALAMBOS [1] and KÁTAI [3], I establish the announced result. It should be noted that the more general result of GALAMBOS and KÁTAI [2] cannot be applied to deficient numbers since $\sigma(n)$ is multiplicative while their result is applicable to certain additive functions only.

Theorem. *For all sufficiently large natural number n, there is a deficient number between n and $n + (\log n)^2$.*

PROOF. Notice that

$$\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d} \tag{1}$$

Hence, it suffices to prove that

$$S = \sum_{n+1 \leq m \leq n+k} \sum_{d|m} \frac{1}{d} \quad < \quad 2k \tag{2}$$

with $k = (\log n)^2$. As in Galambos [2], we split

$$S = S_1 + S_2 \tag{3}$$

where

$$S_1 = \sum_{n+1 \le m \le n+k} \quad \sum_{d|m,\ d<f(n)} \frac{1}{d}$$

$$S_2 = \sum_{n+1 \le m \le n+k} \quad \sum_{d|m,\ d>f(n)} \frac{1}{d}$$

where we choose $f(n)$ later. Now,

$$S_1 = \sum_{1 \le d \le f(n)} \frac{1}{d} \sum_{\substack{n+1 \le m \le n+k \\ d|m}} 1 \quad \le \quad \sum_{1 \le d \le f(n)} \frac{1}{d} \left(\frac{k}{d} + 1 \right) <$$

$$< k \sum_{d=1}^{+\infty} \frac{1}{d^2} \quad + \sum_{1 \le d \le f(n)} \frac{1}{d} < 1.8k + \log \left(f\left(n\right) + 1 \right).$$

On the other hand,

$$S_2 = \sum_{f(n) < d \le n+k} \frac{1}{d} \sum_{\substack{n+1 \le m \le n+k \\ d|m}} 1 < \frac{1}{f(n)} \sum_{f(n) \le d \le n+k} \quad \sum_{\substack{n+1 \le m \le n+k \\ d|m}} 1$$

Let

$$a_n(d) = \sum_{\substack{n+1 \le m \le n+k \\ d|m}} 1$$

Since $d \ge f(n)$, $a_n(d) = 1$ or 0 whenever $f(n)$ is of larger order than $k = (\log n)^2$. Furthermore, a fixed m can contribute 1 to $a_n(d)$ at most as many times as many divisors m has, so,

$$f(n) S_2 < \sum_{n+1 \le m \le n+k} d(m)$$

where $d(m)$ is the number of the divisors of m. It is known that (see [4])

$$\sum_{m \le N} d(m) \quad = N \log N + CN + O\left(N^{1/3} \right),$$

and thus,

$$\begin{aligned} f(n)S_2 <&(n+k)\log(n+k)+C(n+k)+O\left((n+k)^{1/3}\right)-n\log n- \\ &-Cn+O\left(n^{1/3}\right) \quad = n\log\left(1+\frac{k}{n}\right)+k\log(n+k)+Ck+ \\ &+O\left((n+k)^{1/3}\right)= \quad O\left(n^{1/3}\right) \end{aligned}$$

(recall that $k=(\log n)^2$). Now, if we choose $f(n)=\sqrt{n}$, we get

$$S_2 = O\left(n^{-1/6}\right) \tag{4}$$

and our estimate on S_1 becomes

$$S_1 < 1.8k+\frac{1}{2}\log n < 1.9k \tag{5}$$

since $k=(\log n)^2$. The estimates (4) and (5), via (3), imply (2), which completes the proof.

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