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On a Special Superelliptic Equation

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The purpose of this remark is to show that for each given integer $n > 2$, the equation

$$\binom{x}{n} = \binom{y}{2}$$

has at most finitely many solutions in integers x and y. Sierpinski had conjectured to that effect for the case $n = 3$ — thus that there are only finitely many numbers that are both triangular and tetrahedral. This was verified by AVANESOV [1] who determined all the solutions; there are five:

$$(x, y) = (3, 2),\ (5, 5),\ (10, 16),\ (22, 56),\ (36, 120).$$

More recently, KISS [3] showed that, when $n = p$ is prime, the given equation has just finitely many solutions. The critical auxiliary step in his argument consists of showing that a certain polynomial has sufficiently many simple zeros; it makes essential use of the primality of n. Our corresponding argument (Lemma 3, below) is simpler and deals with all n.

In the sequel f_n denotes the polynomial

$$f_n(X) = X(X-1)\cdots\bigl(X-(n-1)\bigr) + \frac{1}{8}n!\,.$$

We employ the notation $C = C(a, b, \dots)$ to indicate that the constant C depends only on the stated parameters $a, b, \dots$.

We prove that

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Theorem. *Suppose $n > 2$ and let $a \neq 0$ be an integer. Then all solutions of the equation*

$$f_n(x) = ay^z$$

in rational integers x, y and z with $|y| > 1$ and $z > 1$ satisfy $\max\{|x|, |y|, z\} < C_1$, where $C_1 = C_1(n, a)$ is an effectively computable constant.

Our original equation is readily reduced to a hyperelliptic equation. We have

$$y^2 - y = 2\binom{x}{n} \quad \text{so} \quad (2y-1)^2 = 8\binom{x}{n} + 1$$

which is

$$x(x-1)\cdots\bigl(x-(n-1)\bigr) + \frac{1}{8}n! = \frac{1}{8}n!(2y-1)^2 .$$

Thus we have, as claimed,

Corollary. *Let $n > 2$ be an integer. Then all the positive integer solutions of the equation*

$$\binom{x}{n} = \binom{y}{2}$$

satisfy $\max\{x, y\} < C_2$, where $C_2 = C_2(n)$ is an effectively computable constant.

Preliminaries

We use the following preliminary results: Let $f(X)$ be a non-constant polynomial with rational coefficients and let b be a nonzero rational number. Denote by $\alpha_1, \dots, \alpha_k$ the distinct zeros of f in $\mathbf{C}$ and let $r_1, \dots, r_k$ be their respective multiplicities. Given a positive integer $m \geq 2$, set

$$q_i = \frac{m}{(m, r_i)} \; ; \quad i = 1, \dots, k .$$

Lemma 1. (Brindza [2]). *Suppose that $(q_1, \dots, q_k)$ is not a permutation of either of the k-tuples $(q, 1, \dots, 1)$ or $(2, 2, 1, \dots, 1)$. Then all solutions of the equation*

$$f(x) = by^m$$

in rational integers x and y satisfy $\max\{x, y\} < C_3$, where $C_3 = C_3(f, m, b)$ is an effectively computable constant.

The actual result of [2] is more general. An elegant proof is given in SHOREY and TIJDEMAN [5] at Theorem 8.3.

Lemma 2. (Schinzel and Tijdeman [4]). *If the polynomial f has at least two distinct zeros, then all solutions of the equation*

$$f(x) = by^z$$

in rational integers x, y and z with $|y| > 1$ satisfy $|z| < C_4$, where $C_4 = C_4(f, b)$ is an effectively computable constant.

To apply these results we need some information on the zeros of the polynomials f_n. We prove more than is needed for the application.

Lemma 3. *If $n > 3$ then all the zeros of the polynomial f_n are simple.*

Clearly $f_n(k) - n!/8 = 0$ for the n integers $k = 0, 1, \dots, n-1$. Thus, its derivative f_n' must have (at least, and of course no more than) $n - 1$ real zeros, so all the zeros of that derivative are real and lie in the interval $(0, n-1)$. Hence it suffices to show that, other than for a simple zero in the interval $(-1, 0)$ when n is odd, f_n has no real zero. But on the left of the interval $(-1, n-1)$ we easily see that $|f_n| \geq n! - n!/8$, and on the right we have $|f_n| \geq n!/8$. For α in the interval $(0, n-1)$

$$|\alpha(\alpha - 1) \cdots (\alpha - (n-1))| \leq \max_{k=1,\dots,n-1} k!(n-k)!/4,$$

whence

$$f_n(\alpha) \geq n!/8 - (n-1)!/4 > 0.$$

Proof of the Theorem

It is easy to confirm explicitly that the polynomial f_3 has no rational zero, is therefore irreducible over $\mathbf{Q}$, and thus has distinct zeros in $\mathbf{C}$. Then the Theorem is just a special case of Lemma 1.

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