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On second order linear divisibility sequences over algebraic number fields

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To the memory of B. Barna, K. Buzási, S. Buzási and M. Erdélyi

1. Introduction

Let R be an integral domain which is finitely generated over $\mathbf{Z}$. (Linear) *divisibility sequences* over R are recurrence sequences $\{u_h\}_{h=0}^{\infty}$, $u_h \in R$, $h = 0, 1, \ldots$ (that is sequences in R satisfying linear homogeneous recurrence relations with constant coefficients) with the property that whenever $h|k$, then $u_h|u_k$ in R. This notation was introduced by M. HALL [4] who described all second order divisibility sequences over $\mathbf{Z}$, as well as the third order divisibility sequences over $\mathbf{Z}$ having irreducible characteristic polynomials.

Let $d > 1$ be an integer. The recurrence sequence $\{u_h\}_{h=0}^{\infty}$ is called d-(linear) *divisibility sequence* if $u_h|u_{hd}$ in R for $h = 0, 1, 2, \ldots$. For $R = \mathbf{Z}$, SOLOMON [6] characterized all 2-divisibility sequences. BÉZIVIN, PETHŐ and VAN DER POORTEN [1] proved for any R, that if $\{u_h\}_{h=0}^{\infty}$ is d-divisible for an integer $d > 1$ then there is a recurrence sequence $\{\bar{u}_h\}_{h=0}^{\infty}$ of the form

$$\bar{u}_h = h^k \prod_i \left(\frac{\alpha_i^h - \beta_i^h}{\alpha_i - \beta_i} \right)$$

with some integer $k \geq 0$ over R such that $u_h|\bar{u}_h$ in R for $h = 0, 1, 2, \ldots$. Thus they confirmed an old conjecture of WARD [7]. For further references concerning divisibility sequences, we refer to [1].

Although the result of Bézivin et al. is very general, it is not straightforward to deduce from it a complet list of d-divisibility sequences over a

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given ring. The aim of this paper is to give a more explicit description of second order d-divisibility recurrence sequences over the ring of integers $\mathbf{Z}_K$ of an algebraic number field K (cf. Theorem 1). In fact, we shall give a criterion for second order non-degenerate recurrence sequences over $\mathbf{Z}_K$ to be d-divisible for some integer $d > 1$. Further, we show (cf. Corollary 2) that a second order non-degenerate recurrence sequence over $\mathbf{Z}_K$ is a divisibility sequence if and only if it is 2-divisible (Corollary 2). Finally, using Theorem 1 we give explicitly all second order recurrence sequences over $\mathbf{Z}$ (cf. Theorem 2) which are d-divisible for some $d > 1$.

2. Results

To state our results we need some notations. Let K be an algebraic number field and denote by $\mathbf{Z}_K$ its ring of integers. Let the sequence $\{u_h\}_{h=0}^{\infty}$ be defined by the initial terms u_0, u_1 and by the recursion

$$u_{n+2} = Au_{n+1} + Bu_n, \quad n \geq 0, \tag{1}$$

where $u_0, u_1, A, B \in \mathbf{Z}_K$ and $u_0^2 + u_1^2 \neq 0$, $B \neq 0$. Denote by α and β the zeros of the polynomial $x^2 - Ax - B$. Then we have (see e.g. [5])

$$u_n = a\alpha^n - b\beta^n \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{2}$$

or

$$u_n = (an + b)\alpha^n \quad \text{for} \quad n = 0, 1, 2, \ldots, \tag{3}$$

according as $\alpha \neq \beta$ or $\alpha = \beta$. Further, we have $a = \dfrac{u_1 - u_0\beta}{\alpha - \beta}$ and $b = \dfrac{u_1 - u_0\alpha}{\alpha - \beta}$ in (2), and $a = \dfrac{u_1 - u_0\alpha}{\alpha}$ and $b = u_0$ in (3).

The sequence $\{u_h\}_{h=0}^{\infty}$ is called *degenerate* if α/β is a root of unity, and *non-degenerate* otherwise.

Theorem 1. *Let $\{u_h\}_{h=0}^{\infty}$ be a second order non-degenerate recurrence sequence over $\mathbf{Z}_K$ with the parameters specified above. (A) If there exists an integer $d > 1$ and an n_0 such that $u_n | u_{nd}$ in $\mathbf{Z}_K$ for all $n \geq n_0$, then $b^{d-1} = a^{d-1}$. (B) Conversely, if $b^{d-1} = a^{d-1}$ for some integer $d > 1$, then $\{u_h\}$ is d-divisible.*

In other words, under the assumptions of Theorem 1 $\{u_h\}$ is d-divisible if and only if $b^{d-1} = a^{d-1}$.

If the assumptions in (A) of Theorem 1 hold for $d = 2$ then we get that $b = a$. Further, in this case $u_0 = 0$, $b = a = \dfrac{u_1}{\alpha - \beta}$ and hence $\{u_h\}$ is divisible. Conversely, if $u_0 = 0$ then $b = a$. Thus we have the following

Corollary 1. *Let $\{u_h\}_{h=0}^{\infty}$ be as in Theorem 1. (A) Assume that there exists an n_0 such that $u_n | u_{2n}$ in $\mathbf{Z}_K$ for all $n \geq n_0$. Then $u_0 = 0$ and*

$$u_n = u_1 \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

(B) Conversely, if $u_0 = 0$ then $\{u_h\}$ is divisible.

This implies the following

Corollary 2. *Let $\{u_h\}_{h=0}^{\infty}$ be a second order non-degenerate recurrence sequence over $\mathbf{Z}_K$. $\{u_h\}$ is a divisibility sequence if and only if it is 2-divisible.*

The degree of $K(\alpha)$ over $\mathbf{Q}$ is at most $2k$, where k denotes the degree of K over $\mathbf{Q}$. Hence, in Theorem 1, there exist only finitely many possibilities for a/b (if $b \neq 0$) and b/a (if $a \neq 0$) which are easily computable if K is given. We shall carry out this explicitly only for $K = \mathbf{Q}$. Using Theorem 1, we shall list in Theorem 2 below all second order (degenerate and non-degenerate) recurrence sequences over $\mathbf{Z}$ which are d-divisible for some $d > 1$.

Theorem 2. *Let $\{u_h\}_{h=0}^{\infty}$ be a second order recurrence sequence over $\mathbf{Z}$ with the parameters $u_0, u_1, A, B \in \mathbf{Z}$, $u_0^2 + u_1^2 \neq 0$, $B \neq 0$ specified above, and let $d > 1$ be an integer. The sequence $\{u_h\}$ is d-divisible if and only if there exist $e, f \in \mathbf{Z}$ such that at least one of the following cases holds:*

(i) $u_0 = 0$, *d arbitrary;*
(ii) $A = 0$, *d odd;*
(iii) $A = 0$, $u_1 | u_0 B^{d/2}$, *d even;*
(iv) $u_0 A = 2u_1$, $A^2 + 4B = 0$, *d arbitrary;*
(v) $u_0 A = 2u_1$, *d odd;*
(vi) $A = 2e$, $B = -2e^2$, $e \neq 0$, $d \equiv 1 \pmod 4$;
(vii) $A = 2e$, $B = -2e^2$, $e \neq 0$, $u_r | B^{2t} u_{r_0}$ *for all integers* r, t, r_0 *with* $1 \leq r \leq 3$, $dr = 4t + r_0$, $t \geq 0$, $0 \leq r_0 \leq 3$ *and* $d \not\equiv 1 \pmod 4$;
(viii) $A = 2e$, $B = -(e^2 + f^2)$, $f \neq 0$, $e \neq \pm f$, $u_1 = u_0(e \pm f)$ *and* $d \equiv 1 \pmod 4$;
(ix) $A = -f$, $B = -f^2$, $f \neq 0$, $d \equiv 1 \pmod 3$;
(x) $A = -f$, $B = -f^2$, $f \neq 0$, $u_r | f^{3t} u_{r_0}$ *for all integers* r, t, r_0 *with* $1 \leq r \leq 2$, $dr = 3t + r_0$, $t \geq 0$, $0 \leq r_0 \leq 2$ *and* $d \not\equiv 1 \pmod 3$;
(xi) $A = 2e - f$, $B = -(e^2 - ef + f^2)$, $e \neq 0$, $f \neq 0$, $e \neq \pm f$, $u_0 \neq 0$, $u_1/u_0 \in \{e - f, e\}$, $d \equiv 1 \pmod 3$;
(xii) $A = 3f$, $B = -3f^2$, $f \neq 0$, $d \equiv 1 \pmod 6$;
(xiii) $A = 3f$, $B = -3f^2$, $f \neq 0$, $u_r | B^{3t} u_{r_0}$ *for all integers* r, t, r_0 *with*

(xiv) $1 \leq r \leq 5$, $dr = 6t + r_0$, $t \geq 0$, $0 \leq r_0 \leq 5$ and $d \not\equiv 1 \pmod 6$;
$A = 2e - f$, $B = -(e^2 - ef + f^2)$, $e \neq \pm f, 2f$, $u_0 \neq 0$,
$u_1/u_0 \in \{e + f, e - 2f\}$ and $d \equiv 1 \pmod 6$.

It follows from Theorem 2 (see also its proof) that in cases (ii) – (iv), (vi), (vii), (ix), (x), (xii) and (xiii), the sequence $\{u_h\}$ is degenerate. In case (i), it is easy to give an example both for the degenerate and for the non-degenerate case. In the other cases $\{u_h\}$ is non-degenerate. More precisely, it is easy to deduce from Theorems 1, 2 and Corollary 2 that there exist only the following five types of second order non-degenerate d-divisibility sequences $\{u_h\}$ over $\mathbf{Z}$;

$\{u_h\}$ is 2-divisible; then it is divisible (case (i));

$\{u_h\}$ is 2-non-divisible but 3-divisible; then it is $(2k+1)$-divisible for every $k \in \mathbf{N}$ (case (v));

$\{u_h\}$ is 2-non-divisible but 4-divisible; then it is $(3k+1)$-divisible for every $k \in \mathbf{N}$ (case (xi));

$\{u_h\}$ is 3-non-divisible but 5 divisible; then it is $(4k+1)$-divisible for every $k \in \mathbf{N}$ (case (viii));

$\{u_h\}$ is 3 and 4 -non-divisible but 7 -divisible; then it is $(6k+1)$-divisible for every $k \in \mathbf{N}$ (case (xiv)).

3. Proofs

PROOF *of Theorem 1.* First we prove assertion (A). Let $\{u_h\}$ be a second order non-degenerate recurrence sequence over $\mathbf{Z}_K$, satisfying the assumptions made in (A) of Theorem 1. Using the notations of Section 2 we have $\alpha \neq \beta$, hence u_n satisfies (2).

An easy computation shows that

$$(4)\quad (a\alpha^n - b\beta^n)\frac{\alpha^{dn} - \beta^{dn}}{\alpha^n - \beta^n} = a\alpha^{dn} - b\beta^{dn} + (a-b)(\alpha\beta)^n \frac{\alpha^{(d-1)n} - \beta^{(d-1)n}}{\alpha^n - \beta^n}$$

for all integers $n, d \geq 1$. Put $L = K(\alpha)$, and denote by $\mathbf{Z}_L$ the ring of integers of L. It is clear that

$$\frac{\alpha^{jn} - \beta^{jn}}{\alpha^n - \beta^n} \in \mathbf{Z}_K \quad \text{for every integer} \quad j \geq 1.$$

Hence, if $u_n | u_{dn}$ then, by (4),

$$(a-b)(\alpha\beta)^n \frac{\alpha^{(d-1)n} - \beta^{(d-1)n}}{\alpha^n - \beta^n} \in \mathbf{Z}_K$$

and

$$u_n \mid (a-b)(\alpha\beta)^n(\alpha^{(d-1)n} - \beta^{(d-1)n}) \quad \text{in } \mathbf{Z}_L. \tag{5}$$

Further, we note that $a - b = u_0$, whence $a - b \in \mathbf{Z}_K$.

For a non-zero ideal $\mathcal{A}$ of $\mathbf{Z}_L$, let $P(\mathcal{A})$ denote the maximum of the rational primes lying below the prime ideal divisors of $\mathcal{A}$. Further, we put $P(0) = P(1) = 1$. By a result of MAHLER [3], $P(u_n)$ is not bounded as $n \to \infty$. Hence there exists a number n_1 such that $P(u_n) > \max\{P(a-b), P(\alpha\beta)\}$ for infinitely many $n \geq n_1$. Let $n \geq \max\{n_0, n_1\}$ with this property, and let $\wp_n$ be a prime ideal divisor of u_n with $P(\wp_n) = P(u_n)$. Then it follows from (5) that

$$\wp_n \mid \alpha^{(d-1)n} - \beta^{(d-1)n} \quad \text{in } \mathbf{Z}_L. \tag{6}$$

Put

$$A_i = (\alpha-\beta)^{i-1}(b^{i-1}\alpha^{(d-i)n} - a^{i-1}\beta^{(d-i)n}) \quad \text{for } i = 1, \ldots, d.$$

Then $A_i \in \mathbf{Z}_L$ for each i. Further, $b(\alpha - \beta) \in \mathbf{Z}_L$. It is easy to see that

$$(\alpha-\beta)^{i-1}\beta^{(d-i)n}a^{i-2}(a\alpha^n - b\beta^n) - b(\alpha-\beta)A_{i-1} = -\alpha^n A_i \tag{7}$$

for each integer i with $2 \leq i \leq d$. We prove now that

$$\wp_n \mid A_i \quad \text{in} \quad \mathbf{Z}_L \tag{8}$$

for each integer i with $1 \leq i \leq d$. By (6), (8) is true for $i = 1$. Assume that it is true for $i - 1 \geq 0$. Then by the definition of $\wp_n$ and by the induction hypothesis, $\wp_n$ divides the element on the left-hand side of (7). Since $\wp_n \nmid \alpha^n$ in $\mathbf{Z}_L$, it must divide A_i in $\mathbf{Z}_L$ on the right-hand side of (7), and (8) is proved.

Setting $i = d$ in (8) we get

$$\wp_n \mid (\alpha-\beta)^{d-1}(b^{d-1} - a^{d-1}) \quad \text{in} \quad \mathbf{Z}_L.$$

But $P(\wp_n)$ can be arbitrarily large, hence $b^{d-1} = a^{d-1}$ which proves the assertion in (A).

Conversely, suppose now that $b^{d-1} = a^{d-1}$ for some integer $d > 1$. Then $b = \zeta_{d-1}a$ with some $(d-1)$-th root of unity ζ_{d-1} from L. Then, by (2), we have

$$(9)\quad \begin{cases} u_{dn} = a(\alpha^{dn} - \zeta_{d-1}\beta^{dn}) = a((\alpha^n)^d - (\zeta_{d-1}\beta^n)^d) = u_n v_n & \text{for all } n \geq 0, \\ \text{where} \\ v_n = \begin{cases} (\alpha^n)^{d-1} + (\alpha^n)^{d-2}(\zeta_{d-1}\beta^n) + \ldots + (\zeta_{d-1}\beta^n)^{d-1} & \text{for } n \geq 1, \\ 1 \quad \text{for } n = 0. \end{cases} \end{cases}$$

In (9), v_n is an algebraic integer. On the other hand, if $u_n \neq 0$ then, by (9), v_n belongs to K, hence $v_n \in \mathbf{Z}_K$. Thus $\{u_h\}$ is d-divisible which proves the assertion of (B).

PROOF *of Theorem 2.* Let now $\{u_h\}_{h=0}^{\infty}$ be a second order recurrence sequence over $\mathbf{Z}$ with initial terms $u_0, u_1 \in \mathbf{Z}$, $u_0^2 + u_1^2 \neq 0$, which satisfies (1) with some $A, B \in \mathbf{Z}$, $B \neq 0$, and suppose that it is d-divisible for some integer $d > 1$. Define α, β, a, b as in Section 2.

Assume first that $\{u_h\}$ is degenerate. Then $\alpha = \zeta\beta$, where ζ is a root of unity belonging to $K = \mathbf{Q}(\alpha)$. The degree of K over $\mathbf{Q}$ is at most two, hence $\zeta \in \mathcal{E} = \{\pm 1, \pm i, \pm\rho, \pm\rho^2\}$ where $\rho = \frac{-1+\sqrt{-3}}{2}$ (see e.g. [2]). We shall distinguish several cases.

Case 1. If $\alpha = \beta$, then $\alpha \in \mathbf{Z}$ and u_n may be written in the form (3). Further, $A^2 + 4B = 0$ and $\alpha = A/2$. If $a = 0$ then $u_0 A = 2u_1$ which corresponds to case (iv). This sequence is indeed d-divisible for every $d > 1$. If $b = 0$ then $u_0 = 0$ which corresponds to case (i). Then the sequence is d-divisible. Finally, suppose that a and b are different from zero. Then $\alpha(and + b)$ and $\alpha(an + b)$ are rational integers and $u_n \mid u_{dn}$ implies that $\alpha(an+b) \mid \alpha(and+b)$. From this it follows that $\alpha(an+b) \mid \alpha b(d-1)$ which is impossible if n is large enough.

Case 2. Let now $\alpha = -\beta$. Then $A = 0$ and $B = \beta^2$. We have $u_{2n} = u_0\beta^{2n}$ and $u_{2n+1} = u_1\beta^{2n}$ for $n = 0, 1, \ldots$. If d is odd then $dn \equiv n \pmod 2$, hence, indeed, $u_n | u_{dn}$ and (ii) follows. While if d is even and n is odd then $u_1\beta^{n-1} \mid u_0\beta^{dn}$ must hold. Then $\{u_h\}$ is indeed d-divisible and $\{u_h\}$ is described in case (iii).

Case 3. Let $\alpha = \pm i\beta$. Then $\alpha + \beta = \beta(1 \pm i) = A$ implies that $\beta = \frac{A}{2}(1 \mp i)$. Since β is an algebraic integer, we have $A = 2e$ and $B = -\alpha\beta = -2e^2$. Let n be an arbitrary non-negative integer, and put $n = 4v + r$ with non-negative integers v, r such that $0 \leq r < 4$. Then, by (2),

$$
\begin{aligned}
u_n &= a\alpha^n - b\beta^n = \beta^n(a(\pm i)^n - b) = \\
&= e^n(1 \mp i)^n(a(\pm i)^n - b) = (-1)^v B^{2v} u_r.
\end{aligned} \tag{10}
$$

If $d \equiv 1 \pmod 4$ then $dn \equiv n \equiv r \pmod 4$ and, by (10), $u_n | u_{dn}$, i.e. $\{u_h\}$ is indeed d-divisible, and this is case (vi). If $d \not\equiv 1 \pmod 4$ then we put $dr = 4t + r_0$ with non-negative integers t, r_0 such that $0 \leq r_0 < 4$. Then $dn = 4vd + 4t + r_0$. By (10), in this case $\{u_h\}$ is d-divisible if and only if

$$
u_r | B^{2(v(d-1)+t)} u_{r_0} \qquad \begin{array}{l}\text{for all } v \geq 0 \text{ and each } r, t, r_0 \text{ such that} \\ 0 \leq r, r_0 < 4 \text{ and } dr = 4t + r_0.\end{array} \tag{11}
$$

But for fixed r, r_0 and t, (11) holds for all $v \geq 0$ if and only if it holds for $v = 0$. Finally, (11) trivially holds for $r = 0$, hence we get case (vii).

Case 4. Next let $\alpha/\beta = \rho^j$ where $j = 1$ or 2. Then α and β belong to the ring of integers of the Eulerian number field $\mathbf{Q}(\rho)$. In this field, $\{1, \rho\}$ is an integral basis, hence we can write $\alpha = e+(-1)^j f\rho$ and $\beta = e+(-1)^j f\rho^2$ with some $e, f \in \mathbf{Z}$. Then, by $\alpha = \beta\rho^j$, we get $A = -f$, $B = -f^2$ and $\beta = \rho^j f$, $j = 1, 2$. Put $n = 3v + r$ with integers v, r such that $v \geq 0$, $0 \leq r < 3$. Then, by (2) and $(\rho^j)^3 = 1$, we have

$$u_n = \beta^n(a\rho^{jn} - b) = f^{3v} \cdot f^r \cdot \rho^{jr}(a\rho^{jr} - b) = f^{3v}u_r.$$

From now on we can proceed in a similar way as in the cases corresponding to $\alpha/\beta = \pm i$, and we get (ix) and (x) in the theorem.

Case 5. Let now $\alpha/\beta = -\rho^j$ where $j = 1$ or 2. Then we get in the same way as in the preceding case that $A = 3f$, $B = -3f^2$ and $\beta = \sqrt{-3} \cdot f(-\rho)^j$ with some rational integer $f \neq 0$. Let $n = 6v + r$ with rational integers v, r such that $v \geq 0$, $0 \leq r < 6$. Then $(-\rho^j)^6 = 1$ and (2) imply again that

$$u_n = \beta^n(a(-\rho^j)^n - b) = (\sqrt{-3}f)^{6v}u_r = B^{3v}u_r.$$

Now we can proceed as in the case above $\alpha/\beta = \pm i$, and we get cases (xii), (xiii) in our theorem.

In the sequel we suppose that $\{u_h\}_{h=0}^{\infty}$ is non-degenerate. Since $\{u_h\}$ is a second order recurrence sequence, we have $ab \neq 0$. By Theorem 1, $\{u_h\}$ is d-divisible if and only if $b = \zeta a$ with some $(d-1)$-th root of unity ζ. Then $\zeta \in \mathbf{Q}(\alpha)$, hence $\zeta \in \mathcal{E}$. Further, $b = \zeta a$ is equivalent to

$$(u_1 - u_0\beta)\zeta = u_1 - u_0\alpha. \tag{12}$$

It suffices to determine all second order non-degenerate recurrence sequences $\{u_h\}$ in $\mathbf{Z}$ having the property (12). We shall distinguish again several cases.

Case 6. Let first $\zeta = 1$. Then (12) implies $u_0 = 0$ and $\{u_h\}$ is d-divisible for every $d > 1$. This corresponds to (i) in the theorem.

Case 7. $\zeta = -1$. This appears only if d is odd. Then we get from (12) that

$$2u_1 = u_0(\alpha + \beta) = u_0 A$$

because α and β are the zeros of $x^2 - Ax - B$. This is the case (v) in the theorem.

Case 8. $\zeta = \pm i$. This is possible only if $d \equiv 1 \pmod 4$.

Then $K = \mathbf{Q}(\alpha) = \mathbf{Q}(i)$ is the Gaussian number field. Hence $\alpha = e + if$ and $\beta = e - if$ with suitable $e, f \in \mathbf{Z}\backslash\{0\}$ for which $e \neq \pm f$.

For $\zeta = i$ and $\zeta = -i$, (12) implies $u_1 = u_0(e - f)$ and $u_1 = u_0(e + f)$, respectively. Furthermore, we have in both cases $A = 2e$ and $B = -(e^2 + f^2)$ which corresponds to case (viii) of the theorem.

Case 9. $\zeta = \rho^j$ where $j = 1$ or 2. Then $d \equiv 1 \pmod 3$ and $K = \mathbf{Q}(\alpha) = \mathbf{Q}(\rho)$ is the Eulerian number field. Thus $\alpha = e + f\rho$ and $\beta = e + f\rho^2$ with suitable integers $e, f \in \mathbf{Z}\backslash\{0\}$ for which $e \neq \pm f, 2f$. Hence $A = 2e - f$ and $B = -(e^2 - ef + f^2)$. Using again (12), we get

$$u_1 = \begin{cases} u_0(e - f) & \text{if } \zeta = \rho \\ u_0 e & \text{if } \zeta = \rho^2, \end{cases}$$

which corresponds to (xi) in the theorem.

Case 10. $\zeta = -\rho^j$ where $j = 1$ or 2. Then $d \equiv 1 \pmod 6$ and $K = \mathbf{Q}(\alpha) = \mathbf{Q}(\rho)$. Taking again $\alpha = e + f\rho$ and $\beta = e + f\rho^2$ with some $e, f \in \mathbf{Z}\backslash\{0\}$, we get for e, f the restrictions $e \neq \pm f, 2f$. A and B have the same form as in case 9. Finally, using (12) we get

$$u_1 = \begin{cases} u_0(e + f) & \text{if } \zeta = -\rho \\ u_0(e - 2f) & \text{if } \zeta = -\rho^2, \end{cases}$$

which corresponds to case (xiv) in the theorem. This completes the proof of Theorem 2.

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