Publ. Math. Debrecen
39 / 3–4 (1991), 185–197

On the geometry of generalized metric spaces II. Spaces of isotropic curvature

By HIDEO IZUMI (Yokosuka) and MAMORU YOSHIDA (Fujisawa)

§0. Introduction

Let M be an n-dimensional differentiable manifold and $T(M)$ its tangent bundle. We consider the bundle M_T which does not contain zero vector of $T(M)$, that is, $M_T := T(M) - \{0\}$. A generalized metric space M_n (or a generalized Finsler space) is a pair $(M_T, g_{ij}(x,y))$, where $g_{ij}(x,y)$ is a symmetric tensor satisfying the following conditions:

(a) $g_{ij}(x,y)$ is positively homogeneous of degree zero in y^i, (b) g_{ij} is positive definite, (c) $g^*{}_{ij} := \frac{1}{2}\partial^2 F^2/\partial y^i \partial y^j$ is non-degenerate, where $F^2 := g_{ij}y^i y^j$.

In the previous paper [1] we introduced three types of connections in M_n (see §1): $C\Gamma(N) = (F_j{}^i{}_k, C_j{}^i{}_k)$, $R\Gamma(N) = (F_j{}^i{}_k, 0)$ and $B\Gamma(G) = (G_j{}^i{}_k, 0)$ and the curvature tensors are $(R_h{}^i{}_{jk}, P_h{}^i{}_{jk}, S_h{}^i{}_{jk})$ for $C\Gamma(N)$, $(K_h{}^i{}_{jk}, F_h{}^i{}_{jk}, 0)$ for $R\Gamma(N)$ and $(H_h{}^i{}_{jk}, G_h{}^i{}_{jk}, 0)$ for $B\Gamma(G)$ respectively.

Let κ_{hijk} be one of the above seven curvature tensors of M_n. For the plane $\pi(X,Y)$ spanned by two independent vectors X^i and Y^i in M_n, the scalar

$$(0.1)\quad \kappa(x,y,\pi(X,Y)) := \kappa_{hijk}X^h Y^i X^j Y^k / (g_{hj}g_{ik} - g_{hk}g_{ij})X^h Y^i X^j Y^k$$

is called the *sectional curvature* for $\pi(X,Y)$ with respect to κ_{hijk}. If the sectional curvature κ is independent of any X^i and Y^i, then the space $M_n(n>2)$ is called to be *isotropic with respect to* κ_{hijk}, or *of κ-isotropic curvature.* We assume that the scalar κ does not vanish.

The purpose of the present paper is to investigate the properties on generalized metric spaces of R-, K- and H-isotropic curvature.

We shall show that a generalized metric space of K-isotropic curvature is a Riemannian space of constant curvature (§3) and a generalized metric space of H-isotropic curvature is a Finsler space of constant curvature (§4).

It seems that R-isotropic curvature on a generalized metric space has not yet been sufficiently investigated.

The terminology and notations are the same as in the paper [1] unless otherwise stated.

The authors wish to express their sincere gratitude to Professor T. SAKAGUCHI for his valuable advices.

§1. Preliminaries

In this section we shall quote some quantities and relations from the paper [1] for later use.

Let M_n be a generalized metric space with tensor $g_{ij}(x,y)$ mentioned in §0. As usual we raise or lower indices by means of g_{ij}.

Three types of connection forms defined on M_n are given as follows:

$$\omega_j^i = F_j{}^i{}_k dx^k + C_j{}^i{}_k \delta y^k, \quad \delta y^k := dy^k + N^k{}_m dx^m \quad \text{for } C\Gamma(N),$$
$$\omega_j^i = F_j{}^i{}_k dx^k \quad \text{for } R\Gamma(N) \quad \text{and} \quad \omega_j^i = G_j{}^i{}_k dx^k \quad \text{for } B\Gamma(G),$$

where

$$F_j{}^i{}_k := \frac{1}{2} g^{ih}(d_j g_{hk} + d_k g_{hj} - d_h g_{jk}),\ d_k := \partial_k - N^h{}_k \dot{\partial}_h, \quad \partial_k := \partial/\partial x^k,$$
$$\dot{\partial}_h := \partial/\partial y^h, \quad C_j{}^i{}_k := \frac{1}{2} g^{ih}(g_{hk(j)} + g_{hj(k)} - g_{jk(h)}), \quad g_{jk(h)} := \dot{\partial}_h g_{jk},$$
$$G_j{}^i{}_k := \dot{\partial}_k G^i{}_j, \quad G^i{}_j := \dot{\partial}_j G^i, \quad G^i := \frac{1}{4} g^{*ih}(y^j \partial_j \dot{\partial}_h F^2 - \partial_h F^2),$$
$$g^{*ih} g^*{}_{hj} = \delta_j^i,$$

and we have $N^i{}_k = F_j{}^i{}_k y^j$.

The covariant derivatives of a vector $v^i(x,y)$ are defined as follows:

$$\text{(1.1)} \quad \begin{aligned} &(a) \quad v^i{}_{/k} := d_k v^i + F_j{}^i{}_k v^j \quad \text{for } C\Gamma(N) \text{ and } R\Gamma(N), \\ &(b) \quad v^i{}_{/(k)} := v^i{}_{(k)} + C_h{}^i{}_k v^h \quad \text{for } C\Gamma(N), \\ &(c) \quad v^i{}_{//k} := \bar{d}_k v^i + G_h{}^i{}_k v^h,\ \bar{d}_k := \partial_k - G^h{}_k \dot{\partial}_h \quad \text{for } B\Gamma(G). \end{aligned}$$

Here the following relations are satisfied:

$$\text{(1.2)} \quad \begin{aligned} &(a)\ C_0{}^i{}_k = C_j{}^i{}_0 = 0, \quad (b)\ C_{jk} := C_j{}^0{}_k = g_{ij(k)} y^i, \\ &(c)\ C_{0k} = C_{j0} = 0, \qquad (d)\ g_{ij(k)} = C_{ijk} + C_{jik}, \end{aligned}$$

where the index 0 denotes transvection by y;

$$(1.3)\qquad v^i{}_{//k} = v^i{}_{/k} + D_h{}^i{}_k v^h - P^h{}_k v^i{}_{(h)},$$

where $D_h{}^i{}_k := G_h{}^i{}_k - F_h{}^i{}_k$ and $P^h{}_k := G^h{}_k - N^h{}_k = D_0{}^h{}_k = D_k{}^h{}_0$;

$$(1.4)\qquad (a)\ P_j{}^0{}_k = P_j{}^i{}_0 = 0,\quad (b)\ P_0{}^i{}_k = 2P^i{}_k,\quad (c)\ P^i{}_0 = P^0{}_k = 0,$$

where $P_j{}^i{}_k := N^i{}_{j(k)} - F_j{}^i{}_k$ and

$$(1.5)\qquad (a)\ D_j{}^0{}_k = -(g_{ik} + C_{ik})P^i{}_j,\quad (b)\ C_{jk/0} = -2D_j{}^0{}_k.$$

The connection $C\Gamma(N)$ is a metrical connection, that is, $g_{ij/k} = 0$, $g_{ij/(k)} = 0$.

It is known if $C_{ij} = 0$ or $C_{ijk} = 0$, then the generalized metric space reduces to a Finsler space or a Riemannian space respectively.

The so-called Ricci formulae for a vector $v^i(x,y)$ are given as follows:

$$(1.6)\qquad \begin{aligned} &(a)\quad v^i{}_{/j/k} - j|k = R_h{}^i{}_{jk} v^h - R^h{}_{jk} v^i{}_{/(h)} = K_h{}^i{}_{jk} v^h - R^h{}_{jk} v^i{}_{(h)},\\ &(b)\quad v^i{}_{//j//k} - j|k = H_h{}^i{}_{jk} v^h - H^h{}_{jk} v^i{}_{(h)}, \end{aligned}$$

where $-j|k$ means the interchange of indices j, k in the foregoing term and subtraction, for instance $A_{jm}B_i{}^m{}_k - j|k = A_{jm}B_i{}^m{}_k - A_{km}B_i{}^m{}_j$. Here the curvature tensors $R_h{}^i{}_{jk}$, $K_h{}^i{}_{jk}$, $H_h{}^i{}_{jk}$ are defined as follows:

$$(1.7)\qquad \begin{aligned} &(a)\quad R_h{}^i{}_{jk} := K_h{}^i{}_{jk} + C_h{}^i{}_m R^m{}_{jk},\\ &(b)\quad K_h{}^i{}_{jk} := d_k F_h{}^i{}_j + F_h{}^m{}_j F_m{}^i{}_k - j|k,\\ &(c)\quad H_h{}^i{}_{jk} := \bar{d}_k G_h{}^i{}_j + G_h{}^m{}_j G_m{}^i{}_k - j|k, \end{aligned}$$

and the torsion tensors $R^i{}_{jk}$ and $H^i{}_{jk}$ are defined by

$$(1.8)\qquad (a)\ R^i{}_{jk} := d_k N^i{}_j - j|k,\quad (b)\ H^i{}_{jk} := \bar{d}_k G^i{}_j - j|k.$$

In this case, the following relations are valid:

$$\begin{aligned} &(a)\ R^i{}_{jk} = R_0{}^i{}_{jk} = K_0{}^i{}_{jk},\ H^i{}_{jk} = H_0{}^i{}_{jk},\\ &(b)\ H_h{}^i{}_{jk} = K_h{}^i{}_{jk} + E_h{}^i{}_{jk},\ E_h{}^i{}_{jk} := D_h{}^i{}_{j/k} + D_h{}^m{}_j D_m{}^i{}_k -\\ &\qquad - P^m{}_k G_h{}^i{}_{jm} - j|k,\ G_h{}^i{}_{jk} := \dot{\partial}_k G_h{}^i{}_j, \end{aligned}$$

$$(1.9)\quad \begin{aligned} &(c)\ H^i{}_{jk} = R^i{}_{jk} + E^i{}_{jk},\\ &\quad E^i{}_{jk} := E_0{}^i{}_{jk} = P^i{}_{j/k} + P^m{}_j D_m{}^i{}_k - j|k,\\ &(d)\ H^i{}_k = R^i{}_k + E^i{}_k,\\ &\quad H^i{}_k := H^i{}_{0k}, R^i{}_k := R^i{}_{0k}, E^i{}_k := E^i{}_{0k},\\ &(e)\ E_0{}^0{}_{jk} = 0,\ E^i{}_k = -P^i{}_{k/0} - P^i{}_m P^m{}_k,\\ &(f)\ H^i{}_{jk(h)} = H_h{}^i{}_{jk},\ H^i{}_{k(j)} - H^i{}_{j(k)} = 3H^i{}_{jk}. \end{aligned}$$

From the Bianchi identities, we shall list the following:

$$(1.10)\quad \begin{aligned} &(a)\ R_h{}^i{}_{jk} + h|j|k = C_h{}^i{}_m R^m{}_{jk} + h|j|k, \quad (b)\ K_h{}^i{}_{jk} + h|j|k = 0,\\ &(c)\ H_h{}^i{}_{jk} + h|j|k = 0, \text{ and consequently } (d)\ E_h{}^i{}_{jk} + h|j|k = 0, \end{aligned}$$

where $+h|j|k$ means the cyclic permutations of indices h, j, k in the foregoing term and summation, for instance $A_{hm}B_j{}^m{}_k + h|j|k = A_{hm}B_j{}^m{}_k + A_{jm}B_k{}^m{}_h + A_{km}B_h{}^m{}_j$.

Moreover we have the useful expressions

$$(1.11)\quad R_{hijk} - R_{jkhi} = B_{hijk},$$

$$(1.12)\quad K_{hijk} - K_{jkhi} = \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m{}_{jk} + g_{jk(m)}R^m{}_{hi}),$$

$$(1.13)\quad H_{hijk} - H_{jkhi} = \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m{}_{jk} + g_{jk(m)}R^m{}_{hi}) + {} + E_{hijk} - E_{jkhi},$$

where

$$\begin{aligned} 2B_{hijk} &:= T_{hijk} + (C_{him} - C_{ihm})R^m{}_{jk} - (C_{jkm} - C_{kjm})R^m{}_{hi},\\ T_{hijk} &:= g_{hj(m)}R^m{}_{ik} + g_{ik(m)}R^m{}_{hj} - j|k, \end{aligned}$$

and they satisfy the following relations:

$$(1.14)\quad 2B_{hijk} + h|j|k = (3C_{him} + C_{ihm})R^m{}_{jk} - {} - (C_{jkm} - C_{kjm})R^m{}_{hi} + h|j|k,$$

$$(1.15)\quad T_{hijk} + h|j|k = 2g_{hi(m)}R^m{}_{jk} + h|j|k.$$

Lastly, we shall prove the following result:

Lemma 1.1. *If a tensor* A_{hijk} *of degree* 4 *satisfies*

$$A_{hijk} + A_{jihk} + A_{hkji} + A_{jkhi} = 0, \tag{1.16}$$

$$A_{hijk} = -A_{hikj}, \tag{1.17}$$

$$A_{hijk} - A_{jkhi} = U_{hijk}, \tag{1.18}$$

$$A_{hijk} + h|j|k = V_{hijk}, \tag{1.19}$$

where U_{hijk} *and* V_{hijk} *are certain tensors, then* A_{hijk} *is expressible as*

$$6A_{hijk} = U_{hijk} + U_{jihk} - U_{kihj} - U_{hikj} + 2V_{hijk}. \tag{1.20}$$

PROOF. Interchanging indices h and j in (1.18), we get

$$A_{jihk} - A_{hkji} = U_{jihk}. \tag{1.18'}$$

If we take the sum of the three equations (1.16), (1.18) and (1.18)' and use (1.17), we obtain

$$2(A_{hijk} - A_{jikh}) = U_{hijk} + U_{jihk}. \tag{1.21}$$

The cyclic change $h \to k \to j \to h$ of indices in (1.21) gives

$$2(A_{kihj} - A_{hijk}) = U_{kihj} + U_{hikj}. \tag{1.21'}$$

Subtracting (1.21)' from (1.21), we have

$$2(2A_{hijk} - A_{jikh} - A_{kihj}) = U_{hijk} + U_{jihk} - U_{kihj} - U_{hikj}. \tag{1.22}$$

Making use of (1.19) on the left hand side of (1.22), we can see (1.20). Q.E.D.

§2. A generalized metric space of R-isotropic curvature

First we consider a generalised metric space of R-isotropic curvature. In this case, from the relation corresponding to (0.1) we have

$$[R_{hijk} - R(x,y)(g_{hj}g_{ik} - g_{hk}g_{ij})]X^hY^iX^jY^k = 0. \tag{2.1}$$

When we put

$$r_{hijk} := R_{hijk} - R(x,y)(g_{hj}g_{ik} - g_{hk}g_{ij}),$$

provided equation (2.1) holds for any X^i and Y^i, then the following equation must hold:

$$r_{hijk} + r_{jihk} + r_{hkji} + r_{jkhi} = 0. \tag{2.2}$$

On the other hand, the tensor r_{hijk} obeys the following relations using the properties of R_{hijk}:

$$r_{hijk} = -r_{hikj}, \tag{2.3}$$

$$r_{hijk} - r_{jkhi} = R_{hijk} - R_{jkhi} = B_{hijk}, \tag{2.4}$$

$$r_{hijk} + h|j|k = R_{hijk} + h|j|k = C_{him}R^m{}_{jk} + h|j|k. \tag{2.5}$$

Consequently, r_{hijk} satisfies the conditions for A_{hijk} in Lemma 1.1, where $U_{hijk} = B_{hijk}$ and $V_{hijk} = C_{him}R^m{}_{jk} + h|j|k$. Therefore r_{hijk} has the form

$$\begin{aligned} 6r_{hijk} &= B_{hijk} + B_{jihk} - B_{kihj} - B_{hikj} + 2(C_{him}R^m{}_{jk} + h|j|k) \\ &= 3B_{hijk} - (B_{hijk} - 2C_{him}R^m{}_{jk} + h|j|k). \end{aligned}$$

Making use of the definition of B_{hijk} and (1.14) in the above equation, we obtain

$$\begin{aligned} 6r_{hijk} =&[(C_{ikm} + 2C_{kim})R^m{}_{hj} + (C_{hjm} + 2C_{jhm})R^m{}_{ik} - j|k]+ \\ &+ 2(C_{him} - C_{ihm})R^m{}_{jk} - (C_{jkm} - C_{kjm})R^m{}_{hi}. \end{aligned} \tag{2.6}$$

Consequently, we have the following

Theorem 2.1. *A generalized metric space of R-isotropic curvature is characterized by* (2.6).

Making use of (2.6), we shall prove the following two propositions.

Proposition 2.2. *In a generalized metric space of R-isotropic curvature, if the relation*

$$R^i{}_{jk} = R(y_j h^i_k - y_k h^i_j) \quad (h^i_k := \delta^i_k - y^i y_k/F^2) \tag{2.7}$$

is satisfied, then the following equation holds:

$$R_{hijk} = R(g_{hj}g_{ik} - g_{hk}g_{ij}). \tag{2.8}$$

PROOF. If we substitute (2.7) into (2.6), then direct calculations show $r_{hijk} = 0$. Hence (2.8) is obtained. Q.E.D.

Proposition 2.3. *In a generalized metric space of R-isotropic curvature, if the symmetric tensor C_{ij} is proportional to h_{ij}, namely*

$$C_{ij} = \lambda h_{ij} \quad (\lambda \neq -1), \tag{2.9}$$

then the following equation holds:

$$R^i{}_{jk} = R(y_j\delta^i_k - y_k\delta^i_j). \tag{2.10}$$

PROOF. Transvecting (1.7) (a) by y_i and using (1.2) (b), we get $K_h{}^0{}_{jk} = -(g_{hm} + C_{hm})R^m{}_{jk}$, transvection of which by y^k gives

$$K_{hj} = (g_{hm} + C_{hm})R^m{}_j, \tag{2.11}$$

where $K_{hj} := K_h{}^0{}_{j0}$. Similarly, if we transvect (1.10) (b) by y_iy^k, then we obtain

$$K_{hj} = K_{jh}. \tag{2.12}$$

On the other hand, transvecting (2.6) by y^h and using (1.2) (a), (b), we get

$$\begin{aligned} 6r_{0ijk} =&(C_{ikm} + 2C_{kim})R^m{}_j - (C_{ijm} + 2C_{jim})R^m{}_k- \\ &- (C_{jkm} - C_{kjm})R^m{}_i - 2(C_{im}R^m{}_{jk}+ \\ &+ C_{jm}R^m{}_{ki} + C_{km}R^m{}_{ij}). \end{aligned} \tag{2.13}$$

Further transvection of (2.13) by y^j yields, with (1.2) (c) in mind,

$$2(R_{ik} - F^2Rh_{ik}) = C^m_k R_{mi} - C^m_i R_{mk}, \tag{2.14}$$

where $R_{ik} := R_{0i0k}$.

Moreover, if we use the hypothesis (2.9), then from (2.11) we have

$$R_{hj} = K_{hj}/(1+\lambda). \tag{2.15}$$

Consequently, substituting (2.15) and (2.9) into (2.14) and noting (2.12), we get

$$R_{hj} = F^2Rh_{hj}. \tag{2.16}$$

Using (2.16), we can easily see that

$$(C_{ikm}+2C_{kim})R^m{}_j-(C_{ijm}+2C_{jim})R^m{}_k-(C_{jkm}-C_{kjm})R^m{}_i = 0, \tag{2.17}$$

because of $C_{ijk} = C_{kji}$.

On the other hand, we see

$$K_{h0jk} = -(g_{hm} + C_{hm})R^m{}_{jk} = -(1+\lambda)R_{0hjk},$$

which yields

$$C_{im}R^m{}_{jk} + i|j|k = -\lambda K_{i0jk}/(1+\lambda) + i|j|k = 0. \tag{2.18}$$

Therefore, if we apply (2.17) and (2.18) to the right hand side of (2.13), then we can conclude $r_{0ijk} = 0$, namely $R_{0ijk} = R(y_jg_{ik} - y_kg_{ij})$, which is equivalent to (2.10). Q.E.D.

Propositions 2.2 and 2.3 yield the following

Theorem 2.4. *If $C_{ij} = \lambda h_{ij}$ $(\lambda \neq -1)$, the generalized metric space of R-isotropic curvature is characterized by*

$$R_h{}^i{}_{jk} = R(g_{hj}\delta^i_k - g_{hk}\delta^i_j).$$

§3. A generalized metric space of K-isotropic curvature

Secondly, we consider a generalized metric space of K-isotropic curvature. In this case, if we put

$$k_{hijk} := K_{hijk} - K(g_{hj}g_{ik} - g_{hk}g_{ij}),$$

then we must have

$$k_{hijk} + k_{jihk} + k_{hkji} + k_{jkhi} = 0. \tag{3.1}$$

It is easily shown that the tensor k_{hijk} satisfies the following relations:

$$k_{hijk} = -k_{hikj}, \tag{3.2}$$

$$\begin{aligned} k_{hijk} - k_{jkhi} &= K_{hijk} - K_{jkhi} \\ &= \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m{}_{jk} + g_{jk(m)}R^m{}_{hi}), \end{aligned} \tag{3.3}$$

$$k_{hijk} + h|j|k = K_{hijk} + h|j|k = 0. \tag{3.4}$$

Consequently, k_{hijk} satisfies the conditions for A_{hijk} in Lemma 1.1, where $U_{hijk} = \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m{}_{jk} + g_{jk(m)}R^m{}_{hi})$ and $V_{hijk} = 0$. Therefore k_{hijk} has the form

$$\begin{aligned} 6k_{hijk} =& \frac{1}{2}(T_{hijk} + T_{jihk} - T_{kihj} - T_{hikj} - g_{hi(m)}R^m{}_{jk} - g_{ji(m)}R^m{}_{hk} + \\ &+ g_{ki(m)}R^m{}_{hj} + g_{hi(m)}R^m{}_{kj} + g_{jk(m)}R^m{}_{hi} + g_{hk(m)}R^m{}_{ji} - \\ &- g_{hj(m)}R^m{}_{ki} - g_{kj(m)}R^m{}_{hi}). \end{aligned}$$

Making use of the definition of T_{hijk} and (1.15) in the above equation, we obtain

$$6k_{hijk} = (2g_{hj(m)}R^m{}_{ik} + g_{ik(m)}R^m{}_{hj} - j|k) - 2g_{hi(m)}R^m{}_{jk}. \tag{3.5}$$

From this equation, we shall derive the following interesting result.

Theorem 3.1. *A generalized metric space of non vanishing K-isotropic curvature is a Riemannian space of constant curvature.*

PROOF. Interchanging indices h and j in (3.3), we get

$$k_{jihk} - k_{hkji} = \frac{1}{2}(T_{jihk} - g_{ji(m)}R^m{}_{hk} + g_{hk(m)}R^m{}_{ji}). \tag{3.3'}$$

Summing the three equations (3.1), (3.3) and (3.3)', we have

$$\begin{aligned} 2(k_{hijk} + k_{jihk}) =& \frac{1}{2}(T_{hijk} + T_{jihk} - g_{hi(m)}R^m{}_{jk} + g_{jk(m)}R^m{}_{hi} - \\ & - g_{ji(m)}R^m{}_{hk} + g_{hk(m)}R^m{}_{ji}). \end{aligned} \tag{3.6}$$

If we consider $+h|j|k$ in (3.6) and use (3.4) and (1.15), we obtain

$$g_{hj(m)}R^m{}_{ki} + h|j|k = 0. \tag{3.7}$$

Transvecting (3.7) by y^h and making use of (1.2) (b), we have

$$C_{jm}R^m{}_{ki} + g_{jk(m)}R^m{}_i + C_{km}R^m{}_{ji} = 0. \tag{3.8}$$

Further transvection of (3.8) by y^k yields, with (1.2) (c) in mind,

$$C_{jm}R^m{}_i = 0. \tag{3.9}$$

Transvecting (3.5) by $y^h y^j$ and using (3.9), we obtain

$$k_{0i0k} = 0, \tag{3.10}$$

which gives, by the definition of k_{hijk},

$$R_{ik} = F^2 K h_{ik}. \tag{3.11}$$

Substituting (3.11) into (3.9), we have

$$F^2 K C_{jk} = 0. \tag{3.12}$$

Since we assume $K \neq 0$, we must have $C_{jk} = 0$. From this result, (3.8) and (3.11), we obtain $g_{jk(i)} = 0$. This means that the space in consideration is a Riemannian space. Q.E.D.

The above proof also yields the following

Corollary 3.2. *A Finsler space of K-isotropic curvature is a Riemannian space of constant curvature.*

§4. A generalized metric space of H-isotropic curvature

Thirdly, we consider a generalized metric space of H-isotropic curvature. In this case, if we put

$$h_{hijk} := H_{hijk} - H(g_{hj}g_{ik} - g_{hk}g_{ij}),$$

then we must have

$$h_{hijk} + h_{jihk} + h_{hkji} + h_{jkhi} = 0. \tag{4.1}$$

On the other hand, the tensor h_{hijk} obeys the following relations:

$$h_{hijk} = -h_{hikj}, \tag{4.2}$$

$$h_{hijk} - h_{jkhi} = H_{hijk} - H_{jkhi} \tag{4.3}$$

$$= \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m{}_{jk} + g_{jk(m)}R^m{}_{hi}) + E_{hijk} - E_{jkhi},$$

$$h_{hijk} + h|j|k = H_{hijk} + h|j|k = 0. \tag{4.4}$$

Consequently, h_{hijk} satisfies the conditions for A_{hijk} in Lemma 1.1, where $U_{hijk} = \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m{}_{jk} + g_{jk(m)}R^m{}_{hi}) + E_{hijk} - E_{jkhi}$, $V_{hijk} = 0$. Therefore h_{hijk} has the form

$$\begin{aligned} 6h_{hijk} =& \frac{1}{2}(T_{hijk} + T_{jihk} - T_{kihj} - T_{hikj} - g_{hi(m)}R^m{}_{jk} - g_{ji(m)}R^m{}_{hk} + \\ &+ g_{ki(m)}R^m{}_{hj} + g_{hi(m)}R^m{}_{kj} + g_{jk(m)}R^m{}_{hi} + g_{hk(m)}R^m{}_{ji} - \\ &- g_{hj(m)}R^m{}_{ki} - g_{kj(m)}R^m{}_{hi}) + E_{hijk} - E_{jkhi} + E_{jihk} - E_{hkji} - \\ &- E_{kihj} + E_{hjki} - E_{hikj} + E_{kjhi}. \end{aligned}$$

Making use of the definition of T_{hijk}, (1.15) and (1.10) (d), we obtain

$$\begin{aligned} 6h_{hijk} =& (2g_{hj(m)}R^m{}_{ik} + g_{ik(m)}R^m{}_{hj} - j|k) - 2g_{hi(m)}R^m{}_{jk} + \\ &+ 3E_{hijk} - E_{jkhi} + E_{kjhi} - E_{hkji} + E_{hjki}. \end{aligned} \tag{4.5}$$

Now we put $E_{ik} := E_{0i0k}$, $\bar{E}_{hj} := E_{h0j0}$, and we first show two lemmas in a generalized metric space.

Lemma 4.1. *In a generalized metric space, we have*

$$\bar{E}_{hj} = \bar{E}_{jh}. \tag{4.6}$$

PROOF. Transvecting (1.10) (d) by $y^i y^k$ and using (1.9) (e), we have (4.6). Q.E.D.

Lemma 4.2. *In a generalized metric space, we have*

$$\bar{E}_{hj} = E_{hj} + C_{hi}E^i{}_j. \tag{4.7}$$

PROOF. Using (1.5), i.e.

$$C_{hj/0} = -2D_{h0j} = 2(g_{ij} + C_{ij})P^i{}_h \tag{4.8}$$

and (1.9) (e), we get

$$\begin{aligned}\bar{E}_{hj} &= D_{h0j/0} - P^i{}_h D_{i0j} = -(g_{ij} + C_{ij})P^i{}_{h/0} - C_{ij/0}P^i{}_h - D_{i0j}P^i{}_h \\ &= -(g_{ij} + C_{ij})P^i{}_{h/0} + D_{m0j}P^m{}_h = -(g_{ij} + C_{ij})(P^i{}_{h/0} + P^i{}_m P^m{}_h) \\ &= (g_{ij} + C_{ij})E^i{}_h = E_{jh} + C_{ij}E^i{}_h = E_{hj} + C_{hi}E^i{}_j.\end{aligned}$$

Q.E.D.

Next, we consider a generalized metric space of H-isotropic curvature.

Lemma 4.3. *In a generalized metric space of H-isotropic curvature, we have*

$$E_{ji} = \bar{E}_{ji} + C_{jm}R^m{}_i. \tag{4.9}$$

PROOF. Interchanging indices h and j in (4.3), we get

$$h_{jihk} - h_{hkji} = \frac{1}{2}(T_{jihk} - g_{ji(m)}R^m{}_{hk} + g_{hk(m)}R^m{}_{ji}) + E_{jihk} - E_{hkji}. \tag{4.3'}$$

Adding (4.1), (4.3) and (4.3)', we obtain

$$\begin{aligned}2(h_{hijk} - h_{jikh}) = &\frac{1}{2}(T_{hijk} + T_{jihk} - g_{hi(m)}R^m{}_{jk} + \\ &+ g_{jk(m)}R^m{}_{hi} - g_{ji(m)}R^m{}_{hk} + g_{hk(m)}R^m{}_{ji}) + \\ &+ E_{hijk} - E_{jkhi} + E_{jihk} - E_{hkji}.\end{aligned} \tag{4.10}$$

Considering $+h|j|k$ in (4.10) and noticing (4.4) and (1.10) (d), we have

$$g_{hk(m)}R^m{}_{ij} + (E_{jkhi} + E_{hkji}) + h|j|k = 0. \tag{4.11}$$

Transvecting (4.11) by y^h, we have

$$\begin{aligned}&C_{km}R^m{}_{ij} + C_{jm}R^m{}_{ik} - g_{jk(m)}R^m{}_i + E_{jk0i} + E_{j0ki} + E_{k0ji} + \\ &+ E_{kj0i} + E_{0jki} + E_{0kji} = 0.\end{aligned} \tag{4.12}$$

Moreover, transvecting (4.12) by y^k and using (1.9) (e), we have (4.9).

Q.E.D.

Lemma 4.4. *In a generalized metric space of H-isotropic curvature, we have*

(4.13) $$H^i{}_k = F^2 H h^i_k.$$

PROOF. Transvecting (4.5) by $y^h y^j$, we see that

$$6h_{0i0k} = 6(H_{ik} - F^2 H h_{ik}) = 2C_{km}R^m{}_i - 3C_{im}R^m{}_k + 3E_{ik} - 2E_{ki} - \bar{E}_{ki}.$$

Lemmas 4.1 and 4.3 tell us that the right hand side of the above equation vanishes and then (4.13) holds. Q.E.D.

Lemma 4.5. *A generalized metric space of H-isotropic curvature is a Finsler space.*

PROOF. From Lemmas 4.2 and 4.3, we see

$$C_{hi}E^i{}_j = -C_{hi}R^i{}_j.$$

Hence, using (1.9) (d) and Lemma 4.4, we get

$$0 = C_{hi}(R^i{}_j + E^i{}_j) = C_{hi}H^i{}_j = F^2 H C_{hi} h^i_j = F^2 H C_{hj},$$

from which we have $C_{hj} = 0$. Therefore the space in consideration reduces to a Finsler space. Q.E.D.

Lemma 4.6. *In a generalized metric space of H-isotropic curvature, we have*

(4.14) $$(a)\ E_h{}^i{}_{jk} = P_h{}^i{}_{j/k} + P_h{}^m{}_j P_m{}^i{}_k - j|k, \quad (b)\ E^i{}_{jk} = 0,\ E^i{}_k = 0,$$
$$(c)\ E_h{}^i{}_{j0} = P_h{}^i{}_{j/0}, \quad E_h{}^i{}_{0k} = -P_h{}^i{}_{k/0}.$$

PROOF. From Lemma 4.5, the space in consideration is a Finsler space. Then, noticing (4.8) we have $P^i{}_k = 0$, which means that $D_j{}^i{}_k = P_j{}^i{}_k + P^i{}_{j(k)} = P_j{}^i{}_k$. Hence (4.14) (a) follows from (1.9) (b) and the other from $P_0{}^i{}_k = 2P^i{}_k = 0$. Q.E.D.

Now we ready to prove the following

Theorem 4.7. *A generalized metric space of H-isotropic curvature is a Finsler space of constant curvature.*

PROOF. Transvecting (4.5) by y^h and using Lemma 4.6, we have

$$6h_{0ijk} = -2C_{ijm}R^m{}_k + 2C_{ikm}R^m{}_j + 3E_{ijk} - E_{jk0i} + E_{kj0i} - E_{kji} + E_{jki}.$$

Noting $R^i{}_k = H^i{}_k + F^2 H h^i_k$ and Lemma 4.6, we obtain

(4.15) $$6h_{0ijk} = 6(H_{ijk} - H(y_j g_{ik} - y_k g_{ij})) = 0, \text{ or } H^i{}_{jk} = H(y_j \delta^i_k - y_k \delta^i_j).$$

By virtue of a well-known theorem ([4], p.133), a Finsler space which satisfies (4.15) is a Finsler space of constant curvature, that is,

$$h_{hijk} = H_{hijk} - H(g_{hj}g_{ik} - g_{hk}g_{ij}) = 0.$$

Q.E.D.

Remark. The right hand side of (4.5) reduces to $2(P_{hij/k} + HC_{hij}y_k) - j|k$ after some calculation using (4.15). This is consistent with the well-known identity in a Finsler space (e.g. [2], (2.7) (b))

$$H_{hijk} + H_{ihjk} = 2(P_{hij/k} - P_{hik/j}) - 2C_{him}H_j{}^m{}_k.$$

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HIDEO IZUMI
DEPARTMENT OF MATHEMATICS
NATIONAL DEFENSE ACADEMY
YOKOSUKA 239, JAPAN

MAMORU YOSHIDA
DEPARTMENT OF MATHEMATICS
SHONAN INSTITUTE OF TECHNOLOGY
FUJISAWA 251, JAPAN

(Received July 12, 1988)