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On the classification of nilpotent Lie algebras of maximal rank

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1. Introduction

In this paper we concern ourselves with maximal rank nilpotent Lie algebras over an algebraically closed field of characteristic 0. After introducing terminology and some known results, we begin by discussing the weight systems of these algebras. We then focus our attention on maximal rank nilpotent Lie algebras on two linearly independent generators. Given a weight system, there exists a projective algebraic variety in which each point corresponds to a class of isomorphic algebras having the given weight system, and such that any two algebras corresponding to distinct points in the variety are nonisomorphic. While there are uncountably many nonisomorphic nilpotent Lie algebras of bounded dimension over a given field, there are only finitely many weight systems, each of which corresponds to a set of isomorphism classes which is a projective algebraic variety.

This work is based on that of G. FAVRE, [2]. For a summary and discussion of varieties corresponding to maximal rank algebras on more than two generators, see [1].

2. Preliminaries

All Lie algebras under discussion here will have a fixed algebraically closed field $\mathbf{K}$ of characteristic 0 as their field of scalars.

Let p and s be positive integers. Following the notation in [2], let $\mathbf{m}(s,p)$ denote the free Lie algebra on s linearly independent generators $E_1, E_2, \dots, E_s$ which is nilpotent of step p. Given an s–tuple $\alpha = (n_1, n_2, \dots, n_s)$ of nonnegative integers, we write $\mathbf{m}^\alpha$ for the subspace of $\mathbf{m} = \mathbf{m}(s,p)$ which is homogeneous of degree n_i in E_i for $i = 1, 2, \dots, s$. Elements of $\mathbf{m}^\alpha$ are said to be of *degree* α and of *total degree* $|\alpha| = n_1 + n_2 + \dots + n_s$. Note that because $\mathbf{m}$ is a Lie algebra, the set

$$R\mathbf{m} = \{\alpha = (n_1, n_2, \dots, n_s) : 0 \le n_i \in \mathbf{Z},\ 1 \le |\alpha| \le p, \text{ and } |\alpha| > 1 \Rightarrow n_i < |\alpha| \text{ for each } i\}$$

consists of all degrees of $\mathbf{m}$.

Any nilpotent Lie algebra of step p on s linearly independent generators may be regarded as a quotient of $\mathbf{m}$ by an ideal $\mathbf{a}$ satisfying $C^p(\mathbf{m}) \nsubseteq \mathbf{a} \subseteq C^2(\mathbf{m})$ (where $C^k(\mathbf{m})$ is the kth ideal in the descending central series of $\mathbf{m}$, i.e., $C^k(\mathbf{m}) = \oplus \mathbf{m}^\alpha$, where the sum is over all α with $k \leq |\alpha| \leq p$). The algebras of prime concern here are those which are of maximal rank:

Definition 2.1. A quotient $\mathbf{g}$ of $\mathbf{m}$ is of *maximal rank* if and only if there exists an ideal $\mathbf{a}$ of $\mathbf{m}$ such that $\mathbf{g} \simeq \mathbf{m}/\mathbf{a}$ and $\mathbf{a} = \oplus(\mathbf{a} \cap \mathbf{m}^\alpha)$, where the sum is over all $\alpha \in R\mathbf{m}$.

The above definition of the maximal rank condition differs from that in [2]. However, the two definitions are easily seen to be equivalent. (For the details, see [4].) Note that if a quotient $\mathbf{g}$ of $\mathbf{m}$ is of maximal rank, then $\dim \mathbf{g}/C^2(\mathbf{g}) = s$.

The main tool used in the study of finite-dimensional nilpotent Lie algebras in [2] is the maximal torus and its corresponding weight system. The standard weight system for $\mathbf{m}$ is $P\mathbf{m} = \{(\alpha,\ \dim \mathbf{m}^\alpha) : \alpha \in R\mathbf{m}\}$. In our study of maximal rank nilpotent (MRN) step p quotients of $\mathbf{m}$, we need not concern ourselves with maximal tori. There is an equivalence relation on the collection of all weight systems which gives rise to the following.

Fact 2.2. *When* $\mathbf{g}$ *is a MRN quotient of* $\mathbf{m}$*, the weight system of* $\mathbf{g}$ *is equivalent to one of the form* $\{(\alpha, d\alpha) : \alpha \in R\}$ *where* $R \subseteq R\mathbf{m}$ *and* $1 \leq d\alpha \leq \dim \mathbf{m}^\alpha$.

Thus weight systems of MRN Lie algebras may be thought of as sets of $(s+1)$–tuples of nonnegative integers. It should be noted, however, that one cannot take an arbitrary subset R of $R\mathbf{m}$ and randomly choose integers $d\alpha$ with $1 \leq d\alpha \leq \dim \mathbf{m}^\alpha$ for each $\alpha \in R$ and expect to produce a weight system. This predicament is the topic of the next section.

Now suppose we are given a weight system. The following algorithm will produce a set $A^\sim$ in which each element corresponds to a class of isomorphic MRN Lie algebras, each having the given weight system. Again the reader is referred to [2] for more details.

Classification algorithm 2.3. Let $P = \{(\alpha, d\alpha) : \alpha \in R\}$ be a weight system. Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ be a basis of R, so each $\alpha \in R$ can be written $\alpha = n_1\alpha_1 + n_2\alpha_2 + \cdots + n_s\alpha_s$ for some $0 \leq n_i \in \mathbf{Z}$.

1. Let $\mathbf{m} = \mathbf{m}(s,p)$, where $p = max\{\,|\alpha| : \alpha \in R\}$. We can regard R as a subset of $R\mathbf{m}$. Let

$$A = \{\text{ ideals } \mathbf{v} \text{ of } \mathbf{m} :\ \dim(\mathbf{v} \cap \mathbf{m}^\alpha) = \dim \mathbf{m}^\alpha - d\alpha \ \text{ for } \alpha \in R, \text{ and } \\ = \dim \mathbf{m}^\alpha \quad \text{ for } \alpha \in R\mathbf{m} - R\}.$$

(Then $\mathbf{m}/\mathbf{v}$ has weight system P for each $\mathbf{v} \in A$.)

2. Let $\mathbf{S}_s$ be the symmetric group on s letters. $\mathbf{S}_s$ acts on $P\mathbf{m}$ by

$$\sigma\,.\,(\alpha, \dim \mathbf{m}^\alpha) = (\alpha^\sigma, \dim \mathbf{m}^\alpha),$$
$$\text{where if } \alpha = \Sigma n_i\alpha_i \text{ then } \alpha^\sigma = \Sigma n_i \alpha_{\sigma(i)}$$

for $\sigma \in \mathbf{S}_s$. Let $\mathbf{G}$ be the subgroup of $\mathbf{S}_s$ which leaves P invariant, i.e., $\mathbf{G} = \{\sigma \in \mathbf{S}_s : d\alpha = d\alpha^\sigma \text{ for each } \alpha \in R\}$.
3. $\mathbf{G}$ acts on $\mathbf{m}$ by $\sigma\,.\,E_i = E_{\sigma(i)}$. This induces an action on A. Let $A^\sim$ be a system of orbit representatives.
4. The map associating each $\mathbf{a}$ in $A^\sim$ with the isomorphism class of $\mathbf{m}/\mathbf{a}$ defines a bijection from $A^\sim$ onto the set of isomorphism classes of MRN Lie algebras with weight systems equivalent to P.

3. Weight Systems

The classification algorithm (2.3) assumes we already have a weight system for a MRN Lie algebra. While we may think of such a system as a set of $(s+1)$–tuples of nonnegative integers, it is difficult to say when a set of $(s+1)$–tuples is a weight system. In this section we provide some necessary conditions.

Given $\mathbf{m} = \mathbf{m}(s,p)$, $R\mathbf{m}$, and $P\mathbf{m}$ as previously described, set $m\alpha = \dim \mathbf{m}^\alpha$ for each $\alpha \in R\mathbf{m}$. Let $\alpha j = (\delta_{1j}, \delta_{2j}, \ldots, \delta_{sj})$ for $j = 1, 2, \ldots, s$ (where δ_{ij} is the Kronecker–δ).

Lemma 3.1. *Let β and $\beta + \alpha j \in R\mathbf{m}$ with $\beta \neq \alpha j$. Let $\{X_1, X_2, \ldots, X_r\}$ be a linearly independent subset of $\mathbf{m}^\beta$. Then $[E_j, X_i] \neq 0$ for $i = 1, 2, \ldots, r$, and $\{[E_j, X_1], [E_j, X_2], \ldots, [E_j, X_r]\}$ is linearly independent.*

PROOF. First note that since $\beta \neq \alpha j$, $X \in \mathbf{m}^\beta \Rightarrow [E_j, X] \neq 0$ unless $|\beta + \alpha j| > p$ or $X = 0$. Thus if $\gamma = \beta + \alpha j \in R\mathbf{m}$, then $\{[E_j, X_1], [E_j, X_2], \ldots, [E_j, X_r]\}$ is a set of nonzero vectors in $\mathbf{m}^\gamma$. The linear independence follows easily. □

Theorem 3.2. *Let $R \subseteq R\mathbf{m}$ and let $P = \{(\beta, d\beta) : \beta \in R$ and $1 \leq d\beta \in \mathbf{Z}\}$. Suppose P is the weight system for a MRN quotient $\mathbf{g}$ of $\mathbf{m}$ which is of step p. Then*
(i) $\beta \in R \Rightarrow 1 \leq d\beta \leq m\beta$.
(ii) $\alpha i \in R$, and $d\alpha i = 1$ *for* $i = 1, 2, \ldots, s$.
(iii) for each integer k with $1 \leq k \leq p$, there exists $\beta \in R$ with $|\beta| = k$.
(iv) $\beta \in R\mathbf{m} \Rightarrow d\beta \leq \Sigma d(\beta - \alpha i)$ $(i = 1, 2, \ldots, s)$, *where we set $d\beta = 0$ if $\beta \in R\mathbf{m} - R$. In particular, $\beta - \alpha i \notin R$ for each $i \Rightarrow \beta \notin R$.*
(v) $\beta \in R \Rightarrow d\beta \leq m\beta - \max\{r(\beta - \alpha i) : 1 \leq i \leq s\}$ *where we set $r\alpha = m\alpha - d\alpha$ for each $\alpha \in R\mathbf{m}$.*

PROOF. Since $\mathbf{g}$ is of maximal rank, there exists an ideal $\mathbf{a}$ of $\mathbf{m}$ satisfying $\mathbf{a} = \oplus(\mathbf{a} \cap \mathbf{m}^\alpha)$ such that $\mathbf{g} \simeq \mathbf{m}/\mathbf{a}$.

(i) This follows from 2.2.

(ii) By (i), if $\alpha i \in R$, then $d\alpha i = 1$ since $m\alpha i = 1$. Since $\mathbf{g}$ is of maximal rank, $\dim \mathbf{g}/C^2(\mathbf{g}) = s$. Thus $\Sigma d\alpha i = s$ (where the sum is over all $\alpha i \in R$). Therefore $\alpha i \in R$ for each $i = 1, 2, \ldots, s$.

(iii) From (ii) we know the result holds when $k = 1$. Fix k, $1 < k \leq p$. Suppose there is no $\beta \in R$ with $|\beta| = k$. Then for every $\alpha \in R\mathbf{m}$ with $|\alpha| = k$, we must have $\mathbf{a} \cap \mathbf{m}^\alpha = \mathbf{m}^\alpha$; this implies $\mathbf{a} \cap C^k(\mathbf{m}) = C^k(\mathbf{m})$, which implies $\mathbf{m}/\mathbf{a}$ (hence $\mathbf{g}$) has step at most $k - 1$. This contradicts our hypothesis that $\mathbf{g}$ is of step p.

(iv) First note that the space $\mathbf{m}^\beta$ is spanned by monomials $[E_\star, [E_\star, \ldots, [E_\star, E_\star] \ldots]]$ where $\star$ represents various subscripts, the subscript j appearing n_j times if $\beta = (n_1, n_2, \ldots, n_s) = \Sigma n_i \alpha i$. Thus we see that $\mathbf{m}^\beta \subseteq \Sigma[\mathbf{m}^{\alpha i}, \mathbf{m}^{\beta - \alpha i}]$. This implies $\pi(\mathbf{m}^\beta) \subseteq \Sigma[\pi(\mathbf{m}^{\alpha i}), \pi(\mathbf{m}^{\beta - \alpha i})]$, where $\pi : \mathbf{m} \to \mathbf{m}/\mathbf{a}$ is the quotient map, i.e., $(\mathbf{m}/\mathbf{a})^\beta \subseteq \Sigma[(\mathbf{m}/\mathbf{a})^{\alpha i}, (\mathbf{m}/\mathbf{a})^{\beta - \alpha i}]$. Hence $d\beta \leq \Sigma \dim [(\mathbf{m}/\mathbf{a})^{\alpha i}, (\mathbf{m}/\mathbf{a})^{\beta - \alpha i}]$. Now $\dim [(\mathbf{m}/\mathbf{a})^{\alpha i}, (\mathbf{m}/\mathbf{a})^{\beta - \alpha i}] \leq \dim (\mathbf{m}/\mathbf{a})^{\beta - \alpha i} = d(\beta - \alpha i)$. Thus $d\beta \leq \Sigma d(\beta - \alpha i)$.

(v) Let $\beta \in R$. We show $d\beta \leq m\beta - r(\beta - \alpha i)$ for each i, $1 \leq i \leq s$. First note that if $r(\beta - \alpha i) = 0$, the inequality reduces to $d\beta \leq m\beta$, which follows from (i). Fix i and suppose $r = r(\beta - \alpha i) > 0$. Then there exist linearly independent vectors $X_1, X_2, \ldots, X_r$ in $\mathbf{a} \cap \mathbf{m}^{\beta - \alpha i}$. Let $Y_j = [E_i, X_j]$. By 3.1, $\{Y_1, Y_2, \ldots, Y_r\}$ is a linearly independent subset of $\mathbf{a} \cap \mathbf{m}^\beta$. Expand to a basis $\{Y_1, Y_2, \ldots, Y_{m\beta}\}$ of $\mathbf{m}^\beta$. Then $\{\pi(Y_j) : 1 \leq j \leq m\beta\}$ spans $\pi(\mathbf{m}^\beta)$, but $\pi(Y_j) = 0$ for $1 \leq j \leq r$. Thus $\dim \pi(\mathbf{m}^\beta) \leq m\beta - r$. □

At this point it might be helpful to recall the formula for the dimension of the weight spaces $\mathbf{m}^\alpha$: let $\alpha = (n_1, n_2, \ldots, n_s) \in R\mathbf{m}$ and let $|\alpha| = n$. Then

$$m\alpha = \dim \mathbf{m}^\alpha = (1/n) \sum_{d | n_i} \mu(d)(n/d)!/[(n_1/d)!(n_2/d)! \cdots (n_s/d)!]$$

where μ is the Möbius function. This formula has been used to compute all values of $m\alpha$ for $\mathbf{m} = \mathbf{m}(2, 7)$. (See Figure 1.)

Before attempting to create a weight system, we make one final observation.

Figure 1. Dimension of $\mathbf{m}^\alpha, \alpha = (n_1, n_2)$.

$n_2 =:$		0	1	2	3	4	5	6
$n_1 =:$	0		1					
	1	1	1	1	1	1	1	1
	2		1	1	2	2	3	
	3		1	2	3	5		
	4		1	2	5			
	5		1	3				
	6		1					

Fact 3.3. *If $P = \{(\beta, d\beta) : \beta \in R$ and $1 \le d\beta \le m\dot{\beta}\}$ is the weight system for a MRN quotient of $\mathbf{m} = \mathbf{m}(s,p)$ which is of step $p > 1$ and rank s, for some $R \subseteq R\mathbf{m}$, then $Q = \{(\beta, d\beta) \in P : |\beta| < p\}$ is the weight system for a maximal rank quotient of $\mathbf{m}$ which is of step $p-1$ and rank s, hence for a maximal rank quotient of $\mathbf{m}(s, p-1)$.*

PROOF. If P is a system, then there is some ideal $\mathbf{a}$ of $\mathbf{m}(s,p)$ such that $\mathbf{m}/\mathbf{a}$ is of maximal rank s and step p, such that $\mathbf{m}/\mathbf{a}$ has system P, and such that $\mathbf{a} = \oplus(\mathbf{a} \cap \mathbf{m}^\alpha)$ $(\alpha \in R\mathbf{m})$. Let $\mathbf{b} = \oplus(\mathbf{a} \cap \mathbf{m}^\alpha)$ $(|\alpha| < p)$. Let $\mathbf{c} = \mathbf{b} \oplus C^p(\mathbf{m})$. Since $\mathbf{a}$ is an ideal, $\mathbf{c}$ is an ideal; $\mathbf{m}/\mathbf{c}$ has system Q, as does $\mathbf{m}(s, p-1)/\mathbf{b}$. □

The theorem and fact together suggest that one possible procedure for creating a weight system is to work recursively, starting with a known system. However, without a known system, it is perhaps more feasible to proceed inductively.

Example 3.4. Let us attempt to create a weight system P for a maximal rank 2, step 7 quotient $\mathbf{m}/\mathbf{a}$ of $\mathbf{m} = \mathbf{m}(2,7)$. We know we must have $(\alpha i, 1) \in P$ for $i = 1$ and 2. Also, since we want $\mathbf{m}/\mathbf{a}$ to be of step 7, we cannot have $\mathbf{m}^{(1,1)} \subseteq \mathbf{a}$. Thus since $m(1,1) = 1$, we need $((1,1),1) \in P$. In other words, $P\mathbf{m}(2,2) \subseteq P$. Indeed, let us take $P\mathbf{m}(2,3) \subseteq P$.

Next let us build on $P\mathbf{m}(2,3)$ to obtain a weight system for a step 4 quotient of $\mathbf{m}(2,4)$. This merely requires that for each $\alpha \in \{(3,1),(2,2),(1,3)\}$, we choose values for $d\alpha$ with $0 \le d\alpha \le m\alpha$. Let us take $d(3,1) = d(1,3) = 1$ but $d(2,2) = 0$, i.e., take $P_4 = P\mathbf{m}(2,3) \cup \{((3,1),1),((1,3),1)\}$ as our system.

Suppose $P_4 \subset P_5$ where P_5 is a weight system for a step 5 quotient of $\mathbf{m}(2,5)$. Then since $d(3,1) = 1, d(2,2) = 0, d(1,3) = 1$ and

$$r(3,1) = 0, r(2,2) = 1, r(1,3) = 0,$$

we see that $0 \le d\alpha \le 1$ for $\alpha = (4,1)$ or $(1, 4)$ by (i) of 3.2, while by either (iv) or (v), $0 \le d\alpha \le 1$ for $\alpha = (3,2)$ or $(2,3)$. Let $P_5 = P_4 \cup \{((4,1),1),((3,2),1)\}$. Is P_5 a weight system? Let us find an ideal $\mathbf{a}$ of $\mathbf{m}(2,5)$ such that $\mathbf{m}/\mathbf{a}$ has system P_5. We will need $\dim(\mathbf{a} \cap \mathbf{m}^\alpha) = r\alpha$ for each $\alpha \in R\mathbf{m}(2,5)$; $r\alpha = 0$ for $|\alpha| \leq 3$ or $\alpha \in \{(3,1),(1,3),(4,1)\}$; $r(2,2) = m(2,2) = 1$, $r(3,2) = m(3,2) - 1 = 1$, $r(2,3) = m(2,3) = 2$ and $r(1,4) = m(1,4) = 1$. Thus let $\mathbf{a}$ be the ideal of $\mathbf{m}(2,5)$ generated by $\mathbf{m}^{(2,2)} \oplus \mathbf{m}^{(2,3)} \oplus \mathbf{m}^{(1,4)}$, i.e., $\mathbf{a} = \mathbf{m}^{(2,2)} \oplus [\mathbf{m}^{(1,0)}, \mathbf{m}^{(2,2)}] \oplus \mathbf{m}^{(2,3)} \oplus \mathbf{m}^{(1,4)}$. Then $\mathbf{m}(2,5)/\mathbf{a}$ is of maximal rank 2 and step 5 with weight system P_5.

Now let us find a weight system $P_6 \supset P_5$ for a step 6 quotient of $\mathbf{m}(2,6)$. Since $d(4,1) = 1, d(3,2) = 1, d(2,3) = 0, d(1,4) = 0$ and

$$r(4,1) = 0, r(3,2) = 1, r(2,3) = 2, r(1,4) = 1,$$

we see that by (v) of 3.2, we must have $d(4,2) \leq 2 - 1 = 1$; by either (iv) or (v) we must have $d(3,3) \leq 1$; by (iv), $d(2,4) = d(1,5) = 0$ and by (i), $d(5,1) \leq 1$. Suppose we choose $P_6 = P_5 \cup \{((5,1),1),((4,2),1),((3,3),1)\}$. Can we find an appropriate ideal $\mathbf{a}'$ of $\mathbf{m}(2,6)$ such that $\mathbf{m}(2,6)/\mathbf{a}'$ has system P_6? We will need $\dim(\mathbf{a}' \cap \mathbf{m}^{\alpha}) = r\alpha$, where $r(4,2) = 1$, $r(3,3) = 2$, $r(2,4) = m(2,4) = 2$ and $r(1,5) = m(1,5) = 1$. We will also need $\mathbf{a} \subset \mathbf{a}'$. However, this presents a problem, since then the subspace $S = [\mathbf{m}^{(0,1)}, [\mathbf{m}^{(1,0)}, \mathbf{m}^{(2,2)}]] + [\mathbf{m}^{(1,0)}, \mathbf{m}^{(2,3)}] \subseteq \mathbf{a}' \cap \mathbf{m}^{(3,3)}$, but it can be shown that $\dim S = 3$ while we need $\dim(\mathbf{a}' \cap \mathbf{m}^{(3,3)}) = 2$. Thus P_6 cannot be a weight system. (Note that this shows that (i) – (v) in 3.2 are not sufficient conditions for a weight system.) However, the set $P_6' = P_5 \cup \{((5,1),1),((4,2),1)\}$ is a weight system: we can take $\mathbf{a}' = \mathbf{a} \oplus [\mathbf{m}^{(1,0)}, [\mathbf{m}^{(1,0)}, \mathbf{m}^{(2,2)}]] \oplus \mathbf{m}^{(3,3)} \oplus \mathbf{m}^{(2,4)} \oplus \mathbf{m}^{(1,5)}$; $\mathbf{a}'$ is then an ideal of $\mathbf{m}(2,6)$ such that $\mathbf{m}(2,6)/\mathbf{a}'$ is of maximal rank 2 and step 6 with weight system P_6'.

Finally, let us build on P_6' to obtain our desired system P. Since $d(5,1) = 1$, $d(4,2) = 1$, $d(3,3) = 0$, $d(2,4) = 0$, $d(1,5) = 0$ and $r(5,1) = 0$, $r(4,2) = 1$, $r(3,3) = 3$, $r(2,4) = 2$, $r(1,5) = 1$, we see that $d(6,1) \leq 1$; by (iv), $d(5,2) \leq 2$, $d(4,3) \leq 1$, $d(3,4) = d(2,5) = d(1,6) = 0$. So, let $P = P_6' \cup \{((6,1),1),((5,2),2),((4,3),1)\}$. Let $\mathbf{a}''$ be any ideal of $\mathbf{m}(2,7)$ satisfying $\mathbf{a}'' = \mathbf{a}' \oplus [\mathbf{m}^{(1,0)}, \mathbf{a}' \cap \mathbf{m}^{(4,2)}] \oplus (\mathbf{a}'' \cap \mathbf{m}^{(4,3)}) \oplus \mathbf{m}^{(3,4)} \oplus \mathbf{m}^{(2,5)} \oplus \mathbf{m}^{(1,6)}$, where $\mathbf{a}'' \cap \mathbf{m}^{(4,3)} \supseteq [\mathbf{m}^{(0,1)}, \mathbf{a}' \cap \mathbf{m}^{(4,2)}] + [\mathbf{m}^{(1,0)}, \mathbf{m}^{(3,3)}]$ and $\dim(\mathbf{a}'' \cap \mathbf{m}^{(4,3)}) = 4$. (Note that $\dim[\mathbf{m}^{(1,0)}, \mathbf{a}' \cap \mathbf{m}^{(4,2)}] = 1 = r(5,2)$ and $3 \leq \dim([\mathbf{m}^{(0,1)}, \mathbf{a}' \cap \mathbf{m}^{(4,2)}] + [\mathbf{m}^{(1,0)}, \mathbf{m}^{(3,3)}]) \leq 4$.) Then $\mathbf{m}/\mathbf{a}''$ is of maximal rank 2 and step 7, and $\mathbf{m}/\mathbf{a}''$ has weight system P.

4. Digraphs

It is helpful to have pictorial representations for $\mathbf{m}(s,p)$ and its ideals which give rise to MRN quotients. The pictures that seem to be the most natural are, in fact, weighted digraphs.

We begin with $\mathbf{m} = \mathbf{m}(s,p)$. Take $R\mathbf{m}$ to be the set of vertices, and $E\mathbf{m} = \{(\alpha, \alpha + \alpha i) : 1 \leq i \leq s \text{ and } \alpha, \alpha + \alpha i \in R\mathbf{m}\}$. The edges symbolize the action of the adjoint maps ad E_i, in that ad $E_i(\mathbf{m}^{\alpha}) \subseteq \mathbf{m}^{\alpha + \alpha_i}$. The digraph $(R\mathbf{m}, E\mathbf{m})$ is connected and cycle–free. The vertices α_i $(1 \leq i \leq s)$ each have in–degree 0 and out–degree $s-1$ (since $2\alpha_i \notin R\mathbf{m}$). The vertices α with $|\alpha| = p$ each have in–degree at least 1 and out–degree 0. Other vertices typically have in–degree between 1 and s and out–degree s. By associating each dimension $m\alpha$ to the vertex α, we make $(R\mathbf{m}, E\mathbf{m})$ into a weighted digraph. See Figure 2.

Now let $\mathbf{a}$ be an ideal of $\mathbf{m}$ with the property that $\mathbf{a} = \oplus(\mathbf{a} \cap \mathbf{m}^{\alpha})$ $(\alpha \in R\mathbf{m})$, so $\mathbf{m}/\mathbf{a}$ is MRN. For each $\alpha \in R\mathbf{m}$, let $r\alpha = \dim(\alpha \cap \mathbf{m}^{\alpha})$. Let $V = \{\alpha \in R\mathbf{m} : r\alpha > 0\}$. Let $E\mathbf{a} = \{(\alpha, \alpha + \alpha_i) \in E\mathbf{m} : \alpha, \alpha + \alpha_i \in V\}$. Then $(V, E\mathbf{a})$ is the digraph associated with $\mathbf{a}$. This digraph is a subgraph of $(R\mathbf{m}, E\mathbf{m})$, so is also cycle-free. It consists of finitely many connected components, and has the property that if $\alpha \in V$ and if there is a directed path in $(R\mathbf{m}, E\mathbf{m})$ from α to β, where $|\beta| = p$, then that directed path is also in $(V, E\mathbf{a})$. (This condition is necessary for $\mathbf{a}$ to be an ideal.) Associate $r\alpha$ with each $\alpha \in V$ to make $(V, E\mathbf{a})$ a weighted digraph. For example, the digraph associated with the ideal $\mathbf{a}''$ of $\mathbf{m}(2, 7)$ described in 3.4 is shown in Figure 3.

Now suppose for some $R \subseteq R\mathbf{m}$, $P = \{(\alpha, d\alpha) : \alpha \in R\}$ is a weight system for MRN quotients of $\mathbf{m}$. Then there exists an ideal $\mathbf{a}$ of $\mathbf{m}$ such that $\mathbf{a} = \oplus(\mathbf{a} \cap \mathbf{m}^{\alpha})$ and $\mathbf{m}/\mathbf{a}$ has system P. We can determine the weighted digraph $(V, E\mathbf{a})$ as follows.

Let $\alpha \in R\mathbf{m}$. If $\alpha \in R$ and $d\alpha = m\alpha$, then $\alpha \notin V$. If $\alpha \in R$ but $d\alpha < m\alpha$, then $\alpha \in V$ and this vertex has weight $r\alpha = m\alpha - d\alpha$. If $\alpha \notin R$, then $\alpha \in V$ and we assign this vertex the weight $m\alpha$. An edge $(\alpha, \alpha + \alpha_i) \in E\mathbf{a}$ if and only if α and $\alpha + \alpha_i$ both belong to V.

Figure 2. The weighted digraph associated with $\mathbf{m}(2, 7)$.

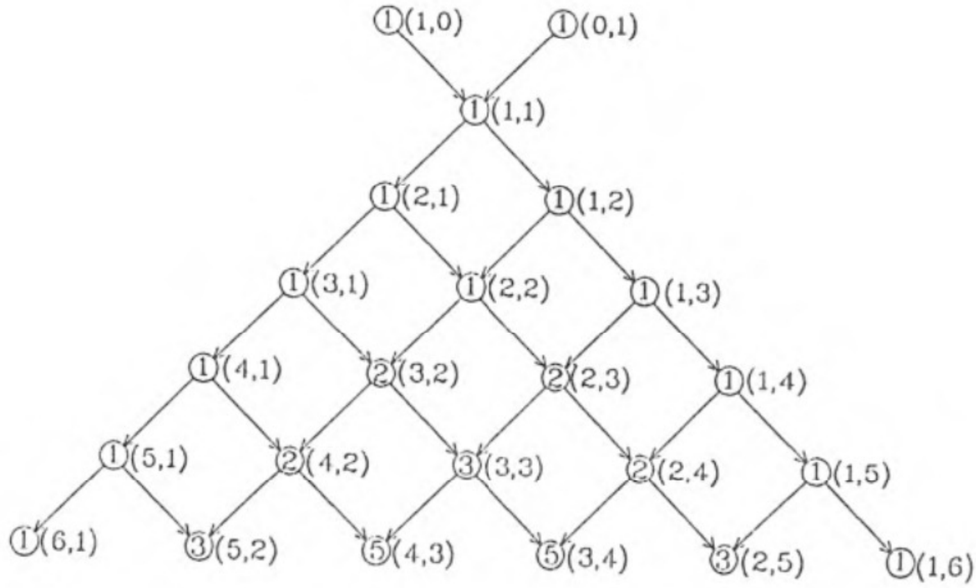
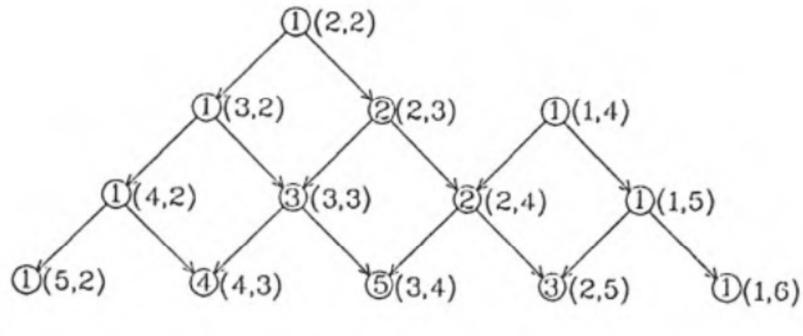


Figure 3. The (weighted) digraph associated with $\mathbf{a}''$.



5. Examples of Classifying Spaces

Recall that each weight system for MRN Lie algebras gives rise to a set $A^\sim$ of isomorphism classes of MRN Lie algebras having the given weight system. More specifically, let $\mathbf{m} = \mathbf{m}(s,p)$ and let $P = \{(\alpha, d\alpha) : \alpha \in R\}$ be a weight system, where R is some subset of $R\mathbf{m}$. Then $A^\sim = A/\mathbf{G} = \{\text{ideals } \mathbf{v} \text{ of } \mathbf{m} : \dim(\mathbf{v} \cap \mathbf{m}^\alpha) = r\alpha \text{ for all } \alpha \in R\mathbf{m}\}/\mathbf{G}$, where $r\alpha = m\alpha - d\alpha$ for $\alpha \in R$ and $r\alpha = m\alpha$ for $\alpha \in R\mathbf{m} - R$, and where the subgroup $\mathbf{G}$ of $\mathbf{S}_s$ is the group of symmetries of P. Now if we write $\mathbf{v}^\alpha$ for $\mathbf{v} \cap \mathbf{m}^\alpha$, then $A = \{\oplus \mathbf{v}^\alpha \ (\alpha \in R\mathbf{m}) : \dim \mathbf{v}^\alpha = r\alpha \text{ and } \oplus \mathbf{v}^\alpha \text{ is an ideal}\} \subseteq V = \Pi Gr_{r\alpha}(\mathbf{m}^\alpha)$ $(\alpha \in R\mathbf{m})$, where $Gr_{r\alpha}(\mathbf{m}^\alpha)$ is the Grassmanian variety consisting of all $r\alpha$–dimensional subspaces of $\mathbf{m}^\alpha$. Moreover, for $\oplus \mathbf{v}^\alpha$ to be an ideal, it is (necessary and) sufficient that $[E_i, \mathbf{v}^\alpha] \subseteq \mathbf{v}^{\alpha+\alpha_i}$ for $i = 1, 2, \ldots, s$. Since these conditions are algebraic, A is an algebraic subvariety of the variety V, and is therefore itself a projective variety. The action of $\mathbf{G}$ on A is also algebraic, so $A^\sim$ is a projective variety. Although we will typically refer to $A^\sim$ as a space, this does not mean we are not concerned with the geometry of $A^\sim$.

Example 5.1. Fix s and p, and consider only those weights α with $|\alpha| = p$. For each such α, choose integers $r\alpha$, $0 \le r\alpha \le m\alpha$, such that some $r\alpha \ne 0$ and some $r\alpha \ne m\alpha$; let $\mathbf{a}^\alpha$ be any $r\alpha$–dimensional subspace of $\mathbf{m}^\alpha$. Then $\mathbf{a} = \oplus \mathbf{a}^\alpha$ $(|\alpha| = p)$ is an ideal $\subset C^p(\mathbf{m})$, and we see that $A = \Pi Gr_{r\alpha}(\mathbf{m}^\alpha)$ $(|\alpha| = p)$. For each α, let $d\alpha = m\alpha - r\alpha$. Then A corresponds to the weight system $P = P\mathbf{m}(s, p-1) \cup \{(\alpha, d\alpha) : |\alpha| = p\}$. $A^\sim = A/\mathbf{G}$ for some $\mathbf{G} < \mathbf{S}_s$ which depends on the choice of the $r\alpha$'s.

In general, the structure of $A^\sim$ is fairly complicated. For this reason, we will describe $A^\sim$ in certain special cases which can be handled by explicit calculations, but yet, which yield interesting varieties. In each example, P is the system of a maximal rank quotient $\mathbf{m}/\mathbf{a}$ of $\mathbf{m}(2,p)$, where $\mathbf{a} \subseteq C^n(\mathbf{m}(2,p))$ with $p-2 \leq n \leq p$. In addition, there are only two possibilities for the symmetry group $\mathbf{G}$: the trivial group and the symmetric group $\mathbf{S}_2 \simeq \mathbf{Z}_2$.

$\mathbf{G}$ is trivial precisely when the weight system P contains an element $((n_1,n_2),d(n_1,n_2))$ such that (i) there is no element $((n_2,n_1),d(n_2,n_1)) \in P$, or (ii) there is such an element, but $d(n_2,n_1) \neq d(n_1,n_2)$. When $\mathbf{G}$ is trivial, the set $A^\sim$ of orbit representatives is identical to the set A of ideals.

$\mathbf{G} = \mathbf{S}_2$ whenever the system P is symmetric in α_1 and α_2, i.e., whenever $((n_1,n_2),d(n_1,n_2)) \in P$ if and only if $((n_2,n_1),d(n_1,n_2)) \in P$. In this case, the orbits of $\mathbf{G}$ consist of pairs $\{\mathbf{a},\mathbf{a}'\}$, where $(12)\,.\,\mathbf{a} = \mathbf{a}'$; recall $(12)\,.\,E_i = E_j$ for $\{i,j\} = \{1,2\}$.

A weight system P is called *complete* if for each $\alpha \in R$, $d\alpha = \dim \mathbf{m}^\alpha$. When a system is complete, there is only one isomorphism class of algebras with that system, namely $A = \{\oplus \mathbf{m}^\alpha \ (\alpha \in R\mathbf{m} - R)\}$. Now for $p < 5$, all systems are complete, since each space $\mathbf{m}^\alpha$ with $|\alpha| < 5$ is one-dimensional.

When $p = 5$, there are two weight spaces of dimension two, so it is possible to have weight systems which are not complete. However, those systems which are not complete still do not yield very complicated varieties: the set A will consist of a single point (when only one algebra, up to isomorphism, has the given system), a projective variety $\mathbf{K}P^1$ (i.e., $Gr_1(\mathbf{m}^\alpha)$, where $\alpha = (3,2)$ or $(2,3)$) contained in $C^5(\mathbf{m})$, or a product of these with up to three factors. (With more than three factors, the quotients would no longer be of step 5.)

That the set A corresponding to a given system consists of a single point is not, however, always obvious, even when $p = 5$.

Example 5.2. Let $P = \{\ ((1,0),1),\ ((0,1),1),\ ((1,1),1),\ ((1,2),1),\ ((1,3),1),\ ((2,3),1),\ ((1,4),1)\ \}$. First note that P is a weight system: let $\mathbf{a} = \mathbf{m}^{(2,1)} \oplus \mathbf{m}^{(3,1)} \oplus \mathbf{m}^{(2,2)} \oplus \mathbf{m}^{(4,1)} \oplus \mathbf{m}^{(3,2)} \oplus [\mathbf{m}^{(0,1)},\mathbf{m}^{(2,2)}]$. Then $\mathbf{a}$ is an ideal of $\mathbf{m}(2,5)$, and $\mathbf{m}(2,5)/\mathbf{a}$ is of maximal rank. Moreover, any ideal $\mathbf{v}$ in the set A must satisfy $\mathbf{v}^{(2,1)} = \mathbf{m}^{(2,1)}$, which implies that $\mathbf{v}^\alpha = \mathbf{m}^\alpha$ for $\alpha = (3,1)$, $(2,2)$, or $(4,1)$, and that $\mathbf{v}^{(2,3)}$ contains $[\mathbf{m}^{(0,1)},\mathbf{m}^{(2,2)}]$ and $[\mathbf{m}^{(1,0)},\mathbf{m}^{(2,2)}]+[\mathbf{m}^{(0,1)},\mathbf{m}^{(3,1)}]$. That the latter subspace is, in fact, $\mathbf{m}^{(3,2)}$ follows from the theorem below. Thus $A = \{\mathbf{a}\}$; since the symmetry group $\mathbf{G}$ is trivial, $A^\sim = A/\mathbf{G}$ consists of a *single point.*

Before discussing any other examples, we prove a needed result. Let $\mathbf{m} = \mathbf{L}(2)$. Let $\mathbf{T} = \mathbf{T}(\mathrm{span}_\mathbf{K}\{E_1,E_2\})$ be the tensor algebra. Make $\mathbf{T}$ into a Lie algebra in the usual way, i.e., set $[X,Y] = X \otimes Y - Y \otimes X$ for all $X, Y \in \mathbf{T}$. Now let $\imath : \{E_1,E_2\} \to \mathbf{T}$ be given by $\imath(E_i) = E_i$ for $i = 1,2$. By the universal property of the free Lie algebra $\mathbf{m}$, $\imath$ extends uniquely to a Lie homomorphism $\imath : \mathbf{m} \to \mathbf{T}$. We identify $\mathrm{span}_\mathbf{K}\{E_1,E_2\}$ with $\mathbf{T}^1$. We will show that for any nonzero $X \in \mathbf{m}$, if $[E_1,[E_2,X]]$ and $[E_2,[E_1,X]]$

are not identical, then they are linearly independent. We will need the following.

Lemma 5.3. *Let $a, b \in \mathbf{K}$ with $a, b, a+b \neq 0$. Let $f = a$ ad E_1 ad E_2+ b ad E_2 ad $E_1 : \mathbf{T} \to \mathbf{T}$. Then $\ker(f) = \mathbf{T}^0$, the scalars.*

PROOF. For $T \in \mathbf{T}$, $f(T) \doteq a[E_1,[E_2,T]] + b[E_2,[E_1,T]] =$ (suppressing $\otimes$) $a(E_1E_2T - E_1TE_2 - E_2TE_1 + TE_2E_1) + b(E_2E_1T - E_2TE_1 - E_1TE_2 + TE_1E_2) = aE_1E_2T + bE_2E_1T - (a+b)E_1TE_2 - (a+b)E_2TE_1 + aTE_2E_1 + bTE_1E_2 = (\star)$.

If $T \in \mathbf{T}^0$, $(\star)$ reduces to 0, so $\mathbf{T}^0 \subseteq \ker(f)$.

Now let $T \in \ker(f)$, so $(\star) = 0$. We may assume $T \in \mathbf{T}^n$ for $n > 0$.

Case 1: $n = 1$. Then $T = c_1E_1 + c_2E_2$, $c_i \in \mathbf{K}$. Combining like terms in $(\star)$, we get

$$\begin{aligned} 0 &= (-(a+b)c_1 + bc_1)E_1E_1E_2 + (ac_2 - (a+b)c_2)E_1E_2E_2 \\ &\qquad + \text{(other monomials)} \\ &= -ac_1E_1E_1E_2 - bc_2E_1E_2E_2 + \text{(other monomials)}, \end{aligned}$$

so $-ac_1 = 0 = -bc_2$, since the distinct monomials are linearly independent. Thus $c_1 = 0 = c_2$, and $T = 0$.

Case 2: $n = 2$. Then $T = c_{11}E_1E_1 + c_{12}E_1E_2 + c_{21}E_2E_1 + c_{22}E_2E_2$, $c_{ij} \in \mathbf{K}$. Again combining like terms in $(\star)$, we get

$$\begin{aligned} 0 =&(-(a+b)c_{11} + bc_{11})E_1E_1E_1E_2 + (ac_{22} - (a+b)c_{22})E_1E_2E_2E_2 \\ &+ (bc_{12} + bc_{21})E_2E_1E_1E_2 + (ac_{12} - (a+b)c_{21} + bc_{12})E_1E_2E_1E_2 \\ &+ \text{(other monomials)} \\ =& - ac_{11}E_1E_1E_1E_2 - bc_{22}E_1E_2E_2E_2 + b(c_{12} + c_{21})E_2E_1E_1E_2 \\ &+ (a+b)(c_{12} - c_{21})E_1E_2E_1E_2 + \text{(other monomials)}. \end{aligned}$$

Since a, b and $a+b$ are nonzero, we get $c_{11} = 0 = c_{22}$, $c_{12} + c_{21} = 0$ and $c_{12} - c_{21} = 0$. The latter two equations together imply that $c_{12} = 0 = c_{21}$, and therefore $T = 0$.

Case 3: $n \geq 3$. Then $T = E_1T_1 + E_2T_2 = T_1'E_1 + T_2'E_2$, with $T_i, T_i' \in \mathbf{T}^{n-1}$ for $i = 1, 2$. From $(\star)$ we get

$$\begin{aligned} 0 =&E_1(aE_2T - (a+b)TE_2) + E_2(bE_1T - (a+b)TE_1) \\ &+ T(aE_2E_1 + bE_1E_2) \\ =&E_1(aE_2T - (a+b)TE_2 + aT_1E_2E_1 + bT_1E_1E_2) \\ &+ E_2(bE_1T - (a+b)TE_1 + aT_2E_2E_1 + bT_2E_1E_2), \end{aligned}$$

which implies

$$(\star\star) \qquad (aE_2T-(a+b)TE_2+aT_1E_2E_1+bT_1E_1E_2)=0$$

and

$$(\star\star\star) \qquad (bE_1T-(a+b)TE_1+aT_2E_2E_1+bT_2E_1E_2=0.$$

First we consider $(\star\star)$:

$$aE_2(T_1'E_1+T_2'E_2)-(a+b)TE_2+aT_1E_2E_1+bT_1E_1E_2=0 \Rightarrow$$
$$(aE_2T_1'+aT_1E_2)E_1+(aE_2T_2'-(a+b)T+bT_1E_1)E_2=0 \Rightarrow$$
$$a(E_2T_1'+T_1E_2)=0 \text{ and } aE_2T_2'-aT-bT+bT_1E_1=0.$$

From the first equation (since $a \neq 0$), we get $E_2T_1'+T_1E_2=0$.
Say $T_i=E_1T_{i1}+E_2T_{i2}$ for $i=1,2$. Then $E_2T_1'+E_1T_{11}E_2+E_2T_{12}E_2 = 0 \Rightarrow E_2(T_1'+T_{12}E_2)=0$ and $E_1T_{11}E_2=0 \Rightarrow T_1'=-T_{12}E_2$ and $T_{11}=0$ (so $T_1=E_2T_{12}$). From the second equation, we get

$$\begin{aligned} 0 &= aE_2T_2'-a(E_1T_1+E_2T_2)-bE_1T_1+E_2T_2)+bE_2T_{12}E_1 \\ &= E_1(-aT_1-bT_1)+E_2(aT_2'-aT_2-bT_2+bT_{12}E_1) \\ &= E_1(-(a+b)T_1)+E_2(a(T_2'-T_2)+b(T_{12}E_1-T_2)), \end{aligned}$$

from which we conclude $-(a+b)T_1=0$, so $T_1=0$.
Now $(\star\star\star)$ is just $(\star\star)$ with E_1 and E_2 exchanged, with a and b exchanged, and with T_2 in place of T_1. Thus $(\star\star\star)$ yields $T_2=0$, so that $T=0$.

Thus we see that $\ker(f)\cap \mathbf{T}^n=0$ unless $n=0$, so $\ker(f)=\mathbf{T}^0$. □

Theorem 5.4. *Let X be any nonzero element of $\mathbf{m}(2,p)$. Then either $[E_1,[E_2,X]]$ and $[E_2,[E_1,X]]$ are identical, or else they are linearly independent.*

PROOF. $(\mathbf{T},\imath)$ is a universal enveloping algebra of $\mathbf{m}$. Let a,b and f be as in the lemma and let $\varphi=a \text{ ad } E_1 \text{ ad } E_2+b \text{ ad } E_2 \text{ ad } E_1 : \mathbf{m}\to\mathbf{m}$. It is then easy to show $\imath\varphi=f\imath$. Now suppose $X\in\ker(\varphi)$. Then $0=\imath\varphi(X)=f\imath(X)\Rightarrow \imath X\in\mathbf{T}^0\Rightarrow X=0$. Thus $\ker(\varphi)=\{0\}$.

We have now shown that if $a+b\neq 0$ and $a\neq 0\neq b$, then $(a \text{ ad } E_1 \text{ ad } E_2)(X)+(b \text{ ad } E_2 \text{ ad } E_1)(X)\neq 0$ for $X\neq 0$. In other words, if $(\text{ad } E_1 \text{ ad } E_2)(X)$ and $(\text{ad } E_2 \text{ ad } E_1)(X)$ are not identical, then they are linearly independent. □

Corollary 5.5. *If $X\in\mathbf{m}$ is homogeneous of degree $n\neq 2$, then $[E_1,[E_2,X]]$ and $[E_2,[E_1,X]]$ are linearly independent.*

PROOF. If $[E_1,[E_2,X]]\neq[E_2,[E_1,X]]$, the conclusion follows from the theorem. Say $[E_1,[E_2,X]]=[E_2,[E_1,X]]$. Then by Jacobi's identity, $[E_1,[E_2,X]]-[E_2,[E_1,X]]+[X,[E_1,E_2]]=0\Rightarrow[X,[E_1,E_2]]=0\Rightarrow X\in\mathbf{m}^{(1,1)}$, which contradicts our hypothesis. Thus $[E_1,[E_2,X]]$ and $[E_2,[E_1,X]]$ are linearly independent. □

Now consider the collection of examples described in the following proposition. We require $p > 2$ so that step p quotients of $\mathbf{m}(2,p)$ will not be abelian of dimension at most 2.

Proposition 5.6. *Let $\mathbf{m} = \mathbf{m}(2,p)$ for some positive integer p; fix n_1 and n_2 so $n = n_1 + n_2 \leq p$. Suppose $n > 2$. Let P be the weight system $\{(\alpha, d\alpha) : \alpha \in R\mathbf{m}\}$ where either*

$$\text{(i)} \quad p = n \quad \text{and} \quad d\alpha = \begin{cases} m\alpha - 1 & \text{for } \alpha = (n_1, n_2), \\ m\alpha & \text{otherwise;} \end{cases}$$

$$\text{(ii)} \quad p = n+1 \quad \text{and} \quad d\alpha = \begin{cases} m\alpha - 1 & \text{for } \alpha = (n_1, n_2), (n_1+1, n_2), \\ & \qquad (n_1, n_2+1), \\ m\alpha & \text{otherwise;} \end{cases}$$

or

$$\text{(iii)} \quad p = n+2 \quad \text{and} \quad d\alpha = \begin{cases} m\alpha - 1 & \text{for } \alpha = (n_1, n_2), (n_1+1, n_2), \\ & \qquad (n_1, n_2+1), (n_1+2, n_2), \\ & \qquad (n_1, n_2+2), \\ m\alpha - 2 & \text{for } \alpha = (n_1+1, n_2+1), \\ m\alpha & \text{otherwise.} \end{cases}$$

Then the space $A^\sim$ of isomorphism classes of MRN quotients of $\mathbf{m}$ having system equivalent to P is $\mathbf{K}P^{m(n_1,n_2)-1}/\mathbf{G}$ for $\mathbf{G} < \mathbf{Z}_2$.

Proof. Let P be as given. To prove that P is indeed a system, we will produce an ideal $\mathbf{a}$ of $\mathbf{m}$ such that $\mathbf{m}/\mathbf{a}$ is of maximal rank with system P. In fact, what the proposition asserts is that any $\mathbf{a}$ generated by a one-dimensional subspace of $\mathbf{m}^{(n_1,n_2)}$ will have system P. Suppose P is a system, and let A be as in 2.3. Then $\mathbf{a} \in A \Rightarrow \mathbf{a} = \oplus \mathbf{a}^\alpha$; $\dim \mathbf{a}^\alpha = 0$ for $|\alpha| < n$ and for $n \leq |\alpha| \leq p$ with $\alpha \neq (n_1 + i, n_2 + j)$ for $i, j \in \{0, 1, 2\}$, while $\dim \mathbf{a}^{(n_1,n_2)} = 1$. Thus any $\mathbf{a} \in A$ satisfies $\mathbf{a} \cap \mathbf{m}(2,n) = \mathbf{K}E$ for some $E \in \mathbf{m}^{(n_1,n_2)}$.

(i) Let $\mathbf{a} = \mathbf{K}E$ for any nonzero $E \in \mathbf{m}^{(n_1,n_2)}$. Then since $n = p$, $\mathbf{a}$ is an ideal in A. Conversely, any $\mathbf{a}$ in A must be of the form $\mathbf{K}E$ for some nonzero $E \in \mathbf{m}^{(n_1,n_2)}$. Thus $A \simeq Gr_1(\mathbf{m}^{(n_1,n_2)}) \simeq Gr_1(\mathbf{K}^{m(n_1,n_2)}) \simeq \mathbf{K}P^{m(n_1,n_2)-1}$, and $A^\sim \simeq \mathbf{K}P^{m-1}/\mathbf{G}$ (for $m = m(n_1, n_2)$).

(ii) Let $\mathbf{a}$ be the ideal generated by $E \in \mathbf{m}^{(n_1,n_2)}$. Then by 3.1, $\dim \mathbf{a}^\alpha = 1$ for $\alpha = (n_1+1, n_2)$ or (n_1, n_2+1). Hence P is a system. Any $\mathbf{a} \in A$ must be generated by a nonzero $E \in \mathbf{m}^{(n_1,n_2)}$, so once again $A \simeq \mathbf{K}P^{m-1}$.

(iii) If $\mathbf{a}$ is the ideal generated by a nonzero element of $\mathbf{m}^{(n_1,n_2)}$, then 3.1 gives us $\dim \mathbf{a}^\alpha = 1$ for $\alpha = (n_1+1, n_2), (n_1, n_2+1), (n_1+2, n_2)$, and (n_1, n_2+2), while 5.5 gives us $\dim \mathbf{a}^{(n_1+1,n_2+1)} = 2$. Thus P is a system and any $\mathbf{a} \in A$ must be generated by a one-dimensional subspace of $\mathbf{m}^{(n_1,n_2)}$, i.e., by a nonzero vector in $\mathbf{m}^{(n_1,n_2)}$. Therefore, $A^\sim \simeq \mathbf{K}P^{m-1}/\mathbf{G}$ as above. □

Corollary 5.7. *Let P be as in the proposition. If $n_1 \neq n_2$, then the symmetry group $\mathbf{G}$ is trivial (and $n_1 = n_2 \Rightarrow \mathbf{G} = \mathbf{Z}_2$).*

PROOF. If $n_1 \neq n_2$, then P contains the pairs $((n_1, n_2), m(n_1, n_2) - 1)$ and $((n_2, n_1), m(n_1, n_2))$ since $m(n_2, n_1) = m(n_1, n_2)$. □

We now present an example which is a combination of (i) and (ii) in 5.6.

Example 5.8. The classifying space for MRN quotients of $\mathbf{m}(2,5)$ having system (equivalent to)

$$P = P\mathbf{m}(2,3) \cup \{((2,2),1),((1,3),1),((3,2),1),((2,3),1),((1,4),1)\}$$

is $\mathbf{K}P^1$:

Any $\mathbf{a} \in A$ satisfies $\mathbf{a} = \mathbf{a}^{(3,1)} \oplus \mathbf{a}^{(4,1)} \oplus \mathbf{a}^{(3,2)} \oplus \mathbf{a}^{(2,3)}$, a direct sum of four one-dimensional subspaces, generated by $\mathbf{a}^{(3,1)} = \mathbf{m}^{(3,1)}$ and a one-dimensional subspace $\mathbf{a}^{(2,3)}$ of $\mathbf{m}^{(2,3)}$. $\mathbf{a}^{(4,1)} = [\mathbf{m}^{(1,0)}, \mathbf{a}^{(3,1)}] = \mathbf{m}^{(4,1)}$ and $\mathbf{a}^{(3,2)} = [\mathbf{m}^{(0,1)}, \mathbf{a}^{(3,1)}]$. Since any such $\mathbf{a}$ is an ideal, P is a system. Also, since any such $\mathbf{a}$ is in A, then $A = \{\mathbf{m}^{(3,1)}\} \times Gr_1(\mathbf{m}^{(2,3)}) \simeq \{\text{point}\} \times \mathbf{K}P^1 \simeq \mathbf{K}P^1$. Here the symmetry group $\mathbf{G}$ is trivial (since, e.g., $((3,1), d(3,1)) \notin P$). Thus $A^\sim \simeq \mathbf{K}P^1$.

We now discuss two fairly simple situations in which the set A is a Grassmanian variety.

Proposition 5.9. *Let $m = m(n_1, n_2)$ where $n = n_1 + n_2 > 4$, and let $k \in \mathbf{Z}$ be such that $2 \leq k \leq m$. Let P be the weight system $\{(\alpha, d\alpha) : \alpha \in R\mathbf{m}(2,p)\}$ where*

$$\text{(i)} \qquad p = n \qquad \text{and} \quad d\alpha = \begin{cases} m - k & \text{for } \alpha = (n_1, n_2), \\ m\alpha & \text{otherwise;} \end{cases}$$

or

$$\text{(ii)} \qquad p = n+1 \quad \text{and} \quad d\alpha = \begin{cases} m\alpha - k & \text{for } \alpha = (n_1, n_2), (n_1+1, n_2), \\ & \qquad (n_1, n_2+1), \\ m\alpha & \text{otherwise.} \end{cases}$$

Then $A^\sim \simeq Gr_k(\mathbf{K}^m)/\mathbf{G}$ for $\mathbf{G} < \mathbf{Z}_2$; unless $n_1 = n_2$, the symmetry group $\mathbf{G}$ is trivial.

PROOF. P is a weight system since if we let $\mathbf{a}$ be an ideal generated by any k–dimensional subspace of $\mathbf{m}^{(n_1,n_2)}$, $\mathbf{m}/\mathbf{a}$ will be MRN of step p with weight system P.

For any $\mathbf{a} \in A$, we need $\dim \mathbf{a}^\alpha = 0$ for $|\alpha| < n$ and for $n \leq |\alpha| \leq p$ with $\alpha \neq (n_1 + i, n_2 + j)$ for $i, j \in \{0,1\}$. Also, $\dim \mathbf{a}^{(n_1,n_2)} = k$ and $\mathbf{a} \cap \mathbf{m}(2,n) = \mathbf{a}^{(n_1,n_2)}$.

(i) For $p = n$, $\mathbf{a} = \mathbf{a}^{(n_1,n_2)} =$ any k–dimensional subspace of $\mathbf{m}^{(n_1,n_2)}$, and $A = Gr_k(\mathbf{m}^{(n_1,n_2)}) \simeq Gr_k(\mathbf{K}^m)$.

(ii) For $p = n+1$, we get $\dim \mathbf{a}^\alpha = k$ for $\alpha = (n_1+1, n_2)$ or (n_1, n_2+1); indeed $\mathbf{a}^{(n_1+1,n_2)} = [\mathbf{m}^{(1,0)}, \mathbf{a}^{(n_1,n_2)}]$ and $\mathbf{a}^{(n_1,n_2+1)} = [\mathbf{m}^{(0,1)}, \mathbf{a}^{(n_1,n_2)}]$, and $\mathbf{a} = \mathbf{a}^{(n_1,n_2)} \oplus \mathbf{a}^{(n_1+1,n_2)} \oplus \mathbf{a}^{(n_1,n_2+1)}$. Since $\mathbf{a}^{(n_1+1,n_2)}$ and $\mathbf{a}^{(n_1,n_2+1)}$ are completely determined by $\mathbf{a}^{(n_1,n_2)}$, once again we have $A \simeq Gr_k(\mathbf{K}^m)$.

Note that if $n_1 \neq n_2$, P is not symmetric, so $A^\sim = Gr_k(\mathbf{K}^m)$. □

Our next proposition lists six situations in which the set A is the product of two projective spaces. See Figures 4 – 9 for digraphs of ideals $\mathbf{a} = \oplus(\mathbf{a} \cap \mathbf{m}^\alpha)$ such that $\mathbf{m}/\mathbf{a}$ has the system P described in (i) – (vi) respectively.

Proposition 5.10. *Let $(n_1, n_2), (n_1', n_2') \in R\mathbf{m}$ be such that $n_1 \geq n_1'$ with $m = m(n_1, n_2) \geq 1$ and $m' = m(n_1', n_2') \geq 1$, $n = n_1 + n_2 > 2$, $n' = n_1' + n_2' > 2$, and $(n_1, n_2) \neq (n_1', n_2')$. Let P be the weight system $\{(\alpha, d\alpha) : \alpha \in R\mathbf{m}(2,p)\}$ where either*

(i) $p = n = n'$ *and*

$$d\alpha = \begin{cases} m\alpha - 1 & \text{for } \alpha = (n_1, n_2), (n_1', n_2'), \\ m\alpha & \text{otherwise;} \end{cases}$$

(ii) $p = n + 1$, $n = n'$ *with* $n_2 + 1 < n_2'$, *and*

$$d\alpha = \begin{cases} m\alpha - 1 & \text{for } \alpha = (n_1, n_2), (n_1', n_2'), (n_1 + 1, n_2), \\ & \qquad (n_1, n_2 + 1), (n_1' + 1, n_2'), (n_1', n_2' + 1), \\ m\alpha & \text{otherwise;} \end{cases}$$

(iii) $p = n + 2$, $n = n'$ *with* $n_2 + 2 < n_2'$, *and*

$$d\alpha = \begin{cases} m\alpha - 1 & \text{for } \alpha = (n_1, n_2), (n_1', n_2'), (n_1 + 1, n_2), \\ & \qquad (n_1, n_2 + 1), (n_1' + 1, n_2'), (n_1', n_2' + 1), \\ & \qquad (n_1 + 2, n_2), (n_1, n_2 + 2), (n_1' + 2, n_2'), \\ & \qquad (n_1', n_2' + 2), \\ m\alpha - 2 & \text{for } \alpha = (n_1 + 1, n_2 + 1), (n_1' + 1, n_2' + 1), \\ m\alpha & \text{otherwise;} \end{cases}$$

(iv) $p = n = n' + 1$ *with* $n_2 < n_2'$, and

$$d\alpha = \begin{cases} m\alpha - 1 & \text{for } \alpha = (n_1', n_2'), (n_1, n_2), (n_1' + 1, n_2'), \\ & \qquad (n_1', n_2' + 1), \\ m\alpha & \text{otherwise;} \end{cases}$$

(v) $p = n + 1 = n' + 2$ *with* $n_2 + 1 < n_2'$, and

$$d\alpha = \begin{cases} m\alpha - 1 & \text{for } \alpha = (n_1', n_2'), (n_1, n_2), (n_1' + 1, n_2'), \\ & \qquad (n_1', n_2' + 1), (n_1 + 1, n_2), (n_1, n_2 + 1), \\ & \qquad (n_1' + 2, n_2'), (n_1', n_2' + 2); \\ m\alpha - 2 & \text{for } \alpha = (n_1' + 1, n_2' + 1), \\ m\alpha & \text{otherwise;} \end{cases}$$

or

(vi) $p = n = n' + 2$ *with* $n_2 < n_2'$, and

$$d\alpha = \begin{cases} m\alpha - 1 & \text{for } \alpha = (n_1', n_2'), (n_1' + 1, n_2'), (n_1', n_2' + 1), \\ & \qquad (n_1, n_2), (n_1' + 2, n_2'), (n_1', n_2' + 2), \\ m\alpha - 2 & \text{for } \alpha = (n_1' + 1, n_2' + 1), \\ m\alpha & \text{otherwise.} \end{cases}$$

Then $A^{\sim} \simeq (\mathbf{K}P^m \times \mathbf{K}P^{m'})/\mathbf{G}$ *for* $\mathbf{G} < \mathbf{Z}_2$. *(Note that we need not separately consider the cases where* $n_1 < n_1'$ *and* $n_2 \geq n_2'$ *by equivalence of weight systems.)*

PROOF. P is a weight system since if we let $\mathbf{a}$ be an ideal generated by any one-dimensional subspace of $\mathbf{m}^{(n_1,n_2)}$ plus any one-dimensional subspace of $\mathbf{m}^{(n_1',n_2')}$, then $\mathbf{m}/\mathbf{a}$ will be MRN of step p with weight system P.

Figure 4. System $P(i)$.

Figure 5. System $P(ii)$.

Figure 6. System $P(iii)$.

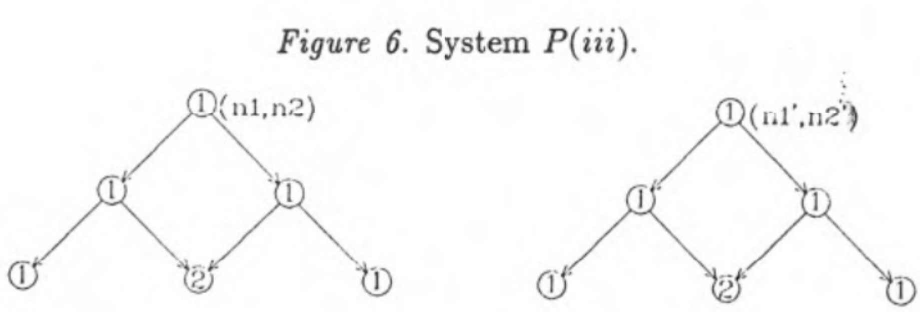


Figure 7. System $P(iv)$.

Figure 8. System $P(v)$.

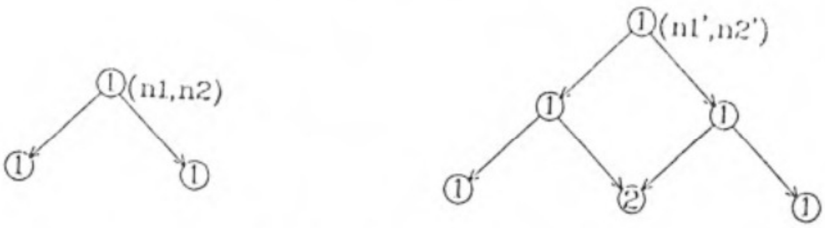
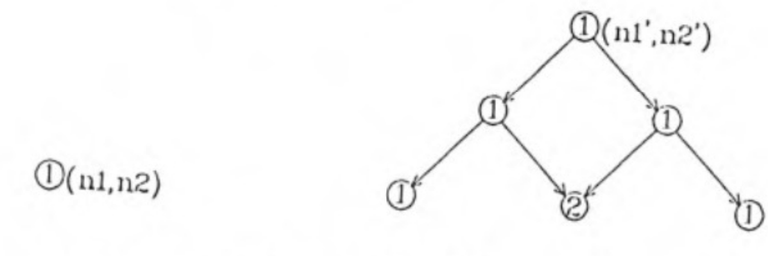


Figure 9. System $P(vi)$.



Let $\mathbf{a} \in A$. Then $\mathbf{a}^{(n_1,n_2)}$ and $\mathbf{a}^{(n_1',n_2')}$ are each one-dimensional (and generate $\mathbf{a}$); moreover, $[\mathbf{m}, \mathbf{a}^{(n_1,n_2)}] \cap [\mathbf{m}, \mathbf{a}^{(n_1',n_2')}] = \{0\}$.
$([C^k(\mathbf{m}), \mathbf{a}^{(n_1,n_2)}] = \{0\}$ if $k > 2$, and likewise for $\mathbf{a}^{(n_1',n_2')}$, since n (and n') $\geq p-2$.) Thus each generating space gives rise to a projective space as a component of A, subject to no additional conditions. $\square$

Corollary 5.11. *Unless $n = n'$ and $n_1' = n_2'$, the symmetry group* $\mathbf{G}$ *is trivial under the conditions of 5.10.*

We now turn our attention to a weight system whose corresponding classifying space is particulary interesting.

Example 5.12. The classifying space for MRN quotients of $\mathbf{m}(2,6)$ having system $P = P\mathbf{m}(2,4) \cup \{((4,1),1),((3,2),2),((2,3),1),((1,4),1), ((5,1),1),((4,2),2),((3,3),1),((1,5),1)\}$ is $\mathbf{K}P^2$ *blown up at a point:*

Any $\mathbf{a} \in A$ would have to satisfy $\mathbf{a} = \mathbf{a}^{(2,3)} \oplus \mathbf{a}^{(3,3)} \oplus \mathbf{a}^{(2,4)}$ with $\dim \mathbf{a}^{(2,3)} = 1$ and $\dim \mathbf{a}^{(3,3)} = 2 = \dim \mathbf{a}^{(2,4)}$. Thus $\mathbf{a}^{(2,4)} = \mathbf{m}^{(2,4)}$ and we see $A \subseteq Gr_1(\mathbf{m}^{(2,3)}) \times Gr_2(\mathbf{m}^{(3,3)}) \times \{\mathbf{m}^{(2,4)}\}$. Also, $\mathbf{G}$ is trivial, so $A^{\sim} = A$.

Now any $\mathbf{a} = \mathbf{a}^{(2,3)} \oplus \mathbf{a}^{(3,3)} \oplus \mathbf{a}^{(2,4)}$ with dimensions as above will be an ideal if and only if $[\mathbf{m}^{(1,0)}, \mathbf{a}^{(2,3)}] \subseteq \mathbf{a}^{(3,3)}$; since such an $\mathbf{a}$ can be chosen, P is a system. Thus we can take $A^{\sim} = \{(V_1, V_2, V_3) \in Gr_1(\mathbf{m}^{(2,3)}) \times Gr_2(\mathbf{m}^{(3,3)}) \times \{\mathbf{m}^{(2,4)}\} : \tau(V_1) \subseteq V_2\}$, where $\tau = \operatorname{ad} E_1$ (restricted to $\mathbf{m}^{(2,3)}$), which implies $A^{\sim} \simeq \{(V_1, V_2) \in \mathbf{K}P^1 \times Gr_2(\mathbf{K}^3) : \tau(V_1) \subseteq V_2\}$. In the latter description, we can define τ (using homogeneous coordinates and a Hall basis for $\mathbf{m}(2,6)$ — see [3] or [4]) by $\tau(a_1, a_2) = (a_1, 2a_1 + a_2, a_2 - a_1)$. Now τ identifies $\mathbf{m}^{(2,3)}$ with a fixed plane in the three–dimensional space $\mathbf{m}^{(3,3)}$, so after changing bases, we may assume $\tau : \mathbf{K}^2 \to \mathbf{K}^3$ with $\tau(x_1, x_2) = (0, -x_2, x_1)$. In place of a plane V_2 in $\mathbf{K}^3$, suppose we take a line orthogonal to V_2, i.e., replace $V_2 \in Gr_2(\mathbf{K}^3)$ with $(y_0, y_1, y_2) \in \mathbf{K}P^2$. Because τ is homogeneous, it defines a map $\mathbf{K}P^1 \to Gr_1(\mathbf{K}^3)$. Since we want $\tau(x_1, x_2) \subseteq V_2$, we need (y_0, y_1, y_2) and $\tau(x_1, x_2)$ to be orthogonal, i.e., we need $0 = (0, -x_2, x_1) \cdot (y_0, y_1, y_2) = -x_2 y_1 + x_1 y_2$. Thus we have
$A^{\sim} \simeq \{(x_1, x_2; y_0, y_1, y_2) \in \mathbf{K}P^1 \times \mathbf{K}P^2 : x_1 y_2 = x_2 y_1\}$. The projection map $A^{\sim} \to \mathbf{K}P^2$ is exactly what Shafarevich calls a *σ–process* in [5], i.e., the map $\mathbf{K}P^2 \to A^{\sim}$ which takes (y_0, y_1, y_2) to $(y_1, y_2; y_0, y_1, y_2)$ for $(y_0, y_1, y_2) \neq (1,0,0)$ and to $(x_1, x_2; 1, 0, 0)$ for arbitrary x_1, x_2 otherwise lets us realize $A^{\sim}$ as $\mathbf{K}P^2$ blown up at a point.

Our final example concerns a weight system whose corresponding MRN algebras are usually generated by two nonzero vectors of total degree 5. It is, however, possible to choose two vectors in such a way that they do not generate an algebra with the given system; in this case, an additional choice (of a vector of total degree 6) must be made.

Example 5.13. The classifying space for MRN quotients of $\mathbf{m}(2,6)$ having system (equivalent to) $P = P\mathbf{m}(2,4) \cup \{(\alpha, 1) : |\alpha| = 5 \text{ or } 6\}$ is $(\mathbf{K}P^2$ *blown up at two points*$)/\mathbf{Z}_2$:

Assuming P is indeed a system, any $\mathbf{a} \in A$ would satisfy $\mathbf{a} = \mathbf{a}^{(3,2)} \oplus \mathbf{a}^{(2,3)} \oplus \mathbf{a}^{(4,2)} \oplus \mathbf{a}^{(3,3)} \oplus \mathbf{a}^{(2,4)}$, where $\dim \mathbf{a}^{(3,2)} = 1 = \dim \mathbf{a}^{(2,3)}$, $\dim \mathbf{a}^{(4,2)} = 1 = \dim \mathbf{a}^{(2,4)}$, and $\dim \mathbf{a}^{(3,3)} = 2$. If we choose one-dimensional subspaces $\mathbf{a}^{(3,2)}$ and $\mathbf{a}^{(2,3)}$, then $\mathbf{a}^{(4,2)} = [\mathbf{m}^{(1,0)}, \mathbf{a}^{(3,2)}]$, and $\mathbf{a}^{(2,4)} = [\mathbf{m}^{(0,1)}, \mathbf{a}^{(2,3)}]$; $[\mathbf{m}^{(1,0)}, \mathbf{a}^{(2,3)}]$ and $[\mathbf{m}^{(0,1)}, \mathbf{a}^{(3,2)}]$ are both one-dimensional subspaces of

$\mathbf{a}^{(3,3)}$. In fact, for every choice of subspaces $\mathbf{a}^{(3,2)}$ and $\mathbf{a}^{(2,3)}$ *except one*, the space $[\mathbf{m}^{(1,0)}, \mathbf{a}^{(2,3)}]+[\mathbf{m}^{(0,1)}, \mathbf{a}^{(3,2)}]$ is two-dimensional, i.e., it is $\mathbf{a}^{(3,3)}$. In any case, $A \subseteq Gr_1(\mathbf{m}^{(3,2)}) \times Gr_1(\mathbf{m}^{(2,3)}) \times Gr_2(\mathbf{m}^{(3,3)}) \simeq \mathbf{K}P^1 \times \mathbf{K}P^1 \times Gr_2(\mathbf{K}^3)$, and more specifically,
$A \simeq \{(V_1, V_2, V_3) \in \mathbf{K}P^1 \times \mathbf{K}P^1 \times Gr_2(\mathbf{K}^3) : \tau(V_1, V_2) \subseteq V_3\}$ where $\tau(V_1, V_2) = \text{span}\,\{\text{ad}\ E_2(V_1),\ \text{ad}\ E_1(V_2)\}$, or using homogeneous coordinates and a Hall basis,
$\tau(a_1, a_2; a_3, a_4) = \text{span}\{(a_1, a_2, -a_2), (a_3, 2a_3 + a_4, a_4 - a_3)\}$. (Note that $\tau : \mathbf{K}P^1 \times \mathbf{K}P^1 \to Gr_1(\mathbf{K}^3) \cup Gr_2(\mathbf{K}^3)$.) We see that $\dim \tau(V_1, V_2) = 1$ if and only if $V_1 = (2, 3)$ and $V_2 = (2, -1)$ (in homogeneous coordinates).

Now since P is symmetric in α_1 and α_2, the symmetry group $\mathbf{G}$ is $\mathbf{S}_2 \simeq \mathbf{Z}_2$. One can check that $(12)\,.\,\text{span}\{(2,3,-3),(b_1,b_2,b_3)\} = \text{span}\{(2,3,-3),(-b_1, b_2 - 3b_1, -b_3)\}$, which is two-dimensional precisely when $\text{span}\{(2,3,-3),(b_1,b_2,b_3)\}$ is; when $V_1 = (2,3)$ and $V_2 = (2,-1) \in \mathbf{K}P^1$, $(1\ 2)\,.\,V_1 = (-2,-3) = V_1$ and $(1\ 2)\,.\,V_2 = (-2,1) = V_2$.

Since the image of $\mathbf{m}^{(3,2)}$ under ad E_2 and the image of $\mathbf{m}^{(2,3)}$ under ad E_1 are each fixed planes in the three-dimensional space $\mathbf{m}^{(3,3)}$, suppose we change bases so that ad $E_1 : \mathbf{K}^2 \to \mathbf{K}^3$ takes (x_1, x_2) to $(0, -x_2, x_1)$ and ad $E_2 : \mathbf{K}^2 \to \mathbf{K}^3$ takes (x_3, x_4) to $(-x_4, 0, x_3)$. Then we see that span $\{(0, -x_2, x_1), (-x_4, 0, x_3)\}$ is two-dimensional unless $x_1 = 0 = x_3$ (assuming $(x_1, x_2) \neq (0,0) \neq (x_3, x_4)$).

Now to represent the plane V_3, choose a line in $\mathbf{K}^3$, i.e., $(y_0, y_1, y_2) \in \mathbf{K}P^2$, which is orthogonal to V_3. Then we have $(0, -x_2, x_1) \in V_3$ if and only if $(0, -x_2, x_1)$ and (y_0, y_1, y_2) are orthogonal, i.e., $x_1 y_2 - x_2 y_1 = 0$. Likewise $(-x_4, 0, x_3) \in V_3$ if and only if $x_3 y_2 - x_4 y_0 = 0$. Thus $A \simeq \{(x_1, x_2; x_3, x_4; y_0, y_1, y_2) \in \mathbf{K}P^1 \times \mathbf{K}P^1 \times \mathbf{K}P^2 : x_1 y_2 = x_2 y_1 \text{ and } x_3 y_2 = x_4 y_0\}$. We have (the restriction of) the projection map $A \to \mathbf{K}P^2$, and the map $\mathbf{K}P^2 \to A$ which maps (y_0, y_1, y_2) to $(y_1, y_2; y_0, y_2; y_0, y_1, y_2)$ if $(y_0, y_1, y_2) \neq (1,0,0)$ or $(0,1,0)$, and maps $(1,0,0)$ to $(x_1, x_2; 1, 0; 1, 0, 0)$ and $(0,1,0)$ to $(1, 0; x_3, x_4; 0, 1, 0)$. This lets us realize A as $\mathbf{K}P^2$ blown up at the two points $(1,0,0)$ and $(0,1,0)$.

6. Conclusion

For each positive integer s, we can consider any weight system for nilpotent Lie algebras of maximal rank s to be a set of $(s+1)$-tuples of nonnegative integers. (For fixed step p, we can even think of each MRN quotient of $\mathbf{m}(s,p)$ as an n-tuple of nonnegative integers for sufficiently large n.) To each such system we associate an algebraic variety, which may be a single point, a projective space, a product of projective spaces, a Grassmanian variety, a projective space blown up at one or more points, etc. Thus one should be able to define a function from the collection of all systems (s and p fixed) to the collection of all varieties, mapping each system to its classifying space. What is not clear, however, is whether or not such a function can be obtained without looking at each system individually. Of course, this question may be premature since we are as yet unable to provide sufficient conditions for a set of $(s+1)$–tuples to be a weight system.

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