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Cauchy and Jensen equations on a restricted domain almost everywhere

By JÓZEF TABOR (Cracow)

Abstract. In the paper we consider the Cauchy equation $f(x+y)=f(x)+f(y)$, where the equality is postulated for "almost all" pairs (x,y) in the triangle $x \in P$, $y \in P$, $x+y \in P$. Under suitable assumptions we obtain an analogue of de Bruijn's theorem. In the similar way the Jensen equation is investigated. We prove, under appropriate assumptions, that if f satisfies the equation $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for "almost all" pairs (x,y) $T \times T$, then f is of the form $f(x)=h(x)+c$ for "almost all" x in T, where h is an additive mapping and c is a constant.

Let X, Y be groups (written additively) and P a subset of X. Under appropriate assumptions on X, Y and P every function $f : P \to Y$ satisfying the condition

$$(1) \qquad f(x+y) = f(x) + f(y) \quad \text{for } x, y \in P,\ x+y \in P$$

can be uniquely extended to the homomorphism $h : X \to Y$ (cf. [2] V §1.4; VII §2.2; [3] Theorem 4.3; [4] Theorems 13.5.3, 13.6.2). On the other hand if $f : X \to Y$ satisfies the equation

$$f(x+y) = f(x) + f(y)$$

"almost everywhere", then there exists a unique homomorphism $h : X \to Y$ such that $f(x) = g(x)$ "almost everywhere" (cf. [4] XVII §6). "Almost everywhere" means here everywhere except elements of some "small" set. We are going to generalize simultanously some results of these two types. Namely, we shall be considering equation (1) almost everywhere. In the similar way also the Jensen equation will be considered.

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1. Introduction

Let $(X, +)$ be a group (not necessarily commutative). A non-empty family $\mathcal{I}$ of subset of X is called a proper linearly invariant ideal (shortly p.l.i. ideal) iff it satisfies the following conditions (cf. [4] p. 437):

(i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$;

(ii) If $A, B \in \mathcal{I}$, then $A \subset B \in \mathcal{I}$;

(iii) $X \notin \mathcal{I}$;

(iv) For every $x \in X$ and $A \in \mathcal{I}$ the set $x - A$ belongs to $\mathcal{I}$.

If condition (ii) is replaced by a stronger condition

$$\text{(ii}'\text{)} \qquad \text{If } A_n \in \mathcal{I},\ n \in N, \text{ then } \bigcup_{n=1}^{\infty} A_n \in \mathcal{I},$$

then $\mathcal{I}$ is called a proper linearly invariant σ–ideal (p.l.i. σ ideal).
If we are given a p.l.i. ideal in X, then we say that a condition is satisfied $\mathcal{I}$–almost everywhere in $X(\mathcal{I}$–(a.e)) iff there exists a set $A \in \mathcal{I}$ such that the condition in question is satisfied for every $x \in X \backslash A$.
Given a p.l.i. ideal (σ–ideal) in X we put

$$\Omega(\mathcal{I}) := \left\{ M \subset X^2 : \underset{U(M)\in\mathcal{I}}{\exists} \ \underset{x\in X\backslash U(M)}{\forall} M[x] := \{y \in X : (x,y) \in M\} \in \mathcal{I} \right\}.$$

$\Omega(\mathcal{I})$ is a p.l.i. ideal ($\sigma -$ ideal) in $X \times X$.

Let us assume that $(X, +)$ is a group in which division by 2 is uniquely performable. A set $A \subset X$ is called convex in the Jensen sense (J–convex) iff it satisfies the following condition (cf. [4] p. 111): If $x, y \in A$, then $\frac{x+y}{2} \in A$.

2. Cauchy equation

Denote by N_0 the set of non negative integers and by Z the set of integers. We shall consider the following hypotheses:

(H_1) $(X, +)$ is an abelian group in which division by 2 is uniquely performable, $\mathcal{I}$ is a σ p.l.i. ideal in X satisfying the following condition:

(v) If $A \in \mathcal{I}$, then $\frac{1}{2}A,\ 2A \in \mathcal{I}$.

(H_2) $S \subset X$ is a subsemigroup of X generating X, $S \notin \mathcal{I}$.

(H_3) $P \subset X$ is a subset of X such that

$$\frac{1}{2}P \subset P \qquad \mathcal{I}-\text{(a.e)},$$
$$S \subset \bigcup_{n \in N_0} 2^n P \qquad \mathcal{I}-\text{(a.e)},$$
$$P_1 := \{2x \in P : P \cap (P-x) \cap (P-2x) \in \mathcal{I}\} \in \mathcal{I}.$$

(H_4) $M \in \Omega(\mathcal{I}),\quad M^{-1} := \{(x,y) : (y,x) \in M\} \in \Omega(\mathcal{I})$.

(H_5) $(Y,+)$ is a group.

(H_6) $f : P \to Y$.
$f(x+y) = f(x) + f(y)$ for $(x,y)\ P^2 \setminus M,\quad x+y \in P$.

Let $A \subset X$ $(A \subset X \times X)$. We shall use the following denotation:

$$A^\star := \bigcup_{n \in \mathbf{Z}} 2^n A.$$

We start our considerations with some auxilliary lemmas.

Lemma 1. *If* (H_1) *holds and* $A \in \mathcal{I}$, *then* $A^\star \in \mathcal{I}$.

PROOF. Obvious.

Lemma 2. *If* (H_1) *holds and* $M \in \Omega(\mathcal{I})$, *then* $M^\star \in \Omega(\mathcal{I})$.

PROOF. Consider an $x \in X \setminus [U(M)]^\star$. We have

$$M^\star[x] = \{y \in X : (x,y) \in M^\star\} = \left\{ y \in X : (x,y) \in \bigcup_{n \in Z} 2^n M \right\} =$$
$$= \left\{ y \in X : \underset{n \in \mathbf{Z}}{\exists} \left(\frac{x}{2^n}, \frac{y}{2^n}\right) \in M \right\} =$$
$$= \left\{ y \in X : \underset{n \in \mathbf{Z}}{\exists} \frac{y}{2^n} \in M\left[\frac{x}{2^n}\right] \right\} = \bigcup_{n \in \mathbf{Z}} 2^n M\left[\frac{x}{2^n}\right].$$

But $\frac{x}{2^n} \in X \setminus [U(M)]^\star$ since $x \in X \setminus [U(M)]^\star$. Hence $M\left[\frac{x}{2^n}\right] \in \mathcal{I}$ for $n \in \mathbf{Z}$. Making use of (v) and of the fact that $\mathcal{I}$ is a σ ideal we obtain that $M^\star[x] \in \mathcal{I}$. Thus $M^\star \in \Omega(\mathcal{I})$.

Lemma 3. *If $(H_1),(H_2),(H_3),(H_4),(H_5)$ and (H_6) hold, then there exists a $W \in \mathcal{I}$ such that*

$$f(2^n x) = 2^n f(x) \text{ for } x \in \left(\frac{1}{2^n}P\right)\setminus W, \qquad n \in N_0. \tag{2}$$

Proof. It follows from (H_3) that there exists an $A \in \mathcal{I}$ such that

$$\left(\frac{1}{2}P\right)\setminus A \subset P. \tag{3}$$

We put

$$W := [U(M) \cup A \cup P_1]^\star. \tag{4}$$

Of course $W \in \mathcal{I}$. The proof of (2) runs by induction. Obviously (2) holds for $n = 0$. Take $n = 1$ and consider an $x \in \left(\frac{1}{2}P\right)\setminus W$. Then $x \notin [U(M)]^\star$ and $2x \notin [U(M)]^\star$. Hence $M[x] \in \mathcal{I}$ and $M[2x] \in \mathcal{I}$. Thus

$$M[x] \cup M[2x] \cup (-x + M[x]) \in \mathcal{I}.$$

Since $x \notin P_1$ (as $x \notin [P_1]^\star$), we have

$$(P - x) \cap (P - 2x) \cap P \notin \mathcal{I},$$

and consequently

$$((P - x) \cap (P - 2x) \cap P) \setminus (M[x] \cup M[2x] \cup (-x + M[x]) \neq \emptyset.$$

Take a y from this set. Then $(x, y) \in P^2\setminus M$ and $x + y \in P$ (as $y \in P - x$), $(2x, y) \in P^2\setminus M$ and $(2x+y) \in P$, $(x, x+y) \in P^2\setminus M$ and $x+x+y \in P$. These conditions and (H_6) imply the following equalities:

$$\begin{aligned} f(x + y) &= f(x) + f(y), \\ f(2x + y) &= f(2x) + f(y), \\ f(2x + y) &= f(x) + f(x + y). \end{aligned}$$

Subtracting the second equality from the third one and substituting the first equality to the resulting equality we get

$$f(2x) = 2f(x).$$

Assume now (2) to hold for an $n \in N$ and consider an $x \in \left(\frac{1}{2^{n+1}}P\right)\setminus W$. Then $2^n x \in \frac{1}{2}P\setminus W$, and hence we have by the first part of the proof and by the induction hypothesis:

$$f(2\,2^n x) = 2f(2^n x) = 2^{n+1} f(x).$$

Induction completes the proof.

The following theorem is the main result of this section.

Theorem 1. *If $(H_1) - (H_6)$ holds, then there exists a unique homomorphism $h : X \to Y$ such that*

$$f = h \quad \mathcal{I} - (a.e) \text{ in } P \cap S^\star. \tag{5}$$

If additionally

$$E := \{x \in P \backslash S^\star : P \cap (P - x) \cap S^\star \cap (S^\star - x) \in \mathcal{I}\} \in \mathcal{I},$$

then

$$f = h \quad \mathcal{I} - (a.e) \text{ in } P. \tag{6}$$

Proof. Since $S \subset \bigcup_{n \in N_0} 2^n P \quad \mathcal{I}$-(a.e), there exists a set $B \in \mathcal{I}$ such that

$$S \backslash B \subset \bigcup_{n \in N_0} 2^n P. \tag{7}$$

We put

$$W_1 := [W \cup B]^\star, \tag{8}$$

where W is defined by (4). By Lemma 1 $W_1 \in \mathcal{I}$. We define now a function $g : S \to Y$. Consider an $x \in S \backslash W_1$. Then there exists an $n \in N_0$ such that $x \in 2^n P$ i.e. $\frac{x}{2^n} \in P$. We set

$$g(x) := 2^n f\left(\frac{x}{2^n}\right). \tag{9}$$

For $x \in S \cap W_1$ $g(x)$ is defined arbitrarily. We must prove that g is "well defined" i.e. that $g(x)$ does not depend on the choice of n. Consider an $x \in S \backslash W_1$ and suppose that $x \in 2^n P$ and $x \in 2^m P$ with $m, n \in N_0$. Then $\frac{x}{2^n} \in P$ and $\frac{x}{2^m} \in P$, whence $\frac{x}{2^{n+m}} \in \frac{1}{2^n} P$ and $\frac{x}{2^{n+m}} \in \frac{1}{2^m} P$. Furthermore $\frac{x}{2^{n+m}} \notin W$, since $x \notin W_1$ and $W \subset W_1$. Applying Lemma 3 we obtain

$$f\left(\frac{x}{2^m}\right) = f\left(2^n \frac{x}{2^{n+m}}\right) = 2^n f\left(\frac{x}{2^{n+m}}\right),$$

$$f\left(\frac{x}{2^m}\right) = f\left(2^m \frac{x}{2^{n+m}}\right) = 2^m f\left(\frac{x}{2^{n+m}}\right).$$

Thus

$$2^m f\left(\frac{x}{2^m}\right) = 2^{n+m} f\left(\frac{x}{2^{n+m}}\right) = 2^n f\left(\frac{x}{2^n}\right),$$

which means that g is "well defined". By the definition of g we have

$$f(x) = g(x) \quad \text{for } x \in P \backslash W_1. \tag{10}$$

We shall prove that

$$g(x+y) = g(x) + g(y, \qquad \Omega(\mathcal{I}) - \text{ (a.e) in } S \times S. \tag{11}$$

Put

$$C := (W_1 \times X) \cup (X \times W_1) \cup \{(x,y) \in X \times X : x+y \in W_1\}$$

and consider an $x \in X \backslash W_1$. Then $C[x] = W_1 \cup (-x + W_1) \in \mathcal{I}$, and hence $C \in \Omega(\mathcal{I})$. Put

$$M_1 := M \cup M^{-1} \qquad C_1 = C \cup M_1^\star. \tag{12}$$

By (H_4) $M_1 \in \Omega(\mathcal{I})$ and so, by Lemma 2, $M_1^\star \in \Omega(\mathcal{I})$. Therefore $C_1 \in \Omega(\mathcal{I})$. Let us consider a pair $(x,y) \in (S \times S) \backslash C_1$. Then

$$x, \ y, \ x+y \in S \backslash W_1 \tag{13}$$

and

$$(x,y) \in (S \times S) \backslash M_1^\star. \tag{14}$$

It follows from (12), taking into account (7) and (8), that there exist $n_1, n_2, n_3 \in N_0$ such that $\frac{x}{2^{n_1}}, \frac{y}{2^{n_2}}, \frac{x+y}{2^{n_3}} \in P$, whence by (13)

$$\frac{x}{2^{n_1}}, \ \frac{y}{2^{n_2}}, \ \frac{x+y}{2^{n_3}} \in P \backslash W_1^\star = P \backslash W_1. \tag{15}$$

We get from (3)

$$\left(\frac{1}{2^n} P\right) \backslash A^\star \subset P \quad \text{for } n \in N_0.$$

But, by (4) and (8), $A^\star \subset W_1$, hence

$$\left(\frac{1}{2^n} P\right) \backslash W_1 \subset P \quad \text{for } n \in N_0. \tag{16}$$

Taking $n \geq \max(n_1, n_2, n_3)$ and making use of (15) and (16) we obtain

$$\frac{x}{2^n}, \ \frac{y}{2^n}, \ \frac{x+y}{2^n} \in P \backslash W_1. \tag{17}$$

We have by (12) and (14)

$$\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \notin M \quad \text{and} \quad \left(\frac{y}{2^n}, \frac{x}{2^n}\right) \notin M, \tag{18}$$

whence further, by (H_6) and (17), we get

$$\begin{aligned} f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) = f\left(\frac{x+y}{2^n}\right) = f\left(\frac{y+x}{2^n}\right) = \\ = f\left(\frac{y}{2^n}\right) + f\left(\frac{x}{2^n}\right), \end{aligned} \tag{19}$$

which means that $f\left(\frac{x}{2^n}\right)$ and $f\left(\frac{y}{2^n}\right)$ commute. Making use of (9), (17) and (19) we obtain

$$\begin{aligned} g(x+y) = 2^n f\left(\frac{x+y}{2^n}\right) = 2^n f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) = \\ = 2^n f\left(\frac{x}{2^n}\right) + 2^n f\left(\frac{y}{2^n}\right) = g(x) + g(y). \end{aligned}$$

Thus (11) is proven. By theorem of GER (cf. [4], p. 491) there exists a homomorphism $h : X \to Y$ such that

$$h\ S = g \qquad \mathcal{I} - \text{ (a.e) in } S.$$

This equality can be rewritten as

$$h(x) = g(x) \text{ for } x \in S \backslash D, \tag{20}$$

where $D \in \mathcal{I}$.

Obviously $W_1 \cup D^\star \in \mathcal{I}$. To prove (5) consider an $x \in (P \cap S^\star) \backslash (W \cup D^\star)$. Then $x \in P$ and, for some $n \in N_0$, $2^n x \in S$, $2^n x \notin W_1$, and $2^n x \notin D$. Making use of (20) and (9) (with x replaced by $2^n x$) we get

$$2^n h(x) = h(2^n x) = g(2^n x) = 2^n f(x).$$

Thus

$$h(x) = f(x) \text{ for } x \in (P \cap S^\star) \backslash (W_1 \cup D^\star), \tag{21}$$

i.e. (5) holds.

To prove (6) consider an $x \in P \backslash (S^\star \cup E \cup W_1)$. Then $M[x] \in \mathcal{I}$ since $x \in X \backslash W_1 \subset X \backslash U(M)$. We have further

$$P \cap (P - x) \cap S^\star \cap (S^\star - x) \backslash (W_1 \cup (W_1 - x) \cup D^\star \cup (D^\star - x) \cup M[x]) \in \mathcal{I},$$

and so there exists an element y in this set. Then

$$y \in (P \cap S^\star)\backslash(W_1 \cup D^\star), \qquad x + y \in (P \cap S^\star)\backslash(W_1 \cup D^\star),$$

$$(x, y) \in (X \times X)\backslash M.$$

Applying (21) and (H_6) we get

$$h(x) + h(y) = h(x + y) = f(x + y) = f(x) + f(y) = f(x) + f(y),$$

and hence $h(x) = f(x)$. Thus

$$h(x) = f(x) \quad \text{for } x \in P\backslash(S^\star \cup E \cup W_1).$$

Since $E \cup W_1 \in \mathcal{I}$, this equality and (5) imply (6). To prove uniqueness of h let us observe that, according to (H_3), there exists an $F \in \mathcal{I}$ such that $S\backslash F \subset \bigcup_{n \in N_0} 2^n P$. But $S \subset S^\star$, so

$$S\backslash F \subset \left(\bigcup_{n \in N_0} 2^n P\right) \cap S^\star = \bigcup_{n \in N_0} (2^n P) \cap S^\star = \bigcup_{n \in N_0} 2^n (P \cap S^\star),$$

which means that $P \cap S^\star$ generates $S\backslash F$. Since moreover $(S\backslash F) - (S\backslash F) = X$ (cf. [4], Lemma 17.5.6), hence $P \cap S^\star$ generates X. Taking into account (5) we obtain that h is unique. This completes the proof.

The assumption that $M^{-1} \in \Omega(\mathcal{I})$ was applied in the proof of Theorem 1 only once to prove that $f\left(\frac{x}{2^n}\right)$ and $f\left(\frac{y}{2^n}\right)$ commute. Therefore if $(Y, +)$ is abelian, then this assumption may be omitted.

Theorem 1 can be generalized. Condition (2) was used in the proof of this theorem only for $x \in S^\star$ (to prove that the definition of g is correct.) Hence it is sufficient to prove (2) for $x \in \left(\frac{1}{2^n} P\backslash W\right) \cap S^\star$. Therefore assumption (H_3) can be weakened. Consider the following hypothesis:

$(H_{3'})$ $P \subset X$ is a subset of X such that
$P_2 := \{2x \in P \cap S^\star : P \cap (P - x) \cap (P - 2x) \notin \mathcal{I}\}$,
$\frac{1}{2} P_2 \subset P_2 \qquad \mathcal{I} - \text{(a.e)}$,
$S \subset \bigcup_{n \in N_0} 2^n P_2$.

Denote

$$P_3 := \{x \in P : P_2 \cap (P_2 - x) \notin \mathcal{I}\}.$$

Modyfying a little bit the proof of theorem 1 one can prove the following theorem.

Theorem 2. *If* $(H_1), (H_2), (H_{3'}), (H_4), (H_5)$ *and* (H_6) *hold, then there exists a unique homomorphism* $h : X \to Y$ *such that*

$$f = h \qquad \mathcal{I}- \text{ (a.e) in } P_2 \cup P_3.$$

It is easy to observe that, under our denotations $P_2 = (P \cap S^\star)\backslash P_1$. Suppose that $(H_{3'})$ holds. Then $P_1 \in \mathcal{I}$, and hence

$$\frac{1}{2}P_2 \underset{\mathcal{I}-\text{ (a.e)}}{=} \frac{1}{2}(P \cap S^\star) = \left(\frac{1}{2}P\right) \cap S^\star \subset P \cap S^\star \underset{\mathcal{I}-\text{ (a.e)}}{=} P_2.$$

Similarly

$$\begin{aligned} S = S \cap S^\star \subset \left(\bigcup_{n\in N_0} 2^n P\right) \cap S^\star = \bigcup_{n\in N_0} 2^n (P \cap S^\star) \underset{\mathcal{I}-\text{ (a.e)}}{=} \\ = \bigcup_{n\in N_0} 2^n (P \cap S^\star) \backslash \bigcup_{n\in N_0} 2^n P_1 \subset \bigcup_{n\in N_0} 2^n [(P \cap S^\star)\backslash P_1] = \\ = \bigcup_{n\in N_0} 2^n P_2. \end{aligned}$$

Thus $(H_{3'})$ results from (H_3) (but not conversely). Furthermore, if (H_3) holds, then $P_2 = (P \cap S^\star)\backslash P_1 \underset{\mathcal{I}-\text{ (a.e)}}{=} P \cap S^\star$, which proves that the condition

$$f = h \qquad \mathcal{I}- \text{ (a.e) in } P_2 \cup P_3$$

is stronger than (5). So the assumptions of Theorem 2 are weaker and the assertion is stronger than of Theorem 1 respectively.

3. Jensen equation

Let us assume that hypotheses $(H_1), (H_2)$ are satisfied and consider the following further hypetheses:

$(H_{3''})$ $T \subset X$ is a $\mathcal{J}$-convex subset of X such that $T \in \mathcal{I}$,
$T_1 := \{(x, y) \in X \times X : x \in T \text{ and } y \in \frac{1}{2}(T - x) \text{ and } (T - x - 2y) \cap (T - x - y) \cap (T - x) \in \mathcal{I}\} \in \Omega(\mathcal{I})$
and

$$T_2 := \left\{ x \in T : X \subset \bigcup_{n\in N_0} 2^n (T - x) \right\} \notin \mathcal{I}.$$

$(H_{4'})$ $M \in \Omega(\mathcal{I})$.

(H_5) $(Y,+)$ is an abelian group in which division by 2 is umiquely performable.

$(H_{6'})$ $f : T \to Y$,

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \quad \text{for } (x,y) \in T \times T \backslash M.$$

The following theorem is an analogue of Theorem 1.

Theorem 3. *If* $(H_1), (H_{3''}), (H_{4'}), (H_{5'})$ *and* $(H_{6'})$ *hold, then there exist a unique homomorphism* $h : X \to Y$ *and a unique constant* $c \in Y$ *such that*

$$f(x) = h(x) + c \qquad \mathcal{I} - \text{(a.e) on } T. \tag{22}$$

PROOF. By $(H_{3''})$ and $(H_{4'})$ $T_2 \backslash (U(T_1) \cup U(M)) \notin \mathcal{I}$. Fix an $x_0 \in T_2 \backslash (U(T_1) \cup U(M))$ and consider the set

$$\begin{aligned} M_0 &:= \{(x,y) \in X \times X : x \in M[x_0] \text{ or } y \in M[x_0] \text{ or } (x,y) \in M \\ &= (M[x_0] \times X) \cup (X \times M[x_0]) \cup M. \end{aligned}$$

This means that $M_0 \in \Omega(\mathcal{I})$.
Now consider a pair $(x,y) \in (T \times T) \backslash M_0$. Then $(x_0, x) \in (T \times T) \backslash M$, $(x_0, y) \in (T \times T) \backslash M$, $(x,y) \in (T \times T) \backslash M$, whence, by $(H_{6'})$, we get

$$\begin{aligned} f\left(\frac{x_0+x}{2}\right) &= \frac{f(x_0)+f(x)}{2}, \\ f\left(\frac{x_0+y}{2}\right) &= \frac{f(x_0)+f(y)}{2}, \\ f\left(\frac{x+y}{2}\right) &= \frac{f(x)+f(y)}{2}. \end{aligned}$$

Subtracting from the last equality the first and the second one we obtain

$$f\left(\frac{x+y}{2}\right) - f\left(\frac{x+x_0}{2}\right) - f\left(\frac{y+x_0}{2}\right) = -f(x_0)$$

i.e. we have

$$f\left(\frac{x+y}{2}\right) - f\left(\frac{x+x_0}{2}\right) - f\left(\frac{y+x_0}{2}\right) = -f(x_0) \qquad \text{for } (x,y) \in (T \times T) \backslash M_0$$

whence substituting $x+x_0$ instead of x and $y+x_0$ instead of y we get

$$f\left(\frac{x+y}{2}+x_0\right)=f\left(\frac{x}{2}+x_0\right)+f\left(\frac{y}{2}+x_0\right)-f(x_0) \tag{23}$$
$$\text{for } (x,y)\in\big[(T-x_0)\times(T-x_0)\big]\setminus\big[M_0+(-x_0,-x_0)\big].$$

Put

$$g(x):=2f\left(\frac{x}{2}+x_0\right)-2f(x_0)\quad\text{for } x\in T-x_0. \tag{24}$$

Since T is $\mathcal{J}$-convex and $x_0\in T$, so, if $x\in T-x_0$, then $\frac{x}{2}+x_0=\frac{x+x_0+x_0}{2}\in T$. This proves that the definition of g is correct. (23) and (24) imply directly that

$$g(x+y)=g(x)+g(y)$$
$$\text{for } (x,y)\in\big[(T-x_0)\times(T-x_0)\big]\setminus\big[M_0+(-x_0,-x_0)\big].$$

Clearly $M_0+(-x_0,-x_0)\in\Omega(\mathcal{I})$. To be able to apply Theorem 1 we must check yet that $T-x_0$ satisfies (H_3). Consider an $x\in T-x_0$. Then $x+x_0\in T$ and hence $\frac{x}{2}+x_0=\frac{x+x_0+x_0}{2}\in T$, i.e. $\frac{x}{2}\in T-x_0$. Thus we have proved that $\frac{1}{2}(T-x_0)\subset T-x_0$. Since $x_0\in T_2$, we have

$$X\subset\bigcup_{n\in N_0}2^n(T-x_0).$$

Now consider the set

$$T_3:=\big\{2x\in T-x_0:(T-x_0)\cap(T-x_0-x)\cap(T-x_0-2x)\in\mathcal{I}\big\}.$$

If $2x\in T-x_0$, then $(x_0,x)\in T_1$, i.e. $x\in T_1[x_0]$. Thus $\frac{1}{2}T_3\subset T_1[x_0]$. But $x_0\notin U(T_1)$, hence $T_1[x_0]\in\mathcal{I}$. This implies that $T_3\in\mathcal{I}$. Now we may apply Theorem 1 (taking $S=X$). By this theorem there exist a homomorphism $h:X\to Y$ and a set $K\in\mathcal{I}$ such that

$$g(x)=h(x)\quad\text{for } x\in(T-x_0)\setminus K. \tag{25}$$

Relations (24) and (25) yield

$$f\left(\frac{x}{2}+x_0\right)=\frac{1}{2}h(x)+f(x_0)\ \text{for } x\in(T-x_0)\setminus K. \tag{26}$$

We prove that

$$f\left(\frac{x}{2}+x_0\right)=\frac{f(x+x_0)+f(x_0)}{2} \quad \text{for } x\in(T-x_0)\backslash(M[x_0]-x_0). \tag{27}$$

Consider an $x\in(T-x_0)\backslash(M[x_0]-x_0)$. Then $x_0+x\in T$ and $(x_0,x_0+x)\notin M$. By $(H_{6'})$ we have

$$f\left(\frac{x}{2}+x_0\right)=f\left(\frac{x_0+x_0+x}{2}\right)=\frac{f(x_0)+f(x_0+x)}{2}.$$

Thus (27) is proven. We obtain from (26) and (27) that

$$f(x+x_0)=h(x)+f(x_0) \quad \text{for } x\in(T-x_0)\backslash(K\cup(M[x_0]-x_0)).$$

Replacing $x+x_0$ by x we get

$$f(x)=h(x)+f(x_0)-h(x_0) \quad \text{for } x\in T\backslash((K+x_0)\cup M[x_0]).$$

Obviously $(K+x_0)\cup M[x_0]\in\mathcal{I}$. Hence (22) is valid. One can easily prove uniqueness of h. Then uniqueness of c follows immediately from (22). This completes the proof.

The assumptions of Theorem 3 can be weakened. It concerns hypothesis $(H_{3''})$. Namely, instead of $\mathcal{J}$–convexity of T we may assume $\mathcal{I}$–(a.e) $\mathcal{J}$–convexity of T and in the definition of T_2 the inclusion

$$X\subset\bigcup_{n\in N_0}2^n(T-x)$$

may be replaced by the relation

$$X\subset\bigcup_{n\in N_0}2^n(T-x) \quad \mathcal{I}-\text{ (a.e)}.$$

For further considerations we need some auxiliary lemma. Denote by μ n–dimensional Lebesgue measure.

Lemma 4. *Let $T\subset R^n$ be a convex set such that int $T\neq\emptyset$. Then* $\mu(\mathrm{cl}T\backslash\mathrm{int}\,T)=0$.

PROOF. Consider an $x_0\in\mathrm{int}\,T$ and put for $n\in N: K_n:=\{x\in R^n: \|x-x_0\|\le n\}$, $S_n:=\{x\in R^n:\|x-x_0\|=n\}$. The set cl T is convex (cf. [4] p. 133), hence the set $K_n\cap$ cl T is convex and bounded. Furthermore int $(K_n \cap \text{cl } T) \neq \emptyset$. Therefore $K_n \cap \text{cl } T$ is measurable in Jordan sense (cf. [1] or [5]) and hence $\mu[(K_n \cap \text{cl } T) \setminus \text{int}(K_n \cap \text{cl } T)] = 0$. But int cl T = int T (cf. [4], p. 133), so int $(K_n \cap \text{cl } T) = \text{int} K_n \cap \text{int cl} T = \text{int} K_n \cap \text{int} T$. Thus

$$\mu[(K_n \cap \text{ cl} T)\setminus(\text{int} K_n \cap \text{int} T)] = 0. \tag{28}$$

We have

$$\begin{aligned} \text{cl} T \setminus \text{int} T &= \left(\bigcup_{n=1}^{\infty} K_n\right) \cap (\text{cl } T \setminus \text{int} T) = \\ &= \bigcup_{n=1}^{\infty} (K_n \cap \text{ cl} T) \setminus (K_n \cap \text{int} T) \subset \\ &\subset \bigcup_{n=1}^{\infty} \{[(K_n \cap \text{ cl } T)\setminus(\text{int} K_n \cap \text{int } T)] \cup S_n\}. \end{aligned}$$

By (28) $\mu\{[(K_n \cap \text{ cl} T)\setminus(\text{int} K_n \cap \text{int} T)] \cup S_n\} = 0$, so μ (cl $T \setminus$int $T) = 0$.

Denote by $\mathcal{I}_\circ^n$ a σ p.l.i. ideal of subsets of R^n of n–dimensional measure zero. From Theorem 3 and Lemma 4 we obtain the following

Corollary 1. *Let $T \subset R^n$ be a $\mathcal{J}$–convex set such that int $T \neq \emptyset$ and let $f : T \to R^m$ satisfy the equation*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \quad \Omega(\mathcal{I}_\circ^n) - \text{ (a.e) in } T \times T.$$

Then there exists a unique homomorphism $h : R^n \to R^m$ and an unique constant $c \in R^m$ such that

$$f(x) = h(x) + c \qquad \mathcal{I}_\circ^n - \text{ (a.e) in } T.$$

Proof. To be able to apply Theorem 3 we verify that $T_1 \in \Omega(\mathcal{I}_\circ^n)$ and $T_2 \in \mathcal{I}_\circ^n$. Put $U(T_1) := T \setminus \text{int} T$. By Lemma 4 $\mu(T\setminus\text{int} T) = 0$ i.e. $T\setminus\text{int} T \in \mathcal{I}_\circ^n$. Let us fix an $x \in \text{int} T$, consider a $y \in \frac{1}{2}(T - x)$ and suppose that

$$(T - x) \cap (T - x - y) \cap (T - x - 2y) \in \mathcal{I}_\circ^n.$$

Then

$$\text{int}(T - x) \cap \text{int}(T - x - y) \cap \text{int}(T - x - 2y) = \emptyset. \tag{29}$$

Since $x \in \text{int} T$ and $x + 2y\text{int} T$, we have by Lemma 5.1.8 of [4]

$$x + y = \frac{1}{2}x + \frac{1}{2}(x + 2y) \in \text{int} T,$$

whence

$$0 \in \operatorname{int} T - x - y = \operatorname{int}(T - x - y).$$

Obviously $0 \in \operatorname{int}(T - x)$. Therefore, in view of (29), $0 \notin \operatorname{int}(T - x - 2y)$ i.e. $y \notin \frac{1}{2}\operatorname{int}(T - x)$, and hence

$$y \in \left(\frac{1}{2}(T - x)\right) \setminus \left(\frac{1}{2}(\operatorname{int} T - x)\right) = \frac{1}{2}(T \setminus \operatorname{int} T) - \frac{1}{2}x.$$

Thus

$$T_1[x] \subset \frac{1}{2}(T \setminus \operatorname{int} T) - \frac{1}{2}x \in \mathcal{I}_\circ^n,$$

which proves that $T_1 \in \Omega(\mathcal{I}_\circ^n)$.

Consider now an $x_0 \in \operatorname{int} T$. Then $0 \in \operatorname{int}(T - x_0)$ and hence $R^n \subset \bigcup_{n \in N_0} 2^n(T - x_0)$. So $\emptyset \neq \operatorname{int} T \subset T_2$ and consequently $T_2 \notin \mathcal{I}_\circ^n$. Applying Theorem 3 we get the conclusion.

Following [4] denote by $\mathcal{I}_f^n$ a σ p.l.i. ideal of subsets of R^n of the first category. It is easy to notice that if $T \subset R^n$ is convex and $\operatorname{int} T \neq \emptyset$, then $T \setminus \operatorname{int} T \in \mathcal{I}_f^n$. Actually, by Lemma 4, we have for such a T

$$\operatorname{int}[\operatorname{cl}(T \setminus \operatorname{int} T)] \subset \operatorname{int}(\operatorname{cl} T \setminus \operatorname{int} T) = \emptyset,$$

i.e. $T \setminus \operatorname{int} T \in \mathcal{I}_f^n$.

The topological analogue of Corollary 1 reads as follows

Corollary 2. *Let $T \subset R^n$ be a convex set such that int $T \neq \emptyset$ and let $f : T \to R^m$ satisfy the equation*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2} \quad \Omega(\mathcal{I}_f^n) - (\text{a.e}) \text{ in } T \times T.$$

Then there exist a unique homomorphism $h : R^n \to R^m$ and a unique constant $c \in R^m$ such that

$$f(x) = h(x) + c \qquad \mathcal{I}_f^n - (\text{a.e}) \text{ in } T.$$

Proof. The proof is analogous to the proof of Corollary 1 (it is only necessary to replace $\mathcal{I}_\circ^n$ by $\mathcal{I}_f^n$).

4. Remarks and examples

We are going to show that Theorem 1 generalizes (in abelian case) algebraic part of Proposition 2 of [2], VII §2,2. Let $(X,+) = (R^n,+)$ and

$$P = \left\{ b + \sum_{i=1}^{n} u_i a_i : -1 \le u_i \le 1,\ i = 1, \dots , n \right\},$$

where $b \in R^n$ and $a_1, \dots , a_n \in R^n$ are linearly independent (i.e. P is a parallelepiped with the centre at b spanned on $a_1, \dots , a_n$). We assume that $0 \in P$. It is easy to observe that then $\frac{1}{2}P \subset P$. Let $S := \bigcup_{n\in N_0} 2^n P$. Clearly S is a subsemigroup of X and S generates X. Since $0 \in P$ there exist $-1 \le u_i^0 \le 1,\ i = 1, \dots , n$ such that

$$b = -\sum_{i=1}^{n} u_i^0 a_i.$$

Now suppose that $2x \in P$ i.e. $2x$ is of the form

$$2x = b + \sum_{i=1}^{n} u_i^1 a_i, \quad -1 \le u_i^1 \le 1 \text{ for } \quad i = 1, \dots , n. \tag{30}$$

Then by simple computation, we obtain

$$\begin{aligned} &P \cap (P - x) \cap (P - 2x) = \\ &= \Big\{ b + \sum_{i=1}^{n} u_i a_i : \max(-1, -1 + u_i^0 - u_i^1) \le u_i \le \\ &\le \min(1, 1 + u_i^0 - u_i^1), \quad i = 1, \dots , n) \Big\}. \end{aligned} \tag{31}$$

Let $\mathcal{I}$ be the trivial ideal consisting only of empty set. Then also $\Omega(\mathcal{I}) = \{\emptyset\}$. Suppose that $P \cap (P - x) \cap (P - 2x) \in \mathcal{I}$, i.e. $P \cap (P - x) \cap (P - 2x) = \emptyset$. Then there exists an $i_0 \in 1, \dots , n$ such that

$$\max\left(-1, -1 + u_{i_0}^0 - u_{i_0}^1\right) > \min\left(1, 1 + u_{i_0}^0 - u_{i_0}^1\right).$$

If $u_{i_0}^0 - u_{i_0}^1 \ge 0$, then we get from the last inequality

$$-1 + u_{i_0}^0 - u_{i_0}^1 > 1,$$

whence

$$-1 \le u_{i_\circ}^1 < -2 + u_{i_\circ}^0 \le -1,$$

which is impossible. Similarly, if $u_{i_\circ}^0 - u_{i_\circ}^1 < 0$, then

$$-1 > 1 + u_{i_\circ}^0 - u_{i_\circ}^1,$$

whence

$$1 \le 2 + u_{i_\circ}^0 < u_{i_\circ}^1 \le 1,$$

which again is impossible. Thus $P_1 = \emptyset$ i.e. $P_1 \in \mathcal{I}$ and hence $(H_1), (H_2)$ and (H_3) are valid. Furthermore (H_4) and (H_6) can be rewritten as follows

$$f : P \to Y, \quad f(x+y) = f(x) + f(y) \quad \text{for } x, y, x+y \in P.$$

Example 1. Let $(X, +)$, P, S have the meaning specified above and let $\mathcal{I} = \mathcal{I}_\circ^n$ be the σ p.l.i. ideal of subsets of n-dimensional measure zero.

Suppose that for given element of form (30) $P \cap (P - x) \cap (P - 2x) \in \mathcal{I}$. Then int $(P \cap (P - x) \cap (P - 2x)) = \emptyset$, and hence, by (31), there exists $i_\circ \in \{1, \dots, n\}$ such that

$$\max\left(-1, -1 + u_{i_\circ}^0 - u_{i_\circ}^1\right) \ge \min\left(1, 1 + u_{i_\circ}^0 - u_{i_\circ}^1\right).$$

This inequality implies that $u_{i_\circ}^1 = -1$ or $u_{i_\circ}^1 = 1$, which means that

$$2x \in A_{i_\circ} := \left\{ b \pm a_{i_\circ} + \sum_{\substack{i=1 \\ i \ne i_\circ}}^{n} u_i a_i, \ -1 \le u_i \le 1 \text{ for } i = 1, \dots, n, \ i \ne i_\circ \right\}.$$

Thus we have proved that $P_1 \subset \bigcup_{i_\circ = 1}^{n} A_{i_\circ}$. But obviously $A_{i_\circ} \in \mathcal{I}$, hence $P_1 \in \mathcal{I}$. This means that $(H_1), (H_2)$ and (H_3) hold.

Example 2. Let $(X, +) = (R^n, +)$, $S = R^n$, $\mathcal{I}$ be a p.l.i. ideal in R^n and let $P \subset R^n$ be an open non-empty set such that $0 \in P$ and $\frac{1}{2}P \subset P$. Obviously $\bigcup_{n \in N_0} 2^n P = R^n$. Consider an element $2x \in P$ and suppose that $P \cap (P - x) \cap (P - 2x) \in \mathcal{I}$. Then $0 \in P \cap (P - x) \cap (P - 2x)$. But $P \cap (P - x) \cap (P - 2x)$ is open. This contradicts to the fact that a σ p.l.i. ideal in R^n does not, contain an open non-empty set. So $P_1 = \emptyset$. We have shown that $(H_1), (H_2)$ and (H_3) are satisfied.

Example 3. Let $(X,+) = (R,+)$, $P = \langle -\frac{1}{2}, \frac{1}{2}\rangle \cup \langle 4,6\rangle$, $S = \langle 1,\infty)$ and let $\mathcal{I}$ be the σ p.l.i. ideal of subsets of R of measure zero. Then

$$P_2 = (0, \frac{1}{2}\rangle, \ P_3 = (-\frac{1}{2}, \frac{1}{2}), \ S^\star = (0,\infty)$$

and, of course, hypotheses $(H_1), (H_2)$ and $(H_{3'})$ are satisfied.

We show now that Theorem 3 generalizes Theorem 13.2.1 of [4]. It is sufficient to prove that, in the case where $(X,+) = (R^n,+)$, $\mathcal{I} = \{\emptyset\}$, if T is a convex subset of X and $\operatorname{int} T \neq \emptyset$, then $T_1 = \emptyset$ and $T_2 \neq \emptyset$. So fix an $x \in T$ and consider a $y \in \frac{1}{2}(T-x)$. Then $0 \in T - x$, $0 \in T - x - 2y$ and $0 \in T - x - y$ (since $x + y = \frac{1}{2}x + \frac{1}{2}(x+2y) \in T$). Thus $(T-x) \cap (T-x-y) \cap (T-x-2y) \neq \emptyset$ i.e. $T_1 = 0$. Further, there exists an $x_0 \in \operatorname{int} T$. Then $0 \in \operatorname{int}(T - x_0)$ and hence $R^n \subset \bigcup_{n \in N_0} 2^n(T - x_0)$, which means that $T_2 \neq \emptyset$.

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JÓZEF TABOR
DEPT. OF MATHEMATICS
PEDAGOGICAL UNIVERSITY OF CRACOW
PODCHORAZYCH 2
30-084 CRACOW
POLAND

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