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On extrinsic spheres in Kähler manifolds

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1. Introduction

An m-dimensional submanifold, $m \geq 2$, of a Riemannian manifold is called to be an extrinsic sphere if it is totally umbilical and its mean curvature vector field is non-zero and parallel. Extrinsic spheres have been geometrically characterized by NOMIZU and YANO [2]. Since they have the same extrinsic properties as ordinary spheres in a Euclidean space, it is natural to ask when an extrinsic sphere is isometric to an ordinary sphere.

When the ambient manifold is a Kähler manifold, it is proved that:

Theorem A [1]. *A complete, connected, simply connected and even-dimensional extrinsic sphere of a Kähler manifold is isometric to an ordinary sphere if its normal connection is flat.*

Theorem B [4]. *A complete, connected and simply connected extrinsic sphere M in a Kähler manifold $\tilde{M}$ is one of the following:*

(1) *M is isometric to an ordinary sphere,*
(2) *M is homothetic to a Sasakian manifold,*
(3) *M is a totally real submanifold and the f-structure is not parallel in the normal bundle.*

Here we note that the cases (2) and (3) occur only when $\dim M = m$ is odd and $\dim \tilde{M} \geq m+1$, respectively.

The main theorem of the present paper is as follows:

Theorem 1.1. *Let $\tilde{M}$ be a Kähler manifold and M a complete, connected and simply connected extrinsic sphere of dimension $m \geq 3$ in $\tilde{M}$. Assume, additionaly, that the curvature tensor of the connexion induced in the normal bundle of M satisfies the condition*

$$\nabla_n \nabla_m R_{jiy}x - \nabla_m \nabla_n R_{jiy}x = 0. \tag{1.1}$$

Then one of the following cases occurs:

1) *M is isometric to an ordinary sphere,*
2) *M is homothetic to a Sasakian manifold.*

Moreover, we give some sufficient conditions for an extrinsic sphere to be homothetic to a Sasakian manifold. In the proofs we use the following well-known theorem of Okumura [3].

Theorem C. *Let M be a Riemannian manifold. If M admits a Killing vector field v of constant lenght satisfying*

$$\nabla_k \nabla_j v_i = h^2(v_j g_{ik} - v_i g_{jk}),$$

for a non-zero constant h, then M is homothetic to a Sasakian manifold.

2. Preliminaries

Let $\tilde{M}$ be a Kähler manifold of dimension n and (F, G) its Kähler structure. Then we have

$$(2.1) \qquad F^\alpha{}_\gamma F^\gamma{}_\beta = -\delta^\alpha_\beta, \quad F^\lambda{}_\alpha F^\gamma{}_\beta G_{\lambda\gamma} = G_{\alpha,\beta}, \quad \nabla_\gamma F^\alpha{}_\beta = 0.$$

Here and in the sequel the indices $\alpha, \beta, \dots$ run over the range $\{1, 2, \dots, n\}$ and ∇ denotes the covariant differentiation in $\tilde{M}$.

Let M ($\dim M = m$, $m < n$) be a submanifold of $\tilde{M}$. Denote by (x^α) and (u^i) local coordinates of $\tilde{M}$ and M, respectively. The continual indices $h, i, j, \dots$ run over the range $\{1, 2, \dots, m\}$.

Let $x^\alpha = x^\alpha(u^i)$ be a local parametric representation of M in $\tilde{M}$. Set $B_i^\alpha = \partial x^\alpha / \partial u^i$. M inherits from $\tilde{M}$ the Riemannian metric g of local components $g_{ij} = G_{\alpha\beta} B_i{}^\alpha B_j{}^\beta$. Next we take $n - m$ mutually orthogonal unit local vector fields normal to M and denote their local components by $N_x{}^\alpha$. The indices $x, y, z, u, w, \dots$ run over the range $\{m+1, \dots, n\}$. Denote by g_{xy} the components of the metric tensor induced on the normal bundle $T^\perp M$ of M from the metric tensor G of $\tilde{M}$, that is $g_{xy} = G_{\alpha\beta} N_x{}^\alpha N_y{}^\beta$.

Let us express FB_i and FN_x as linear combinations of B_i and N_x as follows

$$(2.2) \qquad F_\gamma{}^\alpha B_i{}^\gamma = f_i{}^r B_r^\alpha + f_i{}^x N_x{}^\alpha,$$

$$(2.3) \qquad F_\gamma{}^\alpha N_x{}^\gamma = f_x{}^r B_r{}^\alpha + f_y{}^x N_x{}^\alpha.$$

Then $f_y{}^x$ are components of an f – structure in the normal bundle $T^\perp M$. It can easily be noted that

$$f_{ij} = -f_{ji}, \ f_{ix} = -f_{xi}, \ f_{xy} = -f_{yx},$$

where

$$f_{ij} = f_i{}^r g_{rj}, \ f_{ix} = f_i{}^y g_{xy}, \ f_{xi} = f_x{}^r g_{ri}, \ f_{xy} = f_x{}^v g_{vy}.$$

With the help of (2.2) - (2.3), we find

$$f_j{}^r f_{ri} + f_j{}^x f_{xi} = -g_{ij}, \tag{2.4}$$

$$f_i{}^r f_{rx} + f_i{}^v f_{vx} = 0, \tag{2.5}$$

$$f_x{}^v f_{vy} + f_x{}^r f_{ry} = -g_{xy}. \tag{2.6}$$

If $FT_xM \subset T_x^\perp M$ for any $x \in M$ (i.e. $f_j{}^i = 0$), then M is said to be a totally real submanifold of $\tilde{M}$.

Denoting by $h_{ij}{}^x$ the components of the second fundamental tensor of the submanifold M, we have the following equations of Gauss

$$\nabla_j B_i{}^\alpha = h_{ij}{}^v N_v{}^\alpha, \tag{2.7}$$

and of Weingarten

$$\nabla_j N_x{}^\alpha = -h_j{}^r{}_x B_r{}^\alpha, \; h_j{}^r{}_x = h_{ji}{}^v g^{ir} g_{vx}, \tag{2.8}$$

where ∇ denotes the Van der Waerden – Bortolotti covariant differentiation (see e.g. YANO and ISHIHARA [5]).

The mean curvature vector field of the submanifold M has local components $h^x = 1/m g^{rs} h_{rs}{}^x$ and the function h is such that $h^2 = g_{xy} h^x h^y$ is the mean curvature.

The submanifold M is said to be totally umbilical if $h_{ij}{}^x = g_{ij} h^x$ and totally geodesic if $h_{ij}{}^x = 0$. If the mean curvature vector field of a totally umbilical submanifold is non – zero and parallel ($\nabla_j h^x = 0$), then M is called an extrinsic sphere in $\tilde{M}$ (cf. [2]). The mean curvature h of an extrinsic sphere is a non – zero constant.

Let M be an extrinsic sphere in $\tilde{M}$. Differentiating (2.2) and (2.3) covariantly along the submanifold M we obtain, by virtue of (2.1) – (2.3), (2.7) and (2.8),

$$\nabla_k f_{ij} = v_i g_{jk} - v_j g_{ik}, \tag{2.9}$$

$$\nabla_k f_{ix} = f_{vx} h^v g_{ik} - h_x f_{ik}, \tag{2.10}$$

$$\nabla_k f_{xy} = h_y f_{kx} - h_x f_{ky}, \tag{2.11}$$

where $v_i = f_{ix} h^x$.

The covector field $v = (v_i)$ satisfies

$$\nabla_j v_i = -h^2 f_{ij}. \tag{2.12}$$

Thus it is Killing. Differentiating again (2.12) covariantly and making use of (2.9), we obtain

$$\nabla_k \nabla_j v_i = h^2 (v_j g_{ik} - v_i g_{jk}). \tag{2.13}$$

3. Extrinsic spheres

Lemma 3.1. *For an extrinsic sphere in a Kähler manifold, the following relations hold*

$$v^r T_{kjir} = 0, \tag{3.1}$$

$$f_{rx} T_{kji}{}^r + f_i{}^v R_{kjxv} = 0, \tag{3.2}$$

$$\begin{aligned} &f_{rx}(\nabla_n \nabla_m R_{kji}{}^r - \nabla_m \nabla_n R_{kji}{}^r) + \\ &f_{iv}(\nabla_n \nabla_m R_{kjx}{}^v - \nabla_m \nabla_n R_{kjx}{}^v) = \\ &= h^2(f_{nx} T_{kjim} - f_{mx} T_{kjin} + g_{in} f_{rx} T_{kjm}{}^r - g_{im} f_{rx} T_{kjn}{}^r), \end{aligned} \tag{3.3}$$

where

$$T_{kjih} = R_{kjih} - h^2(g_{kh} g_{ij} - g_{ki} g_{jh}), \tag{3.4}$$

while R_{kjih} and $R_{kjxy} = R_{kjx}{}^v g_{vy}$ are covariant components of the curvature tensor of M and the curvature tensor of the connexion induced in the normal bundle of M, respectively.

PROOF. (3.1) follows from (2.13), the Ricci identity and (3.4). Differentiating now (2.10) covariantly and applying (2.9), (2.11) and the Ricci identity, we obtain

$$f_{rx} R_{kji}{}^r + f_i{}^v R_{kjxv} = h^2(g_{ij} f_{kx} - g_{ik} f_{jx}).$$

This, by making use of (3.4), can be written in the form (3.2). To prove (3.3) we transvect the Ricci identity

$$\begin{aligned} \nabla_n \nabla_m R_{jiyx} - \nabla_m \nabla_n R_{jiyx} = &- R_{riyx} R_{nmy}{}^r - R_{jryx} R_{nmi}{}^r - \\ &- R_{jivx} R_{nmy}{}^r - R_{jiyv} R_{nmx}{}^v, \end{aligned}$$

with $f_k{}^x$ and use (3.2), (3.4) and the known identity

$$\begin{aligned} \nabla_n \nabla_m R_{hkji} - \nabla_m \nabla_n R_{hkji} = &- R_{rkji} R_{nmh}{}^r - R_{hrji} R_{nmk}{}^r - \\ &- R_{hkri} R_{nmj}{}^r - R_{hkjr} R_{nmi}{}^r. \end{aligned}$$

This completes the proof.

The above lemma will be used in the following proof:

PROOF OF THEOREM 1.1. Suppose that M is a complete, connected and simply connected extrinsic sphere in a Kähler manifold M.

By Theorem B, it is sufficient to show that in the case (3) of Theorem B M is isometric to an ordinary sphere if its curvature tensor of the normal connexion satisfies (1.1). Let M satisfy (1.1) and be totally real ($f_{ij}=0$). Then, from (3.3) it follows that

$$(3.5)\quad \begin{aligned} f_{rx}(\nabla_n\nabla_m R_{kji}{}^r - \nabla_m\nabla_n R_{kji}{}^r) =& h^2(f_{nx}T_{kjim} - f_{mx}T_{kjin} + \\ &+ g_{in}f_{rx}T_{kjm}{}^r - g_{im}f_{rx}T_{kjn}{}^r), \end{aligned}$$

and from (2.4)

$$(3.6)\quad f_j{}^x f_{xi} = -g_{ji}.$$

Next, permuting (3.5) cyclically with respect to the indices (k,j,i), adding the resulting equations and using the relation $T_{kjir}+T_{jikr}+T_{ikjr}=0$ and the first Bianchi identity, we find

$$f_x{}^r(g_{in}T_{kjmr} - g_{im}T_{kjnr} + g_{kn}T_{jimr} - g_{km}T_{jinr} + g_{jn}T_{ikmr} - g_{jm}T_{iknr}) = 0.$$

Hence, by transvection with $f_h{}^x g^{im}$ and making use of (3.6), we have

$$(3.7)\quad (m-3)T_{kjnh} = T_{jh}g_{kn} - T_{kh}g_{jn},$$

where $T_{kh} = T_{kjnh}g^{jn}$, which by transvection with g^{jn} yields $T_{kh}=0$ for $m \geq 3$. If $m > 3$, then the relation $T_{kh}=0$ used in (3.7) yields $T_{kjih}=0$. As it is known, if $m=3$, then $T_{kh}=0$ always implies $T_{kjih}=0$. Thus, M is a manifold of constant curvature. As M is complete, connected and simply connected, M is isometric to an ordinary sphere. The proof is complete.

If $\nabla_k f_y{}^x = 0$, then the f-structure in the normal bundle $T^\perp M$ is said to be parallel. It is known that if M is a totally real submanifold, then the f–structure is not parallel (cf. [4], Theorem B). For an f–structure which is parallel, we have the following.

Theorem 3.2. *Let M be an extrinsic sphere in a Kähler manifold. If the f–structure in $T^\perp M$ is parallel, then M is homothetic to a Sasakian manifold.*

PROOF. Suppose that $\nabla_k f_y{}^x = 0$ holds. Then, from (2.11) it follows that $f_{kx}h_y - f_{ky}h_x = 0$. Hence, by covariant differentiation and using (2.10), we have

$$(3.8)\quad h_y f_{zx}h^z - h_x f_{zy}h^z = 0.$$

Now, transvecting (3.8) with h^y, we get $f_{zx}h^z = 0$. On the other hand, by transvection of (2.6) with $h^x h^y$, we obtain, in virtue of the above relation, $v_r v^r = h^2$. The vector field v is Killing (see 2.12), satisfies (2.13) and has non – zero constant length. But if a Riemannian manifold admits such a vector field, then it is homothetic to a Sasakian manifold [3]. The proof is complete.

Corollary 3.3. *If the f-structure in the normal bundle of M vanishes, then M is a hypersurface homothetic to a Sasakian manifold.*

PROOF. By Theorem 3.2, it is sufficient to show that M is a hypersurface. Suppose that $f_{yx} = 0$ holds on M.

Then, (2.11) yields $f_{kx}h_y - f_{ky}h_x = 0$, which, by transvection with $f_z{}^k h^y$ and the use of (2.6) gives $h^2 g_{zx} = h_z h_x$ at each point of M. From this we conclude that $\dim M = n - 1$, which completes the proof.

Finally, note that from (2.9) and (2.11) we can easily obtain the following.

Proposition 3.4. *An extrinsic sphere M in a Kähler manifold is totally real if and only if the tensor field of components f_{ij} is parallel on M.*

Contrary to the above we have

Proposition 3.5. *Let M be an extrinsic sphere in a Kähler manifold. There are no open subsets of M on which $\nabla_k f_{jx} = 0$.*

PROOF. Suppose that U is an open subset of M on which $\nabla_k f_{jx} = 0$. Then, from (2.10) it follows that $f_{ij} = 0$ and $f_{zx}h^z = 0$ on M. From (2.9) we get $v_i = 0$ and from (2.11) $h^2 f_{jx} = v_j h_x$. Consequently, we have $f_{jx} = 0$ on U. This contradicts the equality (2.4). The proof is complete.

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