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Equation solving iterations based on tangential convex functions

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Abstract. For solving nonlinear equations SZABÓ [3] worked out a combined root-finding algorithm which is based on Newton's method and the always convergent method of tangential parabolas. In this paper we are going to give generalizations of this algorithm. To construct our combined methods we shall use the always convergent methods of tangential convex functions which was introduced by us in [4].

§1.

The theory of interval arithmetic [2] was developed by the progress of digital computers and machine-oriented numerical methods, as well as by the demand for automatic registration of error accumulation. Later on, with the help of this theory root-finding algorithms were worked out, which are always convergent under strong conditions. These methods usually generate such a quickly decreasing (in the sense of inclusion) interval-sequence which contains zeros of a real function $f : [a,b] = I \subseteq \mathbf{R} \to \mathbf{R}$. In general, the function f is strictly monotone, differentiable as often as necessary and obeys

$$f(a) \cdot f(b) < 0.$$

For the application of such methods it is necessary to know the intervals containing the range of the derivatives $f^{(k)}$, $k = 1, \dots, p$.

MOORE [2] was the first to give a procedure of this kind, i.e. the interval arithmetic variant of Newton's iteration. Then ALEFELD and HERZBERGER [1] worked out further always convergent methods.

In this paper we present always convergent combined root-finding algorithms which show great similarity with the results in [1]. At the same time, none of the developed methods apply the tools of interval arithmetic. A similar procedure can be found in SZABÓ [3]. Our methods can be considered as generalizations of this procedure.

Definition ([3]). An iteration function $F(x,r)$ and the iteration method given by this function are said to be always convergent with respect to the function $f: I \subseteq \mathbf{R} \to \mathbf{R}$, if we start from an arbitrary point $x_0 \in I,\ f(x_0) \neq 0$, then the iteration sequence $\{x_n\}$ given by

$$x_{n+1} = F(x_n, r), \quad n = 0, 1, \dots \tag{1.1}$$

1° is monotone;

2° converges to the zero $\alpha \in I$ of f lying nearest to the right (or to the left) of x_0 — if such α exists — depending on the fact that during the whole iteration process the direction parameter r is chosen as 1 (or -1) consistently;

3° leaves I, if such a zero α does not exists.

We denote the set of always convergent iteration functions by $\mathcal{A}(f, I)$.

The method of tangential convex functions given by us in [4] is an example for an always convergent iteration:
Suppose that

$$\begin{cases} f: [a,b] = I \subseteq \mathbf{R} \to \mathbf{R} \text{ is twice continuously} \\ \text{differentiable in } I \text{ and the inequalities} \\ |f(x)| \leq M \neq 0, \\ |f'(x)| \leq M_1 \neq 0, \\ |f''(x)| \leq M_2 \neq 0, \qquad x \in I \\ \text{are fulfilled.} \end{cases} \tag{1.2}$$

Furthermore let

$$\begin{cases} g: (-h,h) = H \subseteq \mathbf{R} \to \mathbf{R} \text{ be twice continuously} \\ \text{differentiable in } H, \\ g(0) = g'(0) = 0 \qquad \text{and} \\ g''(x) > 0 \quad \text{for} \quad x \in H. \\ \text{Let still exist } q_2 > 0 \text{ such that for suitable } c > 0 \\ \text{the condition } q_2 \leq g''(x) \text{ is satisfied,} \\ \text{if } x \in \left[a - b + g'^{-1}\left(-\dfrac{M_1}{c}\right), b - a + g'^{-1}\left(\dfrac{M_1}{c}\right)\right] \cap H. \end{cases} \tag{1.3}$$

The Theorem 2.1. in [4] holds for the iteration function

$$F_g(x,r) = x - g'^{-1}\left(-\frac{s}{c} f'(x)\right) + g_r^{-1}\left[\frac{|f(x)|}{c} + g\left(g'^{-1}(-\frac{s}{c} f'(x))\right)\right],$$

$$s \doteq \operatorname{sign}(f(x_0)), \quad x_0 \in I$$

of the tangential convex functions method, i.e. if conditions (1.2) and (1.3) are fulfilled then $F_g(x,r) \in \mathcal{A}(f,I)$. Assume that

$$(1.4) \quad \begin{cases} \alpha \in I \text{ is a simple zero of } f, \\ 0 < m_1 \le |f'(x)|, \quad x \in I \quad \text{and} \\ g''(x) \le Q_2, \\ x \in \left[g_{-1}^{-1}\left(\frac{M}{c} + g(g'^{-1}(-\frac{M_1}{c}))\right), g_{+1}^{-1}\left(\frac{M}{c} + g(g'^{-1}(-\frac{M_1}{c}))\right)\right] \end{cases}$$

If conditions (1.2), (1.3) and (1.4) are fulfilled, for the error estimate of the tangential convex functions method we get

$$(1.5) \qquad |e_{n+1}| \le \frac{\frac{c}{2}Q_2 + M_2}{m_1} |e_n|^2,$$

where $e_n = x_n - \alpha$. (Theorem 2.2 in [4])

§2.

$$(2.1) \qquad \begin{cases} \text{Suppose that the function } f \text{ obeys (1.2)}, \\ f(a) \cdot f(b) < 0 \quad \text{and} \\ f \text{ is either convex or concave in } I. \end{cases}$$

(Consequently f' is monotone and f has an unique, simple zero in I.) Let now, for example,

$$(2.2) \qquad f(a) > 0 \quad \text{and} \quad f''(x) < 0, \ x \in I.$$

Starting from the points $a_0 = a$ and $b_0 = b$ we calculate the iteration sequences $\{a_n\}$ and $\{b_n\}$ by the help of the formulas

$$a_{n+1} = F_g(a_n, 1), \qquad n = 0, 1, \dots$$

and

$$b_{n+1} = b_n - \frac{f(b_n)}{f'(b_n)}, \ n = 0, 1, \dots .$$

So we get the sequence of intervals

$$\mathcal{J}_n = [a_n, b_n], \ n = 0, 1, \dots .$$

Usually if f is concave in I, we apply the tangent method (Newton's method) starting from that endpoint of I in which the value of f is negative. If f is convex, we apply the tangent method starting from f–positive endpoint of I.

Theorem 2.1. *If conditions* (1.3) *and* (2.1) *hold, then the sequence of intervals* $\mathcal{J}_n$ *calculated according to the above described method has the following properties:*

$$1^\circ \quad \alpha \in \mathcal{J}_n, \qquad n = 0, 1, \dots ;$$
$$2^\circ \quad \mathcal{J}_n \supset \mathcal{J}_{n+1}, \quad n = 0, 1, \dots ;$$
$$3^\circ \quad \bigcap_{n=0}^{\infty} \mathcal{J}_n = \alpha, \quad f(\alpha) = 0.$$

PROOF. Suppose that (2.2) is fulfilled. Then because of the Lemma 2.1. in [4] our tangential convex function is under f, so $a_n \le \alpha$, $n = 0, 1, \dots$. Since f is concave, our tangent is above f, that is $\alpha \le b_n$, $n = 0, 1, \dots$. Thus we have $\alpha \in \mathcal{J}_n$ for all $n \ge 0$.

Furthermore, $\{a_n\}$ is monotone increasing (Theorem 2.1. in [4]) and $\{b_n\}$ is monotone decreasing, so $\mathcal{J}_n \supset \mathcal{J}_{n+1}$, $n = 0, 1, \dots$.

Finally, by virtue of (2.1), f has an unique zero α in I and $F_g(x, 1) \in \mathcal{A}(f, I)$. From this it follows that $\{a_n\}$ tends to α. On the other hand the relations

$$f(b) < 0, \qquad f''(x) \le 0, \qquad x \in I$$

are valid, therefore $\{b_n\}$ also tends to α. Because of

$$\lim_{n \to \infty} (b_n - a_n) = 0$$

the sequence of diameters of $\mathcal{J}_n$

$$d(\mathcal{J}_n) \doteq b_n - a_n, \quad n = 0, 1, \dots$$

tends to zero and so we get $\bigcap_{n=0}^{\infty} \mathcal{J}_n = \alpha$.

In other cases proof is similar. □

The decreasing of the diameters $d(\mathcal{J}_n)$ of the intervals $\mathcal{J}_n$ can be estimated as follows:

Theorem 2.2. *If conditions* (1.3), (1.4) *and* (2.1) *are fulfilled and* f' *keeps its sign in* I, *then*

$$d(\mathcal{J}_{n+1}) \le \frac{\frac{c}{2} Q_2 + M_2}{m_1} |d(\mathcal{J}_n)|^2, \qquad n = 0, 1, \dots ,$$

where $m_1 = \min\{|f'(a)|,\ |f'(b)|\}$.

PROOF. Examine first the case (2.2). In this case f' is monotone decreasing and keeps its sign in I, therefore we have

$$0 < m_1 \doteq |f'(a)| \le |f'(x)|, \quad x \in I.$$

Then because of (1.5) we get

$$|\alpha - a_{n+1}| \le \frac{\frac{c}{2}Q_2 + M_2}{m_1}(\alpha - a_n)^2, \; n = 0, 1, \dots .$$

For the error estimate of Newton's method we have

$$|b_{n+1} - \alpha| \le \frac{|f''(\eta)|}{2 \cdot |f'(b_n)|}(b_n - \alpha)^2 \le \frac{M_2}{2m_1}(b_n - \alpha)^2,$$

$\eta \in (\alpha, b_n), \; n = 0, 1, \dots$. As a final result we can write that

$$\begin{aligned} |b_{n+1} - a_{n+1}| &= |b_{n+1} - \alpha| + |\alpha - a_{n+1}| \le \\ &\le \frac{M_2}{2m_1}(b_n - \alpha)^2 + \frac{\frac{c}{2}Q_2 + M_2}{m_1}(\alpha - a_n)^2 \le \\ &\le \frac{\frac{c}{2}Q_2 + M_2}{m_1}\left[(b_n - \alpha)^2 + (\alpha - a_n)^2\right] \le \\ &\le \frac{\frac{c}{2}Q_2 + M_2}{m_1}(b_n - a_n)^2 \end{aligned}$$

as well as

$$d(\mathcal{J}_{n+1}) \le \frac{\frac{c}{2}Q_2 + M_2}{m_1}|d(\mathcal{J}_n)|^2, \quad n = 0, 1, \dots .$$

In the other three cases the proof is quite similar. □

§3.

Now starting from both initial point we apply the methods of tangential convex functions to generate the sequences $\{a_n\}$ and $\{b_n\}$. Let $a_0 = a, \; b_0 = b$ and let us build the iteration sequences according to

$$a_{n+1} = F_{g_1}(a_n, 1), \quad n = 0, 1, \dots,$$

$$b_{n+1} = F_{g_2}(b_n, -1), \quad n = 0, 1, \dots .$$

Then we get the sequence of intervals $\mathcal{J}_n = [a_n, b_n], \quad n = 0, 1, \dots$.

Theorem 3.1. *If conditions* (1.3) *and* (2.1) *are fulfilled then the interval-sequence* $\{\mathcal{J}_n\}$ *derived from our method has the following properties:*

$$1^\circ \quad \alpha \in \mathcal{J}_n, \qquad n = 0, 1, \ldots;$$
$$2^\circ \quad \mathcal{J}_n \supset \mathcal{J}_{n+1}, \quad n = 0, 1, \ldots;$$
$$3^\circ \quad \bigcap_{n=0}^{\infty} \mathcal{J}_n = \alpha, \quad f(\alpha) = 0.$$

PROOF. First suppose that $f(a) > 0$. Then because of the Lemma 2.1. in [4] the curve of the tangential convex function produced by the help of the convex function g_1 is always under f, but the curve of the tangential convex function produced by the help of g_2 goes over f, so $a_n \le \alpha$ and $\alpha \le b_n$, that is $\alpha \in \mathcal{J}_n$, $n \ge 0$. The case $f(a) < 0$ is proved in a completely analogous manner.

It follows that $\mathcal{J}_{n+1} \subset \mathcal{J}_n$ for every $n \ge 0$, since $\{a_n\}$ is monotone increasing and $\{b_n\}$ is monotone decreasing (Theorem 2.1. in [4]), and the relation $\alpha \in \mathcal{J}_n$ is valid for $n \ge 0$.

Finally F_{g_1} and F_{g_2} are always convergent iteration functions and f has an unique zero α in I, from which it follows that

$$\lim_{n\to\infty} a_n = \alpha\,, \quad \text{and} \quad \lim_{n\to\infty} b_n = \alpha\,.$$

So

$$\lim_{n\to\infty} d(\mathcal{J}_n) = \lim_{n\to\infty} (b_n - a_n) = 0,$$

that is

$$\bigcap_{n=0}^{\infty} \mathcal{J}_n = \alpha. \qquad \square$$

In the following theorem we give an error estimate:

Theorem 3.2. *If conditions* (1.3), (1.4) *and* (2.1) *are fulfilled and* f' *keeps its sing in* I *then*

$$d(\mathcal{J}_{n+1}) \le \frac{\frac{c}{2}Q_2 + M_2}{m_1} |d(\mathcal{J}_n)|^2, \; n = 0, 1, \ldots,$$

where $m_1 = \min\{|f'(a)|,\ |f'(b)|\}$,
$c \;\; = \max\{c^{g_1}, c^{g_2}\}$,
and $Q_2 = \max\{Q_2^{g_1}, Q_2^{g_2}\}$.

PROOF. Since $|f'|$ is monotone in I, cleary

$$0 < m_1 \le \; |f'(x)|, \quad x \in I.$$

But according to the error estimate (1.5) we have

$$|\alpha - a_{n+1}| \le \frac{\frac{c^{g_1}}{2} Q_2^{g_1} + M_2}{m_1} (\alpha - a_n)^2, \quad n = 0, 1, \ldots$$

and

$$|b_{n+1} - \alpha| \le \frac{\frac{c^{g_2}}{2} Q_2^{g_2} + M_2}{m_1} (b_n - \alpha)^2, \quad n = 0, 1, \ldots \; .$$

It follows that

$$\begin{aligned} |b_{n+1} - a_{n+1}| &= |b_{n+1} - \alpha| + |\alpha - a_{n+1}| \le \\ &\le \frac{\frac{c^{g_1}}{2} Q_2^{g_1} + M_2}{m_1} (\alpha - a_n)^2 + \frac{\frac{c^{g_2}}{2} Q_2^{g_2} + M_2}{m_1} (b_n - \alpha)^2 \le \\ &\le \frac{\frac{c}{2} Q_2 + M_2}{m_1} \left[(\alpha - a_n)^2 + (b_n - \alpha)^2 \right] \le \\ &\le \frac{\frac{c}{2} Q_2 + M_2}{m_1} (b_n - a_n)^2, \quad n = 0, 1, \ldots \; , \end{aligned}$$

as well as

$$d(\mathcal{J}_{n+1}) \le \frac{\frac{c}{2} Q_2 + M_2}{m_1} |d(\mathcal{J}_n)|^2, \quad n = 0, 1, \ldots \; ,$$

which was to be proved . □

§4.

Finally we shall show several numerical examples:

1.) $f(x) = \exp x + 10x - 2; \quad M_2 = \exp 1;$
$g(x) = x^2; \quad c = M_2/2$

n	a_n	b_n	$b_n - a_n$
0	0	1	1
1	0.0904041752	0.1572539457	10^{-1}
2	0.0905251012	0.0907532514	10^{-4}
3	0.0905251012	0.0905251038	10^{-9}
4	0.0905251012	0.0905251013	10^{-10}

2.) $f(x) = x^2 - x - 1; \quad M_1 = 3; \quad M_2 = 2;$

$$g(x) = \sqrt{1+x^2} - 1; \quad c = \max\left\{\sqrt{2}M_1, \left(\frac{\sqrt{13}}{2}\right)^3 \cdot M_2\right\}$$

n	a_n	b_n	$b_n - a_n$
0	−1	−0.5	1
1	−0.6666666667	−0.6105365195	10^{-2}
2	−0.6190476191	−0.6180006124	10^{-3}
3	−0.6180344478	−0.6180339880	10^{-6}
4	−0.6180339888	−0.6180339889	10^{-10}

3.) $f(x) = \sin x - 0.5x; \quad M = 1.5; \quad M_1 = 1.5; \quad M_2 = 1;$

$$g(x) = 1 - \sqrt{1-x^2};$$

$$c = \max\left\{M_2, \sqrt{\frac{M^2+\sqrt{M^4+4M^2M_1^2+1}}{2}}\right\}$$

n	a_n	b_n	$b_n - a_n$
0	1.5	3	1
1	1.806832324	2.087995413	1
2	1.888838904	1.912229258	10^{-1}
3	1.895403150	1.895652628	10^{-4}
4	1.895494265	1.895494282	10^{-8}
5	1.095494265	1.895494267	10^{-9}

4.) $f(x) = x^3 - x - 1; \quad M_2 = 12;$

$$g(x) = \operatorname{ch} x - 1; \quad c = M_2$$

n	a_n	b_n	$b_n - a_n$
0	1	2	1
1	1.271346645	1.545454545	10^{-1}
2	1.323160837	1.359614916	10^{-2}
3	1.324716597	1.325801345	10^{-3}
4	1.324717957	1.324719049	10^{-6}
5	1.324717957	1.324717957	0

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