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A note on 'optimal measures'

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Abstract. The aim of the paper is to give an elementary proof for the structure theorem of 'optimal measures'.

The structure theorem of 'optimal measures' was proved in [2] using the Zorn lemma. In this paper an elementary proof for that structure theorem is given.

Definition 1 (see [1], [2]). Let $(\Omega, \mathcal{A})$ be a measurable space. A function $\mu : \mathcal{A} \to [0,1]$ is called an 'optimal measure' if

(A1) $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$;

(A2) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for all $A, B \in \mathcal{A}$;

(A3) if $A_n \in \mathcal{A}$, $n = 1, 2, \dots$, with $A_1 \supseteq A_2 \supseteq \dots$, then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$. □

Throughout the paper only measurable subsets of Ω will be used. $A_n \downarrow A$ means that $A_1 \supseteq A_2 \supseteq \dots$ and $\bigcap_{n=1}^{\infty} A_n = A$ ($A_n \uparrow A$ is defined in an analogous way). $a_n \downarrow a$ means that the sequence of numbers $\{a_n\}$ is decreasing and it converges to a ($a_n \uparrow a$ is defined similarly).

Let $(\Omega, \mathcal{A}, \mu)$ with 'optimal measure' μ be fixed. The following sequence of simple remarks leads to the description of the structure of μ. (We give a self-contained proof, some parts of our arguments are parallel to those of [2].)

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Remark 2. Axiom (A3) implies that there is no infinite sequence of pairwise disjoint sets B_n such that $B_n \geq \varepsilon > 0$ for all n. □

μ is monotone. There are no infinite 'properly' increasing chains of sets:

Remark 3. Let $A_i \in \mathcal{A}$, $i = 1, 2, \dots$. If $A_1 \subseteq A_2 \subseteq \dots$, then there exists n_0 such that $\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(A_n)$ if $n \geq n_0$.

Proof. Let $A = \bigcup_{i=1}^{\infty} A_i$, $\varepsilon = \mu(A)$. The case $\varepsilon = 0$ is obvious. Let $\varepsilon > 0$. As $A - A_n \downarrow \emptyset$, if $n \to \infty$, so there exists n_0 such that $\mu(A - A_n) < \varepsilon$ if $n \geq n_0$. As $\varepsilon = \mu(A) = \max\{\mu(A_n), \mu(A - A_n)\}$ we obtain that $\mu(A_n) = \varepsilon$ if $n \geq n_0$. □

It follows from the axioms that $\mu(\bigcup_{i=1}^{n} A_i) = \max_{1 \leq i \leq n} \mu(A_i)$. The same is true for countably infinite unions:

Remark 4 (Lemma 1.4 of [2]). Let $A_i \in \mathcal{A}$, $i = 1, 2, \dots$. Then $\mu(\bigcup_{i=1}^{\infty} A_i) = \max_{1 \leq i < \infty} \mu(A_i)$.

The proof follows from Remark 3. □

Remark 5. Let $\varepsilon > 0$. Then the set of values of $\mu(A)$, $A \in \mathcal{A}$, which are greater than ε is finite.

Proof. If $M = \{\mu(A) : A \in \mathcal{A},\ \mu(A) > \varepsilon\}$ is an infinite set, then there exists a point of accumulation of M. Then there exists either a strictly increasing infinite sequence $\{\varepsilon_i\} \subseteq M$ or a strictly decreasing infinite sequence $\{\varepsilon_i\} \subseteq M$. The first case contradicts to Remark 4. In the second case let $\mu(A_i) = \varepsilon_i$, $i = 1, 2, \dots$, and let $B_i = A_i - (\bigcup_{k>i} A_k)$, $i = 1, 2, \dots$. Then $B_1, B_2, \dots$ are pairwise disjoint. Moreover, the strictly decreasing property imply $\mu(B_i) = \varepsilon_i$, $i = 1, 2, \dots$. This contradicts to Remark 2. □

It is obvious that the number of pairwise disjoint sets A_i with $\mu(A_i) = \varepsilon > 0$ is finite. Moreover, the cardinality of such sets is bounded:

Remark 6. Let $\varepsilon > 0$. Then there exists a finite constant k_ε such that for any sequence of pairwise disjoint sets $A_i \in \mathcal{A}$, with $\mu(A_i) = \varepsilon$, $i = 1, 2, \dots, k$ we have $k \leq k_\varepsilon$.

Proof. Suppose that there exists a sequence $n_i \to \infty$ such that for each $i = 1, 2, \dots$ there exist pairwise disjoint subsets of Ω

$$A_1^{(i)}, A_2^{(i)}, \dots, A_{n_i}^{(i)}$$

with $\mu(A_k^{(i)}) = \varepsilon$, $k = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots$. We can assume that for each i the set $\left\{A_1^{(i)}, \ldots, A_{n_i}^{(i)}\right\}$ is maximal in the sense that there is no subset C_i with $\mu(C_i) = \varepsilon$ and $C_i \cap A_k^{(i)} = \emptyset$ for each k. Let $\left\{\hat{A}_1^{(1)}, \ldots, \hat{A}_{n_1}^{(1)}\right\}$ be defined as $\left\{A_1^{(1)}, \ldots, A_{n_1}^{(1)}\right\}$. It is easy to see that one can substitute the original set $\left\{A_1^{(2)}, \ldots, A_{n_2}^{(2)}\right\}$ with a set $\left\{\hat{A}_1^{(2)}, \ldots, \hat{A}_{n_2}^{(2)}\right\}$ which consists of pairwise disjoint sets with $\hat{A}_k^{(1)} \supseteq \hat{A}_k^{(2)}$ for $k = 1, 2, \ldots, n_1$ and $\mu\left(\hat{A}_k^{(2)}\right) = \varepsilon$ for $k = 1, 2, \ldots, n_2$. Now, apply the above argument for $\left\{\hat{A}_1^{(2)}, \ldots, \hat{A}_{n_2}^{(2)}\right\}$ and $\left\{A_1^{(3)}, \ldots, A_{n_3}^{(3)}\right\}$. By induction we get an array of sets for which $\bigcap_i \hat{A}_k^{(i)} = B_k$, $i = 1, 2, \ldots$, is a sequence of pairwise disjoint sets with $\mu(B_k) = \varepsilon$ for each k. It is a contradiction. □

The main feature of an atom (in terminology of [2] indecomposable atom) that it can not be split into two 'essential' parts. More precisely:

Definition 7. A set $A \in \mathcal{A}$ is called an atom if $\mu(A) > 0$ and and for any $B \in \mathcal{A}$ either $\mu(A \cap B) = \mu(A)$, $\mu(A - B) = 0$, or $\mu(A - B) = \mu(A)$, $\mu(A \cap B) = 0$. A set of pairwise disjoint atoms is called maximal if there is no set of positive μ measure which is disjoint to each member of the set. □

Remark 8. Let the values of the function $\mu : \mathcal{A} \to [0, 1]$ be $\delta_1 > \delta_2, \ldots$, $\delta_k \downarrow 0$ (otherwise the sequence $\delta_1, \delta_2, \ldots$ is finite). For each δ_k let

$$B_1^{(k)}, B_2^{(k)}, \ldots, B_{n_k}^{(k)}$$

be pairwise disjoint subsets with $\mu\left(B_l^{(k)}\right) = \delta_k$, for $l = 1, 2, \ldots, n_k$, for which the cardinality n_k is maximal. Then the sets

$$\hat{B}_i^{(k)} = B_i^{(k)} - \left[\bigcup_{l>k} \left(\bigcup_{j=1}^{n_l} B_j^{(l)}\right)\right],$$

$i = 1, 2, \ldots, n_k$, $k = 1, 2, \ldots$, is a maximal set of pairwise disjoint atoms. □

Proof. It is obvious that the sets $\hat{B}_i^{(k)}$ are disjoint. As the sequence $\{\delta_k\}$ is strictly decreasing we have $\mu\left(\hat{B}_i^{(k)}\right) = \delta_k$ for all i and k. As the cardinality of each set $\left\{B_1^{(k)}, B_2^{(k)}, \ldots, B_{n_k}^{(k)}\right\}$ is maximal therefore each $\hat{B}_i^{(k)}$ is an atom and there is no set of positive μ-measure outside $\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{n_k} \hat{B}_i^{(k)}$. □

The above remarks imply the following structure theorem.

Theorem 9 (see [2], Theorem 1.2). *There exists a countable set of atoms A_i, $i = 1, 2, \ldots$, such that for each $B \in \mathcal{A}$ we have $\mu(B) = \max\{\mu(B \cap A_i) : i = 1, 2, \ldots\}$.*

Remark 10. The maximal set of disjoint atoms is unique in the following sense. If $\{A_i : i = 1, 2, \ldots\}$ and $\{B_i : i = 1, 2, \ldots\}$ are two maximal sets of pairwise disjoint atoms then there exists a one-to-one correspondence $A_i \to B_i$, say, of the first set onto the second such that $\mu(A_i \cap B_i) = \mu(A_i) = \mu(B_i)$ and $\mu(A_i \cap B_j) = 0$ for $i \neq j$.

Remark 11. Atoms in the 'optimal measure theory' are the same as in the Lebesgue measure theory in the following sense. If A is an atom with $0 < \mu(A) \leq 1$ where μ is a measure in Lebesgue measure theory then A is an atom endowed with the same 'optimal measure' μ and vice versa.

Remark 12. The structure of 'optimal measures' is far from being as rich as measures in Lebesgue's theory. Optimal measure can be described as follows. Choose a sequence $a_n \downarrow 0$ with $0 < a_n \leq 1$, for all n, and choose atoms A_n (in the sense of the Lebesgue measure theory) with $\mu_n(A_n) = a_n$, $n = 1, 2, \ldots$. Let $\Omega = \bigcup_{n=1}^{\infty} A_n$, and let $\mathcal{A}$ be the σ-algebra generated by the σ-algebras on atoms A_n, and let $\mu(A) = \max_{1 \leq n < \infty} \mu_n(A_n \cap A)$ for each $A \in \mathcal{A}$.

Furthermore, it is easy to see that any measurable function is almost surely constant on an atom. Moreover, if $f^{(n)}$, $n = 1, 2, \ldots$, is a sequence of measurable functions on $(\Omega, \mathcal{A}, \mu)$, where μ is an 'optimal measure', then one can find a maximal set A_k, $k = 1, 2, \ldots$, of pairwise disjoint atoms such that each function is constant on each atom. Therefore up to a set of zero measure the sequence of functions can be described with the sequence of numerical sequences $\left\{\varphi_k^{(n)}, k = 1, 2, \ldots\right\}$, $n = 1, 2, \ldots$, where $\varphi_k^{(n)} = f^{(n)}(\omega)$ if $\omega \in A_k$, for each k and n. □

References

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