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On k-Lagrange geometry

By MAGDALEN SZ. KIRKOVITS (Sopron)

Abstract. In this paper we describe the theory of k-Lagrange-geometry. It is a suitable geometrical model for studying variational problems of multiple integrals in a geometrical manner. We consider the vector bundle $\eta = \left(\bigoplus_1^k TM, \pi, M\right)$ and study the geometry of the total space $E = \bigoplus_1^k TM$.

1. Introduction

Preliminaires and motivations.

It is known that the fundamental problem in the calculus of variations for multiple integrals can be briefly formulated as follows ([11]). Let R be some domain of the space of the variables x^i. (In the sequel Latin and Greek indices assume the values $1, \dots, n$ and $1, \dots, k$ $(k < n)$, respectively, the summation convention being used in both cases.) We will call *admissible* the class of functions $x^i(t^\alpha), \overline{x}^i(t^\alpha), \dots$ defined on the same domain G_t of t^α if they are of class C^2 and coincide with each other on the boundary ∂G_t of G_t.

Suppose that we have a function $\mathcal{L}(x^j(t^\beta), \dot{x}^j_\alpha(t^\beta))$; $\dot{x}^j_\alpha := \partial x^j / \partial t^\alpha$, also of class C^2 and defined over each space $C_k : x^i = x^i(t^\alpha)$ of the admissible class. Moreover, let G_t be a fixed, bounded and simply connected domain in the k-dim. space of t^α. One can then form the following k-fold integral

$$(1.1) \qquad I(C_k) = \int_{G_t} \mathcal{L}\big(x^j(t^\beta), \dot{x}^j_\alpha(t^\beta)\big)\, d(t); \qquad d(t) := dt^1 \dots dt^k.$$

The fundamental variational problem for a multiple integral (1.1) is to establish necessary and sufficient conditions for an admissible set of functions $x^i(t^\alpha)$ in order that it gives an extreme of (1.1) relative to other admissible sets. A necessary condition for this is that the first variation δI of the fundamental integral (1.1) should vanish. This implies that $x^i(t^\alpha)$ must satisfy the system of n second order partial differential equations:

$$(1.2) \qquad \varepsilon_i(\mathcal{L}) := \frac{d}{dt^\alpha}\frac{\partial \mathcal{L}}{\partial \dot{x}^i_\alpha} - \frac{\partial \mathcal{L}}{\partial x^i} = 0 \qquad (\textit{summation over } \alpha).$$

Remark. ε_i are the components of the covariant Euler–Lagrange vector ([12]).

A question of the variational calculus for single or multiple integrals is the *equivalence* between two variational problems of the same type. This was studied by C. CARATHÉODORY ([1]), A. MOÓR ([9], [10]) and also by H. RUND ([11], [12]).

In [4] the authoress has considered a generalized version of the equivalence of two variational problems for single integrals treated by MOÓR ([9]). This problem has the following form in Lagrange spaces $(M, \mathcal{L}^*(x,y))$ and $(M, \mathcal{L}(x,y))$ ([6], [7])

$$(1.3) \qquad \varepsilon_i(\mathcal{L}^*(x,y)) = \lambda(x,y)\, \varepsilon_i(\mathcal{L}(x,y)); \quad \lambda(x,y) \neq 0,$$

where λ depends not only on x but on y too. In [4] two geometrical conditions were found which are equivalent to (1.3). Moreover, necessary and sufficient conditions for this equivalence were established. We note that in the proofs *only geometrical methods* of the theory of Lagrange spaces were used.

A. MOÓR ([10]) gave the most general definition of the *equivalence of two variational problems* for *multiple integrals* and he investigated it in some important cases but he did not investigate its geometrical meanings.

Our purpose is to construct a *geometrical model* for multiple integrals in the calculus of variations, then to study the MOÓR equivalence in a geometrical manner and to give other applications. In a joint paper ([8]) we have briefly sketched the first results. Now we describe the theory of k–Lagrange geometry by using as a model the geometry of the total space of a vector bundle developed by R. MIRON ([5]). We remark that our theory is based on the study of a metric which is derived from a Lagrangian and thus it differs from Günther's theory ([3]).

2. Vector bundles, differential structure on $E = \bigoplus_1^k TM$

We consider the 1–*jet bundle* $\mathcal{J}^1(R^k, TM)$ $(k < n)$ over an n–dimensional manifold M. This bundle is isomorphic to the vector bundle $\mathrm{Hom}(R^k, TM) \to M$. Moreover, if we fix a basis $(e_1, \ldots, e_k)$ of R^k there is the isomorphy: $\mathrm{Hom}(R^k, TM) \simeq \bigoplus_1^k TM = \overbrace{TM \oplus TM \oplus \ldots \oplus TM}^{k\text{-times}}$ ([3]). We shall systematically use the latter fact. Denoting $E = \bigoplus_1^k TM$ and by π its projection on M, we shall study the vector bundle $\eta = (E, \pi, M)$ and the geometry of the total space E. Clearly $\dim E = nk$.

Let (U, ψ) be a chart on M. Then $(U \times R^{kn}, \varphi)$ is a vector chart of the vector bundle η where $\varphi : \pi^{-1}(U) \to U \times R^{kn}$. For any $X_x \in \bigoplus_1^k T_x M$, $x \in M$ we get

$$\varphi(X_x) = (y^i_\alpha), \qquad y^i_\alpha \in \overset{\alpha}{T}M. \tag{2.1}$$

Hence we have for every fixed α

$$\underset{\alpha}{X}_x = y^i_\alpha (\partial/\partial x^i)_x. \tag{2.2}$$

This means that any vector $X_x \in \bigoplus_1^k T_x M$ is determined by the following components

$$X_x = (y^i_1 \partial/\partial x^i, \ldots, y^i_k \partial/\partial x^i)\ . \tag{2.3}$$

We put $(x^i) = \psi(x)$ and define

$$h : \pi^{-1}(U) \to \psi(U) \times R^{kn} \tag{2.4}$$

by

$$h(\underset{1}{X}_x, \ldots, \underset{k}{X}_x) = (x^i, y^i_\alpha) \in R^n \times R^{kn} \tag{2.5}$$

which are *canonical coordinates* on $\pi^{-1}(U)$. The set of charts $(\pi^{-1}(U), h)$ defines a vector atlas on $E = \bigoplus_1^k TM$.

Denote $\psi_j \circ \psi_i^{-1}(x^s) = (\overline{x}^s)$, then

$$(h_j \circ h_i^{-1})(x^s, y^s_\alpha) = (\overline{x}^s(x^1, \ldots, x^n),\ \partial_k \overline{x}^s(x) y^k_\alpha) \quad (\partial_k := \partial/\partial x^k)$$

i.e. for a coordinate transformation on U the corresponding coordinate transformation on $\pi^{-1}(U)$ is

$$\begin{aligned} \overline{x}^s &= \overline{x}^s(x^1, \dots, x^n); \quad \mathrm{rank}(\partial_k \overline{x}^s) = n \\ \overline{y}^s_\alpha &= \partial_k \overline{x}^s y^k_\alpha \qquad (\alpha = \overline{1,k}). \end{aligned} \tag{2.6}$$

The transformation law shows that (y^i_α) can be considered as a *contravariant vector.* In the sequel we denote y^i_α by y^a where $\binom{i}{\alpha} := a$ and use a shorter notation $a, b, c, \dots (a', b', c', \dots)$ instead of double contravariant (or covariant) indices $\binom{i}{\alpha}$ (or $\binom{\alpha}{i}$) and $\binom{i'}{\alpha}$ (or $\binom{\alpha}{i'}$) respectively, if the computation allows it.

Let $T_u E$ be the tangent space of E at u. Its basis is $(\partial_i, \partial^\alpha_i) := (\partial_i, \partial_a)$, where $\partial_i := \partial / \partial x^i$ and $\partial_a := \partial^\alpha_i = \partial / \partial y^i_\alpha$. Hence a tangent vector $X_u \in T_u E$ looks locally as follows

$$\begin{aligned} &X_u = X^i \partial_i + \dot{X}^i_\alpha \partial^\alpha_i := X^i \partial_i + \dot{X}^a \partial_a \\ &(a = \overline{1, nk},\ (X^i) \in R^n,\ (\dot{X}^a) \in R^{nk}). \end{aligned} \tag{2.7}$$

The change of the local basis on $E = \bigoplus\limits_1^k TM$ is given by

$$\begin{aligned} \partial_i &= \partial_i \overline{x}^k \overline{\partial}_k + \partial_j \partial_i \overline{x}^k y^j_\alpha \overline{\partial}^\alpha_k; \quad (\overline{\partial}_k := \partial / \partial \overline{x}^k)\ (\overline{\partial}^\alpha_k := \partial / \partial \overline{y}^k_\alpha) \\ \partial^\alpha_i &= \partial_i \overline{x}^k \overline{\partial}^\alpha_k \quad (\alpha = \overline{1,k}). \end{aligned} \tag{2.8}$$

The dual basis is denoted by (dx^i, dy^i_α). Its transformation law follows from (2.6):

$$\begin{aligned} d\overline{x}^i &= \partial_k \overline{x}^i dx^k; \\ d\overline{y}^i_\alpha &= \partial_j \partial_k \overline{x}^i y^j_\alpha dx^k + \partial_k \overline{x}^i dy^k_\alpha, \quad (\alpha = \overline{1,k}). \end{aligned} \tag{2.9}$$

Let us consider the vector bundle $\left(\bigoplus\limits_1^k TM, \pi, M \right) := (E, \pi, M)$. Then $D\pi : TE \to TM$ is the *differential* map of π. The mapping $D\pi$ is a π–morphism which maps the tangent bundle (TE, π_E, E) of E into the tangent bundle (TM, π, M) of M. Here $\pi_E : TE \to E$ and $\pi : TM \to M$ are the projections. Put $(VE, \pi_V, E) := \ker D\pi$. $VE = (VE, \pi_V, E)$ is called *vertical subbundle* over E. Its total space is $VE = \bigcup\limits_{u \in E} V_u E$. The vertical subspace $V_u E$ of $T_u E$ is spanned by $\{\partial_a\}$. It is easy to see that

$X_u \in V_uE$ iff $X^i = 0$. The map $u \to V_uE$ over E determines the vertical distribution. Since $[\partial_a, \partial_b] = 0$ one gets that this distribution is integrable.

Now we define for each α an *operator*

$$\overset{\alpha}{\mathcal{J}} : T_uE \to T_uE \tag{2.10}$$

by

$$\overset{\alpha}{\mathcal{J}}(\partial_i) = \partial_i^\alpha, \qquad \overset{\alpha}{\mathcal{J}}(\partial_i^\beta) = 0 \qquad (\beta = \overline{1,k}). \tag{2.11}$$

It is easy to check that

$$\overset{\alpha}{\mathcal{J}}{}^2 = 0 \quad \text{and} \quad \ker \overset{\alpha}{\mathcal{J}} = \mathrm{Im} \overset{\alpha}{\mathcal{J}} = V_uE \tag{2.12}$$

hold for every α.
So we have obtained that the manifold E can be endowed with k–*different almost tangent structures.*

It is not difficult to see that the Nijenhuis tensor associated to $\overset{\alpha}{\mathcal{J}}$ vanishes for every α, i.e. the almost tangent structures are integrable.

3. Nonlinear connection on $E = \bigoplus\limits_1^k TM$

By the general theory the following sequence of vector bundles

$$0 \to VE \underset{C}{\overset{\iota}{\rightleftarrows}} TE \xrightarrow{\pi!} \pi^*TM \to 0 \tag{3.1}$$

is an exact sequence. Here $\pi^*(TM)$ is the pull-back of TM over E by π, ι is the inclusion map and $\pi!(X)$ is given by $\pi!(X) = (\pi_E(X), D\pi(X))$ where $\pi_E : TE \to E$ is the projection.

Definition 3.1. A nonlinear connection on E is a splitting at the left of the sequence (3.1), i.e. a map $C : TE \to VE$ such that $C \circ \iota = \mathrm{id}|_{VE}$. The kernel $HE = (HE, \pi_H, E)$ of the morphism C is a subbundle of $TE \xrightarrow{\pi_E} E$ which will be called the horizontal bundle over E.

One gets for the total spaces

$$TE = HE \oplus VE \quad (\textit{Whitney sum}). \tag{3.2}$$

Conversely, the existence of a subbundle HE of $TE \xrightarrow{\pi_E} E$ which satisfies (3.2) implies the existence of a morphism like C, i.e. a nonlinear connection on E.

Remark 3.1. By a general result there exist always nonlinear connections on E provided M is paracompact ([5]).

The exact sequence (3.1) looks locally as follows

$$0 \to \psi(U) \times R^{kn} \times \{0\} \times R^{kn} \xrightarrow{\iota} \psi(U) \times R^{kn} \times R^{n} \times R^{kn} \to$$

$$\to \psi(U) \times R^{kn} \times R^{n} \to 0,$$

where

$$\iota(x, y, 0, \dot{X}) = (x, y, 0, \dot{X}); \quad \pi!(x, y, X, \dot{X}) = (x, y, X).$$

Here $x := (x^i)$, $y := (y^a)$ and $X := (X^i)$, $\dot{X} := (\dot{X}^a)$.
The map C is described locally as

$$(x, y, X, \dot{X}) \to \big(x, y, 0, C_\varphi(x, y, X, \dot{X})\big),$$

where C_φ is a map linear in X and $\dot{X}$. The condition $C \circ \iota = \mathrm{id}|_{VE}$ implies that

$$\text{(3.3) (a)} \qquad C_\varphi(x, y, X, \dot{X}) = \dot{X}^i_\alpha + N^i_{\alpha j}(x, y) X^j := \dot{X}^a + N^a{}_j X^j.$$

This shows that C can locally be written as

$$\text{(3.3) (b)} \qquad (x, y, X, \dot{X}) \to (x, y, 0, \dot{X}^a + N^a{}_j(x, y) X^j).$$

We have obtained a set of real functions $N^a{}_j$ defined on the domain of the local charts of E. These functions determine a nonlinear connection N.

It is not difficult to check that if $\overline{N}^a{}_j$ is a similar set of functions on $\pi^{-1}(V)$ with $U \cap V \neq \emptyset$ then on $\pi^{-1}(U \cap V)$ we have the following transformation law

$$\text{(3.4)} \qquad \overline{N}^i_{\alpha j}(\overline{x}, \overline{y}) \partial_s \overline{x}^j = \partial_k \overline{x}^i N^k_{\alpha s}(x, y) - y^k_\alpha \partial_s \partial_k \overline{x}^i.$$

Conversely, a set of functions $N^a{}_j$ which transform by (3.4) when the local chart is changed, defines a nonlinear connection on E.

In view of (3.2) the existence of a nonlinear connection implies the existence of a π–isomorphism between $HE \xrightarrow{\pi_H} E$ and $E \xrightarrow{\pi} M$. It follows that every tangent vector field Z on M determines a horizontal vector Z^h on E such that $D\pi(Z^h) = Z$. Z^h is the horizontal lift of Z. Taking $\delta_i = (\partial_i)^h$ one obtains a local basis of $H_u E$ ($u \in E$). Generally we have

$\delta_i = A_i^j\partial_j + B_{\alpha i}^j\partial_j^\alpha$. From $D\pi(\delta_i) = \partial_i$ and $C(\delta_i) = 0$ we get $A_i^j = \delta_i^j$ and $B_{\alpha i}^j = -N_{\alpha i}^j$, respectively. Thus we have

$$\delta_i = \partial_i - N_{\alpha i}^j\partial_j^\alpha := \partial_i - N^a{}_i\partial_a. \tag{3.5}$$

It is clear that (δ_i, ∂_a) is an *adapted basis* for the decomposition (3.2). Its dual basis is $(dx^i, \delta y^a)$, where

$$\delta y^a = dy^a + N^a{}_j dx^j. \tag{3.6}$$

Their transformation laws are

$$\begin{array}{c}(\text{a})\ \delta_i = \partial_i\overline{x}^s\overline{\delta}_s\ (\overline{\delta}_s := \delta/\delta\overline{x}^s)\quad (\text{b})\ \partial_i^\alpha = \partial_i\overline{x}^s\overline{\partial}_s^\alpha\ (\overline{\partial}_s^\alpha := \partial/\partial\overline{y}_\alpha^s)\\ (\text{c})\ d\overline{x}^i = \partial_j\overline{x}^i dx^j\quad (\text{d})\ \delta\overline{y}_\alpha^i = \partial_j\overline{x}^i\delta y_\alpha^j.\end{array} \tag{3.7}$$

By a direct calculation we obtain

$$(\text{a})\ [\delta_j, \delta_k] = R^a{}_{jk}\partial_a\quad (\text{b})\ [\delta_j, \partial_b] = \partial_b N^a{}_j\partial_a \tag{3.8}$$

where

$$R^a{}_{jk} = \delta_k N^a{}_j - \delta_j N^a{}_k. \tag{3.9}$$

Thus the horizontal distribution $u \to H_uE$ is integrable iff $R^a{}_{jk} = 0$.

Definition 3.2. The tensor algebra spanned by 1, δ_i, ∂_a, dx^i, δy^a is called the algebra of d–tensor fields over $\mathcal{F}(E)$.

For convenience we give examples of tensor fields of type (1, 1), (2, 0) and (0, 2): $t^i{}_j\delta_i \otimes dx^j$, $t^a{}_b\partial_a \otimes \delta y^b$, $t^{ia}\delta_i \otimes \partial_a$, $t^{ab}\partial_a \otimes \partial_b$, $t_{aj}\delta y^a \otimes dx^j$.

Remark 3.2. All the coefficients of these tensor products change like the coefficients of a tensor field on M with respect to the Latin indices, the Greek indices being unchanged.

Remark 3.3. The functions $R^a{}_{jk}$ define the d–tensor field $R = R^a{}_{jk}dx^j \otimes dx^k \otimes \partial_a$ which is called the *integrability* tensor of the horizontal distribution (cf. [6]).

4. Tensorial structures on $E = \bigoplus_1^k TM$

Let us suppose that there exists on E a nonlinear connection such that (3.1) holds. Then two *supplementary projectors* ν, h and an *almost* *product structure* $P = h - \nu$ can be considered. Locally these operators are as follows:

$$(4.1)\qquad \begin{array}{llll} \text{(a)} & \nu(\delta_i) = 0 & \text{(b)} & \nu(\partial_a) = \partial_a \\ \text{(c)} & h(\delta_i) = \delta_i & \text{(d)} & h(\partial_a) = 0 \\ \text{(e)} & P(\delta_i) = \delta_i & \text{(f)} & P(\partial_a) = -\partial_a. \end{array}$$

It is easy to check that the following equalities hold

$$(4.2)\qquad \overset{\alpha}{\mathcal{J}} P = \overset{\alpha}{\mathcal{J}}; \qquad P\overset{\alpha}{\mathcal{J}} = -\overset{\alpha}{\mathcal{J}}$$

for every α.

For P we have

Theorem 4.1. *If $P : T_uE \to T_uE$ $(u \in E)$ is an endomorphism satisfying (4.2) then $P^2 = I$ and the eigenspace corresponding to the eigenvalue -1 is a vertical subspace.*

PROOF.

A. P can be expressed locally as follows:

$$P(\partial_i) = P^j{}_i\,\partial_j + P^a{}_i\,\partial_a; \quad P(\partial_a) = P_a{}^j\partial_j + P_a{}^b\partial_b.$$

Then $\overset{\alpha}{\mathcal{J}} P = \overset{\alpha}{\mathcal{J}}$ yields $P^j{}_i = \delta_i^j$ and $P_a{}^j = 0$. Moreover, $P\overset{\alpha}{\mathcal{J}} = -\overset{\alpha}{\mathcal{J}}$ implies that $P(\partial_a) = -\partial_a$. Hence $P(\partial_i) = \partial_i + P^a{}_i\partial_a$ and $P(\partial_a) = -\partial_a$. Using these expressions a short calculation shows that $P^2 = I$.

B. Suppose that $PX = -X$ $(X \in T_uE)$. From this and $\overset{\alpha}{\mathcal{J}} P = \overset{\alpha}{\mathcal{J}}$ we obtain $\overset{\alpha}{\mathcal{J}} X = \overset{\alpha}{\mathcal{J}} PX = \overset{\alpha}{\mathcal{J}}(-X) = -\overset{\alpha}{\mathcal{J}} X$ and hence $\overset{\alpha}{\mathcal{J}} X = 0$. This implies $X^i = 0$ in $X = X^i\partial_i + \dot{X}^a\partial_a$, i.e. $X \in V_uE$.

To a nonlinear connection we can associate a set of *F–structures* in K. Yano's sense ([13]). Indeed, if we put

$$(4.3)\qquad \begin{array}{l} \overset{\alpha}{F}(\delta_i) = -\partial_i^\alpha \\ \overset{\alpha}{F}(\partial_i^\beta) = \delta^{\alpha\beta}\delta_i \end{array} \qquad \text{where } \delta^{\alpha\beta} = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

we get k tensor fields of type $(1,\,1)$ on E which satisfy

$$(4.4)\qquad \overset{\alpha}{F}{}^3 + \overset{\alpha}{F} = 0 \qquad (\alpha = \overline{1,k})$$

as it is easy to check.

Another set of *tensorial structures* on E can be defined as follows:

$$(4.5)\qquad \overset{\alpha}{Q}(\delta_i) = \partial_i^\alpha, \quad \overset{\alpha}{Q}(\partial_i^\alpha) = \delta_i, \quad \overset{\alpha}{Q}(\partial_i^\beta) = 0 \quad (\alpha \neq \beta).$$

We can easily calculate that

$$(4.6)\qquad \overset{\alpha}{Q}{}^3 - \overset{\alpha}{Q} = 0.$$

In the last part of the next section we study the *integrability* of these structures.

5. d–connections on $E = \bigoplus_{1}^{k} TM$

Let us suppose that E is endowed with a nonlinear connection.

Definition 5.1. A linear connection on E is a map $C : TE \to E$ for which

$$C(x, y, X, \dot{X}) = (x, \dot{X}^a + K^a_{bi}(x)X^i y^b) \qquad (a, b = \overline{1, nk})$$

holds.

Remark 5.1. The real functions $K^a_{bi}(x)$ defined on M determine a linear connection D on E ([6]).

Definition 5.2. A linear connection D on E is said to be a distinguished connection (shortly d–connection) if D preserves by parallelism the vertical distribution $u \to V_u E$ as well as the horizontal distribution $u \to H_u E$.

Theorem 5.1. *A linear connection D on E is a d–connection iff one of the following conditions holds:*

$$(5.1)\quad \begin{aligned} &(a)\quad vD_X(hY) = 0,\ \ hD_X(vY) = 0\ \ (X, Y \in \mathcal{X}(E))\\ &(b)\quad Dv = 0,\ \ Dh = 0\\ &(c)\quad DP = 0\\ &(d)\quad D_X Y = hD_X(hY) + vD_X(vY)\ \ (X, Y \in \mathcal{X}(E))\\ &(e)\quad vD_X(h\omega) = 0,\ \ hD_X(v\omega) = 0\ \ (\omega \in \Lambda^1(E),\ \ X \in \mathcal{X}(E))\\ &(f)\quad D_X\omega = hD_X(h\omega) + vD_X(v\omega). \end{aligned}$$

PROOF.

A. Suppose that D is a d–connection. The Definition 5.2. gives $D_X(hY) \in HE$ and $D_X(vY) \in VE$. Hence we directly get the condition (5.1) (a). The conditions $Dv = 0$ and $Dh = 0$ are equivalent to $D_X(vY) = vD_X Y$ and to $D_X(hY) = hD_X Y$ respectively. Since we have an almost product structure $P = h - v$ the condition (c) is equivalent to (b). Moreover, since $D_X Y = hD_X(hY) + vD_X(hY) + hD_X(vY) + vD_X(vY)$ (cf. [6]) and D is a d–connection we get the condition (d). The conditions (e) and (f) are analogous to (a) and (d).

B. A direct calculus shows that any condition in (5.1) is sufficient in order that D should be a d–connection.

The following decomposition holds and is unique for every $X, Y \in \mathcal{X}(E)$:

$$D_X Y = D_{hX} Y + D_{vX} Y. \qquad (5.2)$$

Putting

$$D_X^h Y = D_{hX} Y,\ D_X^h \omega = D_{hX}\omega,\ D_X^h(f) = (hX)f \tag{5.3}$$

for $X, Y \in \mathcal{X}(E)$, $\omega \in \Lambda^1(E)$, $f \in \Lambda^0(E)$ (cf. [6]) and extending this operator by linearity and Leibniz rule one obtains an *operator* of *covariant derivation* in the algebra of the d-tensor fields over E called the *h-covariant derivation.* Similarly putting

$$D_X^v Y = D_{vX} Y,\ D_X^v \omega = D_{vX}\omega,\ D_X^v(f) = (vX)f \tag{5.4}$$

we obtain the operator of the *v-covariant derivation* in the same algebra.

In local coordinates D^h and D^v, respectively appear as follows:

$$\begin{aligned} &\text{(a)}\quad D^h_{\delta_k}\delta_j = L^i_{jk}\delta_i;\ D^h_{\delta_k}\partial^\beta_j = L^{i\beta}_{\alpha jk}\partial^\alpha_i;\ D^h_{\delta_k} f = \delta_k f \\ &\text{(b)}\quad D^v_{\partial^\beta_k}\delta_j = C^{i\beta}_{jk}\delta_i;\ D^v_{\partial^\beta_k}\partial^\gamma_j = C^{\gamma i\beta}_{j\alpha k}\partial^\alpha_i;\ D^v_{\partial^\beta_k} f = \partial^\beta_k f. \end{aligned} \tag{5.5}$$

So we obtain a set of functions defined locally on E

$$D\Gamma = \left(L^i_{jk}(x,y),\ L^{i\beta}_{\alpha jk}(x,y),\ C^{i\beta}_{jk}(x,y),\ C^{\gamma i\beta}_{j\alpha k}(x,y)\right) \tag{5.6}$$

which gives a d-connection D.

Let $x^i = x^i(\overline{x})$ and $\overline{x}^i = \overline{x}^i(x)$ respectively be a transformation of the local coordinates on a neighbourhood of M. Then the above coefficients change as follows:

$$\begin{aligned} \overline{L}^i_{jk} &= \partial_s\overline{x}^i\overline{\partial}_j x^m\overline{\partial}_k x^n L^s_{mn} - \partial_m\partial_n\overline{x}^i\overline{\partial}_j x^m\overline{\partial}_k x^n; \\ \overline{L}^{i\beta}_{\alpha jk} &= \partial_s\overline{x}^i\overline{\partial}_j x^m\overline{\partial}_k x^n L^{s\beta}_{\alpha mn} - \partial_m\partial_n\overline{x}_i\overline{\partial}_j x^m\overline{\partial}_k x^n; \\ \overline{C}^{i\beta}_{jk} &= \partial_s\overline{x}^i\overline{\partial}_j x^m\overline{\partial}_k x^n C^{s\beta}_{mn}; \\ \overline{C}^{i\gamma\beta}_{\alpha jk} &= \partial_s\overline{x}^i\overline{\partial}_j x^m\overline{\partial}_k x^n C^{s\gamma\beta}_{\alpha mn}. \end{aligned} \tag{5.7}$$

Theorem 5.2. *The formulae (5.7) characterize the coefficients of a d-connection. If a set of functions $D\Gamma$ satisfying (5.7) is given on E then by (5.5) (a), (b) and (5.7) we obtain h-and v-covariant derivatives, and by (5.2) a d-connection on E.*

We give the local form of the h- and v-covariant derivatives of some tensor fields:
h-covariant derivative:

$$t^i_{a|k} = \delta_k t^i{}_a + L^i_{sk} t^s{}_a - L^b_{ak} t^i{}_b.$$

v-covariant derivative:

$$t^a{}_b|_c = \partial_c t^a{}_b + C^a_{dc} t^d{}_b - C^d_{bc} t^a{}_d.$$

It is obvious that the vector field $C = y^a\partial_a$ is globally defined on E.

Definition 5.3. A d–connection D on E is said to be of Cartan type if

$$\text{(a)} \quad D^h_X C = 0, \qquad \text{(b)} \quad D^v_X C = vX$$

hold for every $X \in \mathcal{X}(E)$ and

$$(5.8) \qquad \text{(c)} \quad D^a{}_k = y^b L^a_{bk} - N^a{}_k = 0.$$

$D^a{}_k$ is called the *deflection tensor* of D.

Theorem 5.3. *A d–connection D on E is of Cartan type iff*

$$(5.9) \qquad D^a{}_k = 0 \text{ and } y^a C^c_{ab} = 0.$$

Indeed, the conditions in (5.9) are equivalent to (5.8) (a), (b), (c).

The *torsion* of a d–connection D on E is defined as usual:

$$\mathbf{T}(X,Y) := D_X Y - D_Y X - [X,Y]; \quad (X, Y \in \mathcal{X}(E)).$$

Decomposition into vertical and horizontal parts leads to the following *fiv* d–tensor fields which will be called the *torsion tensors* of D :

$$(5.10) \quad \begin{aligned} T(X,Y) &= h\mathbf{T}(hX,hY) = D^h_X(hY) - D^h_Y(hX) - h[hX,hY];\\ R(X,Y) &= v\mathbf{T}(hX,hY) = -v[hX,hY];\\ C(X,Y) &= h\mathbf{T}(hX,vY) = -D^v_Y(hX) - h[hX,vY];\\ P(X,Y) &= v\mathbf{T}(hX,vY) = D^h_X(vY) - v[hX,vY];\\ S(X,Y) &= v\mathbf{T}(vX,vY) = D^v_X(vY) - D^v_Y(vX) - v[vX,vY].\\ &(X, Y \in \mathcal{X}(E)). \end{aligned}$$

In local coordinates we get

$$T(\delta_j,\delta_k) = T^i{}_{kj}\delta_i; \;\; R(\delta_k,\delta_j) = R^a{}_{jk}\partial_a; \;\; C(\delta_k,\partial_b) = C^i_{kb}\delta_i$$
$$P(\delta_k,\partial_b) = P^a{}_{bk}\partial_a; \;\; S(\partial_b,\partial_c) = C^a_{bc}\partial_a$$

and the *torsion tensor fields* of the d–connection D are

$$(5.11) \quad \begin{aligned} &\text{(a)} \quad T^i{}_{kj} = L^i_{kj} - L^i_{jk} \qquad \text{(b)} \quad R^a{}_{jk} = \delta_j N^a{}_k - \delta_k N^a{}_j\\ &\text{(c)} \quad C^i_{kb} \quad \text{(d)} \quad P^a{}_{bk} = \partial_b N^a{}_k - L^a{}_{bk} \quad \text{(e)} \quad S^a{}_{bc} = C^a_{bc} - C^a_{cb}. \end{aligned}$$

In the usual way we get six *curvature tensors*. These are the followi ones in local form:

$$(5.12) \quad \begin{aligned} R(\delta_k,\delta_j)\delta_l &= R_l{}^i{}_{jk}\delta_i, & R(\delta_k,\delta_j)\partial_b &= \tilde{R}_b{}^a{}_{jk}\partial_a,\\ R(\partial_c,\delta_k)\delta_j &= P_j{}^i{}_{kc}\delta_i, & R(\partial_c,\delta_k)\partial_b &= \tilde{P}_b{}^a{}_{kc}\partial_a,\\ R(\partial_c,\partial_b)\delta_j &= S_j{}^i{}_{bc}\delta_i, & R(\partial_d,\partial_c)\partial_b &= \tilde{S}_b{}^a{}_{cd}\partial_a, \end{aligned}$$

Next we define a *particular case* of d–connection.

Definition 5.4. A linear d–connection $D\Gamma = (L^i_{jk},\ L^{i\beta}_{\alpha kj},\ C^{i\beta}_{jk},\ C^{i\gamma\beta}_{\alpha jk})$ on E is normal if $D\overset{\alpha}{F} = 0$, i.e. the F–structures $\overset{\alpha}{F}$ are absolute parallel by D for every α.

Since $D_X(\overset{\alpha}{F}Y) = (D_X\overset{\alpha}{F})Y + \overset{\alpha}{F}(D_XY)$ the condition $D\overset{\alpha}{F} = 0$ is equivalent to

$$D_X(\overset{\alpha}{F}Y) = \overset{\alpha}{F}(D_XY). \tag{5.13}$$

From this we obtain in local coordinates for h–covariant derivatives:

$$\text{(5.14)} \qquad \text{(a)} \qquad D^h_{\delta_k}\overset{\alpha}{F}(\delta_j) = \overset{\alpha}{F}(D^h_{\delta_k}\delta_j),$$

$$\text{i.e.} \qquad \text{(b)} \qquad L^{i\alpha}_{\beta jk}\partial^\beta_i = L^i_{jk}\partial^\alpha_i.$$

Moreover, we have

$$\text{(5.15)} \qquad \text{(a)} \qquad D^h_{\delta_k}\overset{\alpha}{F}(\partial^\alpha_j) = \overset{\alpha}{F}(D^h_{\delta_k}\partial^\alpha_j),$$

$$\text{i.e.} \qquad \text{(b)} \qquad \delta^\alpha_\beta L^s_{jk}\delta_s = L^{\beta s}_{j\alpha k}\delta_s.$$

So we obtain

$$L^{\beta s}_{j\alpha i} = \delta^\beta_\alpha L^s_{ji}. \tag{5.16}$$

If $\alpha \neq \beta$ then

$$D^h_{\delta_k}\overset{\alpha}{F}(\partial^\beta_j) = \overset{\alpha}{F}(D^h_{\delta_k}\partial^\beta_j) = 0. \tag{5.17}$$

We can carry out similar calculations for v–covariant derivatives:

$$\text{(5.18)} \qquad \text{(a)} \qquad D^v_{\partial^\beta_i}\overset{\alpha}{F}(\delta_j) = \overset{\alpha}{F}(D^v_{\partial^\beta_i}\delta_j),$$

$$\text{i.e.} \qquad \text{(b)} \qquad C^{\alpha l\beta}_{j\gamma i}\partial^\gamma_l = C^{s\beta}_{ji}\partial^\alpha_s.$$

The next step is the following

$$\text{(5.19)} \qquad \text{(a)} \qquad D^v_{\partial^\beta_i}\overset{\alpha}{F}(\partial^\gamma_j) = \overset{\alpha}{F}(D^v_{\partial^\beta_i}\partial^\gamma_j) \quad (\alpha = \gamma),$$

$$\text{i.e.} \qquad \text{(b)} \qquad \delta^\gamma_\varepsilon C^{s\beta}_{ji}\delta_s = C^{\gamma s\beta}_{j\varepsilon i}\delta_s.$$

Finally we get

$$C^{\gamma s\beta}_{j\varepsilon i} = \delta^\gamma_\varepsilon C^{s\beta}_{ji}. \tag{5.20}$$

Hence we have

Theorem 5.4. *A linear d–connection $D\Gamma$ is normal iff*

$$(5.21) \qquad \text{(a)} \qquad L^{s\beta}_{\alpha ji} = \delta^{\beta}_{\alpha} L^{s}_{ji}$$

$$\text{and} \qquad \text{(b)} \qquad C^{\gamma s\beta}_{j\varepsilon i} = \delta^{\gamma}_{\varepsilon} C^{s\beta}_{ji}$$

hold.

Thus a normal d–connection is completely determined by $(L^{i}_{jk}, C^{i\beta}_{jk})$.

Its torsions are as follows:

$$(5.22) \qquad \begin{array}{l} \text{(a)} \quad T^{i}{}_{kj} = L^{i}_{kj} - L^{i}_{jk} \quad \text{(b)} \quad T^{i}_{\alpha kj} = R^{i}_{\alpha kj} \quad \text{(c)} \quad C^{i\beta}_{jk} \\ \text{(d)} \quad P^{\beta\alpha}_{jik} = \partial^{\beta}_{j} N^{i}_{\alpha k} - \delta^{\beta}_{\alpha} L^{i}_{jk} \qquad \text{(e)} \quad S^{\beta k\alpha}_{j\gamma i} = \delta^{\beta}_{\gamma} C^{k\alpha}_{ji} - \delta^{\alpha}_{\gamma} C^{k\beta}_{ij}. \end{array}$$

The number of curvatures reduces to three instead of six.

In the previous section we have defined a set of F–structures $\overset{\alpha}{F}$.

Remark 5.2. Because of

$$(5.23) \qquad \begin{array}{ll} \text{(a)} \quad \overset{\alpha}{F}\overset{\beta}{F}(\delta_i) \underset{(\alpha\neq\beta)}{=} -\delta^{\alpha\beta}\delta_i = 0, & \text{(b)} \quad \overset{\beta}{F}\overset{\alpha}{F}(\delta_i) \underset{(\alpha\neq\beta)}{=} -\delta^{\alpha\beta}\delta_i = 0, \\ \text{(c)} \quad \overset{\alpha}{F}\overset{\beta}{F}(\partial^{\gamma}_{i}) = -\delta^{\beta\gamma}\partial^{\alpha}_{i}, & \text{(d)} \quad \overset{\beta}{F}\overset{\alpha}{F}(\partial^{\gamma}_{i}) = -\delta^{\alpha\gamma}\partial^{\beta}_{i} \end{array}$$

we get

$$(5.24) \qquad \overset{\alpha}{F}\overset{\beta}{F} \neq \overset{\beta}{F}\overset{\alpha}{F} \quad \text{if} \quad \alpha \neq \beta.$$

On the other hand we have

$$(5.25) \qquad \overset{\alpha}{F}{}^{4} = -\overset{\alpha}{F}{}^{2}.$$

(5.25) shows that the operators $P_1 = -\overset{\alpha}{F}{}^2$, $P_2 = I + \overset{\alpha}{F}{}^2$ are two supplementary projectors, and taking their kernels one obtains two distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ which are spanned locally by $\{\partial^{\beta}_{i}\}$ $(\beta \neq \alpha)$ and $(\delta_i, \partial^{\alpha}_{i})$, respectively.

We define in the sense of V. Duc ([2]) that $\overset{\alpha}{F}$ is integrable if $\mathcal{D}_1$ and $\mathcal{D}_2$ are involutive. He treated a general Fstructure and proved that it is integrable iff the *Nijenhuis tensor* of *its square* vanishes.

Next we prove

Proposition 5.1. *The Nijenhuis tensor of* $\overset{\alpha}{F}{}^2$ *is equal to zero iff*

$$\text{(5.26)} \qquad \text{(a)} \quad R^k_{\beta ij} = 0 \quad \text{and} \quad \text{(b)} \quad \partial^\alpha_j (N^s_{\beta i}) = 0$$

for any $\beta \neq \alpha$.

PROOF. The Nijenhuis tensor of $\overset{\alpha}{F}{}^2$ is

$$\text{(5.27)} \quad \begin{aligned} N_{\overset{\alpha}{F}{}^2}(X,Y) = \left[\overset{\alpha}{F}{}^2 X, \overset{\alpha}{F}{}^2 Y\right] - \overset{\alpha}{F}{}^2 \left[\overset{\alpha}{F}{}^2 X, Y\right] - \overset{\alpha}{F}{}^2 \left[X, \overset{\alpha}{F}{}^2 Y\right] + \\ + \overset{\alpha}{F}{}^4 [X,Y] \qquad (X,Y) \in \mathcal{X}(E)). \end{aligned}$$

Hence by using (5.10), (5.24) and (5.25) we get for the adapted basis $(\delta_i, \partial^\alpha_i)$ the following equalities:

$$\begin{aligned} \text{(a)} \ N_{\overset{\alpha}{F}{}^2}(\delta_i, \delta_j) &= \left[\overset{\alpha}{F}{}^2 \delta_i, \overset{\alpha}{F}{}^2 \delta_j\right] - \overset{\alpha}{F}{}^2 \left[\overset{\alpha}{F}{}^2 \delta_i, \delta_j\right] - \overset{\alpha}{F}{}^2 \left[\delta_i, \overset{\alpha}{F}{}^2 \delta_j\right] + \\ &+ \overset{\alpha}{F}{}^4 [\delta_i, \delta_j] = [\delta_i, \delta_j] - \overset{\alpha}{F}{}^2 \left[-\overset{\alpha}{F} \partial^\alpha_i, \delta_j\right] - \overset{\alpha}{F}{}^2 \left[\delta_i, -\overset{\alpha}{F} \partial^\alpha_j\right] - \\ &- \overset{\alpha}{F}{}^2 [\delta_i, \delta_j] = R^k_{\beta ij} \partial^\beta_k + \overset{\alpha}{F}{}^2 [\delta_i, \delta_j] + \overset{\alpha}{F}{}^2 [\delta_i, \delta_j] - \overset{\alpha}{F}{}^2 [\delta_i, \delta_j] = \\ &= R^k_{\beta ij} \partial^\beta_k + \overset{\alpha}{F}{}^2 (R^k_{\beta ij} \partial^\beta_k) = R^k_{\beta ij} \partial^\beta_k + R^k_{\beta ij} \left(\overset{\alpha}{F}{}^2 \partial^\beta_k\right) = \\ &= R^k_{\beta ij} \partial^\beta_k + R^k_{\beta ij} \overset{\alpha}{F} (\delta^{\alpha\beta} \delta_k) = R^k_{\beta ij} \partial^\beta_k - R^k_{\alpha ij} \partial^\alpha_k = R^k_{\beta ij} \partial^\beta_k, \\ &\qquad \text{(not summing over } \alpha, \ \beta \neq \alpha), \end{aligned}$$

$$\begin{aligned} \text{(b)} \ N_{\overset{\alpha}{F}{}^2}(\delta_i, \partial^\alpha_j) &= [\delta_i, \partial^\alpha_j] - \overset{\alpha}{F}{}^2 \left[\delta_i, \overset{\alpha}{F} (\delta^{\alpha\beta} \delta_j)\right] - \\ &- \overset{\alpha}{F}{}^2 \left[\overset{\alpha}{F} (-\partial^\alpha_i), \partial^\alpha_j\right] - \overset{\alpha}{F}{}^2 [\delta_j, \partial^\alpha_j] = [\delta_i, \partial^\alpha_j] + \overset{\alpha}{F}{}^2 [\delta_i, \partial^\alpha_j] - \\ &- \overset{\alpha}{F}{}^2 [-\delta^{\alpha\beta} \delta_i, \partial^\alpha_j] - \overset{\alpha}{F}{}^2 [\delta_i, \partial^\alpha_j] = \partial^\alpha_j N^s_{\beta i} \partial^\beta_s + \overset{\alpha}{F}{}^2 [\delta_i, \partial^\alpha_j] = \\ &= (\partial^\alpha_j N^s_{\beta i}) \partial^\beta_s + (\partial^\alpha_j N^s_{\beta i}) \overset{\alpha}{F} (\delta^{\alpha\beta} \delta_s) = (\partial^\alpha_j N^s_{\beta i}) \partial^\beta_s - (\partial^\alpha_j N^s_{\alpha i}) \partial^\alpha_s = \\ &= (\partial^\alpha_j N^s_{\beta i}) \partial^\beta_s \qquad (\beta \neq \alpha, \text{ not summing over } \alpha). \end{aligned}$$

If $\beta \neq \alpha$ then we have

$$\text{(c)} \quad N_{\overset{\alpha}{F}{}^2}(\delta_i, \partial^\beta_j) = [-\delta_i, 0] + \overset{\alpha}{F}{}^2 [\delta_i, \partial^\beta_j] - \overset{\alpha}{F}{}^2 [\delta_i, 0] - \overset{\alpha}{F}{}^2 [\delta_i, \partial^\beta_j] = 0.$$

Moreover, for $\beta \neq \alpha$ and $\gamma \neq \alpha$ respectively we get

$$(5.28)\ (d)\quad N_{\overset{\alpha}{F}^2}(\partial_i^\alpha, \partial_j^\beta) = [-\partial_i^\alpha, 0] + \overset{\alpha}{F}{}^2[\partial_i^\alpha, \partial_j^\beta] - \overset{\alpha}{F}{}^2[-\partial_i^\alpha, 0] - \overset{\alpha}{F}{}^2[\partial_i^\alpha, \partial_j^\beta] = 0,$$

$$(e)\quad N_{\overset{\alpha}{F}^2}(\partial_i^\beta, \partial_j^\gamma) = 0.$$

By linearity of the Nijenhuis tensor these equalities establish our assertion.

Remark 5.2. The condition (5.26) (b) shows that the functions $N^s_{\alpha i}$ do not depend on y^j_β if $\beta \neq \alpha$.

Let us consider the formulae (4.5) and (4.6). We define *three supplementary projectors.* $P_1 = I - \overset{\alpha}{Q}{}^2$, $P_2 = \frac{1}{2}\left(\overset{\alpha}{Q}{}^2 + \overset{\alpha}{Q}\right)$ and $P_3 = \frac{1}{2}\left(\overset{\alpha}{Q}{}^2 - \overset{\alpha}{Q}\right)$. So we have three *distributions* $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$. (V. Duc ([2]) treated a general structure K with $K^3 = K$.) In his sense $\overset{\alpha}{Q}$ is integrable if the distributions $\mathcal{D}_i + \mathcal{D}_j$ $(i, j = 1, 2, 3)$ are involutive. By this theorem $\overset{\alpha}{Q}$ is integrable iff $N_{\overset{\alpha}{Q}} = 0$.

Now we prove

Proposition 5.2. *The Nijenhuis tensor of $\overset{\alpha}{Q}$ is equal to zero iff*

$$(5.29)\quad \begin{array}{llll} (a) & \partial_i^\alpha N^s_{\alpha j} = \partial_j^\alpha N^s_{\alpha i} & (b) & R^k_{\alpha ij} = 0 \\ (c) & \partial_i^\alpha N^s_{\beta j} = 0 \quad (\beta \neq \alpha) & (d) & \partial_j^\beta N^s_{\alpha i} = 0 \quad (\beta \neq \alpha). \end{array}$$

PROOF. By using the relations (4.5) and (5.27) for $N_{\overset{\alpha}{Q}}$ it will be sufficient to calculate that

$$(a)\quad N_{\overset{\alpha}{Q}}(\delta_i, \delta_j) = \left[\overset{\alpha}{Q}\delta_i, \overset{\alpha}{Q}\delta_j\right] - \overset{\alpha}{Q}\left[\overset{\alpha}{Q}\delta_i, \delta_j\right] - \overset{\alpha}{Q}\left[\delta_i, \overset{\alpha}{Q}\delta_j\right] + \overset{\alpha}{Q}{}^2[\delta_i, \delta_j] = [\partial_i^\alpha, \partial_j^\beta] - \overset{\alpha}{Q}[\partial_i^\alpha, \delta_j] - \overset{\alpha}{Q}[\delta_i, \partial_j^\alpha] + \overset{\alpha}{Q}{}^2\left(R^k_{\gamma ij}\partial_k^\gamma\right) = \overset{\alpha}{Q}((\partial_i^\alpha N^s_{\beta j})\partial_s^\beta) - \overset{\alpha}{Q}((\partial_j^\alpha N^s_{\beta i})\partial_s^\beta) + R^k_{\gamma ij}\overset{\alpha}{Q}(\delta^{\alpha\gamma}\delta_k) = (\partial_i^\alpha N^s_{\alpha j} - \partial_j^\alpha N^s_{\alpha i})\delta_s + R^k_{\alpha ij}\partial_k^\alpha,$$

$$(b)\quad N_{\overset{\alpha}{Q}}(\delta_i, \partial_j^\alpha) = \left[\overset{\alpha}{Q}\delta_i, \overset{\alpha}{Q}\partial_j^\alpha\right] - \overset{\alpha}{Q}\left[\overset{\alpha}{Q}\delta_i, \partial_j^\alpha\right] - \overset{\alpha}{Q}\left[\delta_i, \overset{\alpha}{Q}\partial_j^\alpha\right] +$$

$$+ \overset{\alpha}{Q}^2[\delta_i, \partial_j^\alpha] = [\partial_i^\alpha, \delta_j] - \overset{\alpha}{Q}[\partial_i^\alpha, \partial_j^\alpha] - \overset{\alpha}{Q}[\delta_i, \delta_j] +$$
$$+ \overset{\alpha}{Q}^2((\partial_j^\alpha N_{\beta i}^s)\partial_s^\beta) = -(\partial_i^\alpha N_{\beta i}^s)\partial_s^\beta - R_{\gamma ij}^k \left(\overset{\alpha}{Q}\partial_k^\gamma\right) +$$
$$+ (\partial_j^\alpha N_{\beta i}^s)\overset{\alpha}{Q}(\delta^{\alpha\beta}\delta_s) = (\partial_j^\alpha N_{\alpha i}^s)\partial_s^\alpha - (\partial_i^\alpha N_{\beta j}^s)\partial_s^\beta - R_{\alpha ij}^k \delta_k =$$
$$= -\partial_i^\alpha N_{\beta j}^s \partial_s^\beta - R_{\alpha ij}^k \delta_k \text{ (not summing over } \alpha;\ \beta \neq \alpha); \tag{5.30}$$

for $\beta \neq \alpha$ we have

$$\text{(c)} \qquad N_{\overset{\alpha}{Q}}(\delta_i, \partial_j^\beta) = \left[\overset{\alpha}{Q}\delta_i, \overset{\alpha}{Q}\partial_j^\beta\right] - \overset{\alpha}{Q}\left[\overset{\alpha}{Q}\delta_i, \partial_j^\beta\right] - \overset{\alpha}{Q}\left[\delta_i, \overset{\alpha}{Q}\partial_j^\beta\right] +$$
$$+ \overset{\alpha}{Q}^2\left[\delta_i, \partial_j^\beta\right] = [\partial_i^\alpha, 0] - \overset{\alpha}{Q}\left[\partial_i^\alpha, \partial_j^\beta\right] - \overset{\alpha}{Q}[\delta_i, 0] +$$
$$+ \overset{\alpha}{Q}^2((\partial_j^\beta N_{\gamma i}^s)\partial_s^\gamma) = (\partial_j^\beta N_{\gamma i}^s)\overset{\alpha}{Q}(\delta^{\alpha\gamma}\delta_s) = (\partial_j^\beta N_{\alpha i}^s)\partial_s^\alpha$$
$$\text{(not summing over } \alpha),$$

and

$$\text{(d)} \quad N_{\overset{\alpha}{Q}}(\partial_i^\alpha, \partial_j^\beta) = 0, \qquad \text{(e)} \quad N_{\overset{\alpha}{Q}}(\partial_j^\beta, \partial_k^\gamma) = 0 \quad (\gamma \neq \alpha).$$

These establish our assertion. We can summarize our results in

Theorem 5.5. *Any* $\overset{\alpha}{F}$ *and any* $\overset{\alpha}{Q}$ *respectively is integrable iff the conditions* (5.26) (a), (b) *and* (5.29) (a), (b), (c), (d) *respectively hold.*

Remark 5.2. Integrability of all $\overset{\alpha}{Q}$ implies integrability of all $\overset{\alpha}{F}$. The converse is not true.

6. Geometry of Lagrangians on $E = \bigoplus_1^k TM$

Let M be endowed with a nonlinear connection.

Definition 6.1. A function $\mathcal{L} : \bigoplus_1^k TM \backslash \{0\} \to R$ is said to be a regular Lagrangian on E if the matrix with the elements

$$g_{ab} = \partial_a \partial_b \mathcal{L}(x^i, y^c) \tag{6.1}$$

is of rank nk.

Definition 6.2. A k–Lagrange space L^n_k is a pair $(M, \mathcal{L}(x^i, y^a))$ where M is an n–dim. manifold and $\mathcal{L}(x^i, y^a)$ is a regular Lagrangian defined over $E = \bigoplus\limits_1^k TM$.

Proposition 6.1. *The set of functions $g_{ab}(x,y)$ and $g_{ij}(x,y)$ defines v–and h–Riemannian structures in the vertical and horizontal bundles as follows:*

$$\text{(a)} \quad g : u \to g_u : VE \times VE \to R \quad (u \in E)$$

$$g_u(X,Y) = X^a Y^b g_{ab}(x,y) \quad (X, Y \in \mathcal{X}(VE))$$

$$\text{(6.2)} \quad \text{(b)} \quad g : u \to g_u : HE \times HE \to R \qquad (u \in E)$$

$$g_u(X,Y) = g(X^i\delta_i, Y^j\delta_j) = X^iY^ig(\delta_i, \delta_j) = X^iY^ig_{ij}(x,y);$$

$$det\|g_{ij}\| \neq 0, \ (X, Y \in \mathcal{X}(HE)).$$

Proof. Under the conditions rank$\|g_{ab}\| = nk$ and rank$\|g_{ij}\| = n$, for each $u \in E$, g_u is a nondegenerate bilinear form on $VE \times VE$ and $HE \times HE$, respectively. Moreover, $g(X,Y) = g(Y,X)$ for all $X, Y \in \mathcal{X}(VE)$ and $\mathcal{X}(HE)$, respectively. These prove the Proposition.

We shall denote by g^{cd} the *inverse* of the matrix g_{ab}, i.e. the following relations hold:

$$g_{ab}g^{bc} = \delta^c_a, \qquad g^{cd}g_{da} = \delta^c_a \tag{6.3}$$

where δ^c_a are the components of the Kronecker tensor on E.

Let us suppose that we have h– and v–*Riemannian structures*, i.e. $g_{ij}(x,y)$ and $g_{ab}(x,y)$. Then the following *metric* can be considered on E

$$G = g_{ij}(x,y)dx^i \otimes dx^j + g_{ab}(x,y)\delta y^a \otimes \delta y^b. \tag{6.4}$$

Definition 6.3. A d–connection $LD = (L^i_{jk}, L^{\alpha j}_{i\beta k}, C^{i\alpha}_{jk}, C^{\gamma i\beta}_{k\alpha j})$ is called metrical with respect to G if

$$g_{ij|k} = 0, \quad g_{ij}|^{\alpha}_{k} = 0, \quad g_{ab|k} = 0, \quad g_{ab}|^{\alpha}_{k} = 0, \tag{6.5}$$

where the short and long bars mean h– and v–covariant derivatives.

Theorem 6.1. *The following d–connection is metrical and its torsion tensors T and S vanish:*

$$(6.6)\quad \begin{aligned} &\text{(a)} && L^i_{jk} = \frac{1}{2} g^{il}(\delta_j g_{kl} + \delta_k g_{jl} - \delta_l g_{jk}) \\ &\text{(b)} && L^{\alpha j}_{i\beta k} = \partial^\alpha_i (N^j_{\beta k}) + \frac{1}{2} g^{js}_{\beta\gamma}(\delta_k g^{\alpha\gamma}_{is} - \partial^\alpha_i (N^r_{\varepsilon k}) g^{\varepsilon\gamma}_{rs} - \partial^\gamma_s (N^r_{\varepsilon k}) g^{\varepsilon\alpha}_{ri}) \\ &\text{(c)} && C^{i\beta}_{jk} = \frac{1}{2} g^{is} \partial^\beta_k (g_{js}) \\ &\text{(d)} && C^{\gamma i\beta}_{k\alpha j} = \frac{1}{2} g^{is}_{\alpha\varepsilon}(\partial^\gamma_k g^{\beta\varepsilon}_{js} + \partial^\beta_j g^{\gamma\varepsilon}_{ks} - \partial^\varepsilon_s g^{\gamma\beta}_{kj}). \end{aligned}$$

PROOF. We can easily see that L^i_{jk} and $C^{\gamma i\beta}_{k\alpha j}$ are *symmetric* in j, k and $\binom{\gamma}{k}, \binom{\beta}{j}$, respectively. Moreover, taking into account the relations (5.11) (a) and (e), we get $T = S = 0$. The equalities in (6.5) can be checked by a direct calculus.

Remark 6.1. If M is endowed with a metric $\tilde{g}_{ij}(x)$ then we can define its horizontal lift $g_u(x, y) = X^i Y^j g_{ij}(x)$. In this case the coefficients of LD are simpler, since the $\tilde{g}_{ij}$ depend on x only not on y. We want to investigate this case. We obtain $C^{i\beta}_{jk} = 0$. On the other hand

$$C^{\gamma i\beta}_{k\alpha j} = \frac{1}{2} g^{is}_{\alpha\varepsilon} \frac{\partial^3 \mathcal{L}}{\partial y^j_\beta \partial y^k_\gamma \partial y^s_\varepsilon}; \quad g^{\varepsilon\alpha}_{si} C^{\gamma i\beta}_{k\alpha j} := \tilde{C}^{\gamma\varepsilon\beta}_{ksj} = \frac{1}{2} \frac{\partial^3 \mathcal{L}}{\partial y^j_\beta \partial y^k_\gamma \partial y^s_\varepsilon}.$$

It follows that $\tilde{C}^{\gamma\varepsilon\beta}_{ksj}$ is symmetric in the pair of indices $\binom{\varepsilon}{s}, \binom{\gamma}{k}, \binom{\beta}{j}$. Moreover we have $L^i_{jk} = \frac{1}{2} g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk})$.

Remark 6.2. The torsion tensors T, S, C of LD considered in Remark 6.1. vanish.

Remark 6.3. Generally $R \neq 0$, and from (5.11) (a) and (6.6) (b) we get

$$P^{\beta i}_{j\alpha k} = \partial^\beta_j (N^i_{\alpha k}) - L^{\beta i}_{j\alpha k} = -\frac{1}{2} g^{is}_{\alpha\gamma}(\delta_k g^{\beta\gamma}_{js} - \partial^\beta_j (N^r_{\varepsilon k}) g^{\varepsilon\gamma}_{rs} - \partial^\gamma_s (N^r_{\varepsilon k}) g^{\varepsilon\beta}_{rj}).$$

This implies

$$g^{\gamma\alpha}_{si} P^{\beta i}_{j\alpha k} = -\frac{1}{2}(\delta_k g^{\beta\gamma}_{js} - \partial^\beta_j (N^r_{\varepsilon k}) g^{\varepsilon\gamma}_{rs} - \partial^\gamma_s (N^r_{\varepsilon k}) g^{\varepsilon\beta}_{rj}).$$

Denote $\tilde{P}^{\beta\gamma}_{jsk} = g^{\gamma\alpha}_{si} P^{\beta i}_{j\alpha k}$ and we see that $\tilde{P}$ is symmetric in $\binom{\beta}{j}$ and $\binom{\gamma}{s}$.

Let D be a d–connection on $E = \bigoplus_1^k TM$. C_t a curve of M, $\tilde{C} : C_t \to \bigoplus_1^k TM$ a section of E over C_t and $X(x^i, y^i_\alpha)$ a k–Lagrangian vector field on $E = \bigoplus_1^k TM$.

Definition 6.3. The covariant derivative of the vector field X on $\tilde{C}$ with respect to D is

$$\frac{DX}{dt^\alpha} := \left(D^h_{\delta_i} X\right) \frac{dx^i}{dt^\alpha} + \left(D^v_{\partial^\beta_k} X\right) \frac{\delta y^k_\beta}{dt^\alpha}. \tag{6.7}$$

Definition 6.4. A k–Lagrangian vector field X in $\mathcal{X}(E)$ is called parallel on the section $\tilde{C} : C_t \to E = \bigoplus_1^k TM$ if

$$\frac{DX}{dt^\alpha} = 0 \qquad (\alpha = \overline{1, k}) \tag{6.8}$$

(cf. [7]).

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MAGDALEN SZ. KIRKOVITS
UNIV. OF FORESTRY AND WOOD SCIENCES
DEPARTMENT OF MATHEMATICS
9400—SOPRON
HUNGARY

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