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Notes on topological properties of $\varphi(L)$ spaces

By DANUTA STACHOWIAK - GNIŁKA (Poznań)

Abstract. The purpose of this paper is to investigate some topological properties of the space $\varphi(L)$ with the topology determined by a symmetric. Also, the connections between this topology and other topologies in $\varphi(L)$ are examined.

1. Let E be a nonempty set and let Σ be a σ–algebra of subsets of E. Moreover, let μ be a nonnegative, nontrivial and finite measure on Σ. Let φ be an even, nonnegative, finite function on $(-\infty,\infty)$, nondecreasing on $\langle 0,\infty)$, for which $\lim_{u\to\infty}\varphi(u)=\infty$, $\varphi(0)=0$ and $\varphi(u)>0$ for $u>0$. We denote by $\varphi(L)$ the set of all real–valued, μ–mesurable functions f defined on E with equality μ–almost everywhere for which

$$\int_E \varphi(f(x))d\mu<\infty.$$

If $f,g\in\varphi(L)$, then the number

$$d_\varphi(f,g)=\int_E \varphi(f(x)-g(x))d\mu$$

is called the φ–distance between f and g.

Further, we denote by

$$A_\varphi(f,\varepsilon)=\{g\in\varphi(L): d_\varphi(f,g)<\varepsilon\}$$

the ε–neighbourhood of $f\in\varphi(L)$, where $\varepsilon>0$ is arbitrary. By $\mathcal{T}_\varphi$ we denote the topology generated by the subbase $\{A_\varphi(f,\varepsilon)\}_{\substack{f\in\varphi(L)\\ \varepsilon>0}}$. The topological space obtained in this manner we denote $(\varphi(L),\mathcal{T}_\varphi)$.

By $\mathcal{T}_{d_\varphi}$ we denote the topology determined by the symmetric d_φ (see [1] or [3]). Hence $U \in \mathcal{T}_{d_\varphi}$ if and only if for any $f \in U$ there is an $\varepsilon > 0$ such that $A_\varphi(f,\varepsilon) \subset U$. The topological space $\varphi(L)$ with this topology is denoted by $(\varphi(L), \mathcal{T}_{d_\varphi})$.

Now, we define the operator p as follows

$$p(A) = \{f \in \varphi(L) : A_\varphi(f,\varepsilon) \cap A \neq \emptyset \text{ for every } \varepsilon > 0\},$$

where $A \subset \varphi(L)$ is an arbitrary set.

The purpose of this paper is to investigate conditions under which the operator p is a closure operator for $(\varphi(L), \mathcal{T}_{d_\varphi})$. We examine the problem of metrizability of a space $(\varphi(L), \mathcal{T}_{d_\varphi})$. Further, we introduce a base in this space and we compare the $\mathcal{T}_{d_\varphi}$ topology with the $\mathcal{T}_\varphi$ topology and with the Orlicz topology (see [2]).

In the sequel we shall need the following definitions:

We say that a function φ satisfies the condition (Δ_2) if there exist constants $C > 0$, $u_0 > 0$ such that

$$\varphi(2u) \leq C\varphi(u) \text{ for } u \geq u_0.$$

We say that a sequence $(f_n)_{n\geq 1}$, $f_n \in \varphi(L)$ for $n \geq 1$ is convergent to $f \in \varphi(L)$ in the sence of the φ–distance if and only if for every $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that $d_\varphi(f_n, f) < \varepsilon$ for $n > N(\varepsilon)$.

2. In this section we show under which conditions the operator p is a Kuratowski operator.

One can easily show that the following lemma holds.

Lemma 2.1. *The operator p has the following properties:*

1° $A \subset p(A)$ *for every* $A \subset \varphi(L)$,

2° $p(\emptyset) = \emptyset$,

3° $p(A \cup B) = p(A) \cup p(B)$ *for every* $A, B \subset \varphi(L)$,

4° *if* $A \subset B$, *then* $p(A) \subset p(B)$ *for every* $A, B \subset \varphi(L)$.

Lemma 2.2. *Let $A \subset \varphi(L)$ be an arbitrary set. Then $p(p(A)) \subset p(A)$ if and only if the following property*

(2.1) *for each $f \in \varphi(L)$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for each $g \in A_\varphi(f,\delta)$ there exists a $\gamma > 0$ such that $A_\varphi(g,\gamma) \subset A_\varphi(f,\varepsilon)$*

is fulfilled.

PROOF. *Sufficiency.* Let us suppose that $f \in p(p(A))$ and $f \notin p(A)$. Then there is an $\alpha > 0$ such that $A_\varphi(f,\alpha) \cap A = \emptyset$. Further $A_\varphi(f,\delta) \cap p(A) \neq \emptyset$, where a $\delta = \delta(f,\alpha) > 0$ is chosen accordingly to (2.1). Hence there is $g \in A_\varphi(f,\delta)$ such that $g \in p(A)$. From (2.1) there is a $\gamma = \gamma(g) > 0$ such that $A_\varphi(g,\gamma) \subset A_\varphi(f,\alpha)$. Clearly $A_\varphi(g,\gamma) \cap A \neq \emptyset$. This implies that $A_\varphi(f,\alpha) \cap A \neq \emptyset$, a contradiction.

Necessity. Let $p(p(A)) \subset p(A)$ for all $A \subset \varphi(L)$ and let us suppose that the property (2.1) is not fulfilled. Then there is $f_0 \in \varphi(L)$ and $\varepsilon_0 > 0$ such that for all $\delta > 0$ there is $g \in A_\varphi(f_0, \delta)$ such that $g \in p(\varphi(L) \setminus A_\varphi(f_0, \varepsilon_0))$. This implies that $A_\varphi(f_0, \delta) \cap p(\varphi(L) \setminus A_\varphi(f_0, \varepsilon_0)) \neq \emptyset$ and so $f_0 \in p(p(\varphi(L) \setminus A_\varphi(f_0, \varepsilon_0)))$. Hence $f_0 \in p(\varphi(L) \setminus A_\varphi(f_0, \varepsilon_0))$. That is $A_\varphi(f_0, \varepsilon_0) \cap (\varphi(L) \setminus A_\varphi(f_0, \varepsilon_0)) \neq \emptyset$, a contradiction.

Now, we shall show the sufficient and necessary condition under which the property (2.1) is fulfilled.

Lemma 2.3. *If $\varphi(+0) = 0$ and if φ statisfies the condition (Δ_2), then the property (2.1) is fulfilled.*

PROOF. There is an $\alpha > 0$ such that $\varphi(\alpha) < \dfrac{\varepsilon}{3\mu E}$. There exists a constant $C_\alpha > 0$ such that $\varphi(2u) \le C_\alpha \varphi(u)$ for $u \ge \frac{\alpha}{2}$ and hence

$$\varphi(u+v) \le C_\alpha(\varphi(u) + \varphi(v)) \quad \text{if } \max(u,v) \ge \frac{\alpha}{2}.$$

Let $A_\varphi(f, \varepsilon)$ be a given neighbourhood and let $0 < \delta < \dfrac{\varepsilon}{3C_\alpha}$. Let $g \in A_\varphi(f, \delta)$. Now, we choose a $\gamma > 0$ such that $\gamma \le \delta$. We shall prove that $A_\varphi(g, \gamma) \subset A_\varphi(f, \varepsilon)$. Let $h \in A_\varphi(g, \gamma)$. Then

$$d_\varphi(f,h) \le \int_{E_1} \varphi(\mid f(x) - g(x) \mid + \mid g(x) - h(x) \mid) d\mu +$$

$$\int_{E_2} \varphi(\mid f(x) - g(x) \mid + \mid g(x) - h(x) \mid) d\mu \le \varphi(\alpha)\mu E_1 + C_\alpha(\delta + \gamma) < \varepsilon,$$

where $E_1 = \{x \in E \ : \mid f(x) - g(x) \mid < \frac{\alpha}{2} \text{ and } \mid g(x) - h(x) \mid < \frac{\alpha}{2}\}$ and $E_2 = E \setminus E_1$. This implies that $h \in A_\varphi(f, \varepsilon)$.

Lemma 2.4. *Let μ be an atomless measure. If the property (2.1) is fulfilled, then the condition (Δ_2) holds.*

PROOF. Let $f \in \varphi(L)$ and $\varepsilon > 0$ be arbitrarily chosen. We choose a natural number $n \ge 1$ such that

$$\int_F \varphi(f(x)) d\mu < \delta \quad \text{if } \mu F \le \frac{\mu E}{n},$$

where a $\delta = \delta(f, \varepsilon) > 0$ is chosen accordingly to the property (2.1).

We define for $1 \le k \le n$

$$g_k(x) = \begin{cases} 0 & \text{if } x \in E_k\ , \\ f(x) & \text{if } x \in E \setminus E_k\ , \end{cases}$$

where $E = \bigcup_{k=1}^{n} E_k\,,\ \ E_i \cap E_j = \emptyset$ for all $i \neq j$ and $0 < \mu E_k = \dfrac{\mu E}{n}$ for all $1 \le k \le n$.

Clearly $g_k \in \varphi(L)$ and $g_k \in A_\varphi(f,\delta)$ for $1 \le k \le n$. From (2.1) for each $1 \le k \le n$ there is a $\gamma_k > 0$ such that $A_\varphi(g_k,\gamma_k) \subset A_\varphi(f,\varepsilon)$.

Now, we choose natural numbers $m_k \ge 1$ $(1 \le k \le n)$ such that

$$\int_G \varphi(f(x))d\mu < \gamma_k \ \text{ if } \ \mu G \le \frac{\mu E_k}{m_k}.$$

We define for $1 \le i \le m_k$ $(1 \le k \le n)$

$$f_i(x) = \begin{cases} f(x) & \text{if } x \in E \setminus E_k\ , \\ -f(x) & \text{if } x \in F_i\ , \\ 0 & \text{if } x \in E_k \setminus F_i\ , \end{cases}$$

where $E_k = \bigcup_{i=1}^{m_k} F_i$ $(1 \le k \le n)$, $F_i \cap F_j = \emptyset$ for all $i \neq j$, $0 < \mu F_i = \frac{\mu E_k}{m_k}$ for all $1 \le i \le m_k$ $(1 \le k \le n)$.

It is clear that $f_i \in \varphi(L)$ and $f_i \in A_\varphi(g_k,\gamma_k)$ for $1 \le i \le m_k$ $(1 \le k \le n)$. Hence $f_i \in A_\varphi(f,\varepsilon)$ for $1 \le i \le m_k$ $(1 \le k \le n)$. Thus $\varepsilon > \int_E \varphi(f_i(x) - f(x))d\mu \ge \int_{F_i} \varphi(2f(x))d\mu$ for $1 \le i \le m_k$ $(1 \le k \le n)$ and so

$$\int_{E_k} \varphi(2f(x))d\mu = \sum_{i=1}^{m_k} \int_{F_i} \varphi(2f(x))d\mu < m_k \cdot \varepsilon \quad \text{for } 1 \le k \le n.$$

Further

$$\int_E \varphi(2f(x))d\mu = \sum_{k=1}^{n} \int_{E_k} \varphi(2f(x))d\mu < \varepsilon \cdot \sum_{k=1}^{n} m_k < \infty.$$

We conclude that $2f \in \varphi(L)$ if $f \in \varphi(L)$. But this implies (compare [5], Lemma 2.3) that φ satisfies the condition (Δ_2).

Lemmas 2.1 – 2.4 immediately imply

Theorem 2.1. *If $\varphi(+0) = 0$, then the condition (Δ_2) is sufficient, and if μ is an atomless measure, it is also necessary for the operator p to be a Kuratowski operator.*

3. Here we examine the connection between the operator p and the closure operator for $\mathcal{T}_{d_\varphi}$.

Proposition 3.1. *A set $U \subset \varphi(L)$ is open in the $\mathcal{T}_{d_\varphi}$ topology if and only if $\varphi(L) \setminus U = p\big(\varphi(L) \setminus U\big)$.*

PROOF. $U \in \mathcal{T}_{d_\varphi}$ if and only if for every $f \in U$ there is an $\varepsilon > 0$ such that $A_\varphi(f,\varepsilon) \cap (\varphi(L) \setminus U) = \emptyset$ and so $f \in \varphi(L) \setminus p(\varphi(L) \setminus U)$. Hence $p(\varphi(L) \setminus U) = \varphi(L) \setminus U$.

Now, we define the operator I_p as follows

$I_p(A) = \varphi(L) \setminus p(\varphi(L) \setminus A)$, for an arbitrary set $A \subset \varphi(L)$.

Lemma 3.1. *The operator I_p has the following properties:*
1° *$I_p(A) \subset A$ for every $A \subset \varphi(L)$,*
2° *$f \in I_p(A_\varphi(f,\varepsilon))$ for every $f \in \varphi(L)$ and $\varepsilon > 0$,*
3° *if $\varphi(+0) = 0$, then the condition (Δ_2) is sufficient and if μ is an atomless measure, then it is also necessary in order that $I_p(A) \in \mathcal{T}_{d_\varphi}$ for every $A \subset \varphi(L)$.*

PROOF. 1° Apply Lemma 2.1.1°; 2° Evident.

3° *Sufficiency.* By Lemmas 2.2 and 2.3 we obtain:
$\varphi(L) \setminus I_p(A) = p(\varphi(L) \setminus A) = p\big(p(\varphi(L) \setminus A)\big) = p\big(\varphi(L) \setminus I_p(A)\big)$ for every set $A \subset \varphi(L)$. Proposition 3.1 gives $I_p(A) \in \mathcal{T}_{d_\varphi}$.

Necessity. Let $I_p(A) \in \mathcal{T}_{d_\varphi}$. Then (see Proposition 3.1) $\varphi(L) \setminus I_p(A) = p\big(\varphi(L) \setminus I_p(A)\big)$ for an arbitrary set $A \subset \varphi(L)$. Hence $p(\varphi(L) \setminus A) = p\big(p(\varphi(L) \setminus A)\big)$ and thus φ satisfies the condition (Δ_2) (see Lemmas 2.2 and 2.4).

In the sequel we shall denote the $\mathcal{T}_{d_\varphi}$–closure of a set A by $\overline{A}$.

Theorem 3.1. *The $\mathcal{T}_{d_\varphi}$–closure operator has the following properties:*
1° *$p(A) \subset \overline{A}$ for every set $A \subset \varphi(L)$,*
2° *$A = \overline{A}$ if and only if $A = p(A)$,*
3° *if $\varphi(+0) = 0$, then the condition (Δ_2) is sufficient, and if μ is an atomless measure, then it is also necessary in order that $\overline{A} = p(A)$.*

PROOF. 1° and 2° are trivial.

3° *Sufficiency.* Let us suppose that there is a set $A \subset \varphi(L)$ such that $p(A) \subsetneq \overline{A}$. Then there is $f \in \varphi(L)$ such that $f \in \overline{A}$ but $f \notin p(A)$. Hence (see Lemma 3.1) we have $I_p\big(A_\varphi(f,\varepsilon)\big) \cap A \neq \emptyset$ and $A_\varphi(f,\varepsilon) \cap A = \emptyset$ for some $\varepsilon > 0$, a contradiction.

Necessity. Let $p(A) = \overline{A}$ for every $A \subset \varphi(L)$ and suppose φ does not satisfy the condition (Δ_2). Then from Lemmas 2.2 and 2.4 there is a set $A \subset \varphi(L)$ such that $p(A) \subsetneq p(p(A))$. Hence $\overline{A} \subsetneq \overline{\overline{A}}$, a contradiction.

Corollary 3.1. *Let* $\mathrm{Int}A = \varphi(L) \setminus \overline{\varphi(L) \setminus A}$ *for* $A \subset \varphi(L)$. *Then*
1° $\mathrm{Int}A \subset I_p(A)$ *for every set* $A \subset \varphi(L)$,
2° $A = \mathrm{Int}A$ *if and only if* $A = I_p(A)$,
3° *if* $\varphi(+0) = 0$, *then the condition* (Δ_2) *is sufficient, and if* μ *is an atomless measure, then it is also necessary in order that* $\mathrm{Int}A = I_p(A)$.

Remark 3.1. If $\varphi(+0) = 0$, then the condition (Δ_2) is sufficient, and if μ is an atomless measure, then it is also necessary in order that $f \in \mathrm{Int}A_\varphi(f,\varepsilon)$ for all $f \in \varphi(L)$ and $\varepsilon > 0$.

PROOF. *Sufficiency* follows from Lemma 3.1.2° and Corollary 3.1.3°.
Necessity. Let us suppose that $f \in \mathrm{Int}A_\varphi(f,\varepsilon)$ for every $f \in \varphi(L)$ and $\varepsilon > 0$ and suppose φ does not satisfy the condition (Δ_2). Then (see Corollary 3.1.3°) there is a set $A \subset \varphi(L)$ such that $\mathrm{Int}A \subsetneq I_p(A)$. Hence there is $g \in \varphi(L)$ such that $g \in I_p(A)$ and $g \notin \mathrm{Int}A$. This implies that $A_\varphi(g,\varepsilon) \cap (\varphi(L) \setminus A) = \emptyset$ and $\mathrm{Int}A_\varphi(g,\varepsilon) \cap (\varphi(L) \setminus A) \neq \emptyset$ for some $\varepsilon > 0$, a contradiction.

4. In this section we shall give the conditions under which the space $(\varphi(L), \mathcal{T}_{d_\varphi})$ is metrizable.

Lemma 4.1. *If* $\varphi(+0) = 0$ *and* φ *satisfies the condition* (Δ_2), *then the symmetric* d_φ *satisfies the following condition: if* $\lim_{n\to\infty} d_\varphi(f_n, f) = 0$ *and* $\lim_{n\to\infty} d_\varphi(f_n, g_n) = 0$ *then,* $\lim_{n\to\infty} d_\varphi(g_n, f) = 0$.

PROOF. Let $\varepsilon > 0$ be arbitrary. There is a $\delta > 0$ such that $\varphi(\delta) < \dfrac{\varepsilon}{2\mu E}$. There exists a constant $C_\delta > 0$ such that $\varphi(u+v) \le C_\delta(\varphi(u) + \varphi(v))$ if $\max(u,v) \ge \frac{\delta}{2}$.

Now, let $f, f_n, g_n \in \varphi(L)$ for $n \ge 1$ be such that $\lim_{n\to\infty} d_\varphi(f_n, f) = 0$ and $\lim_{n\to\infty} d_\varphi(f_n, g_n) = 0$. Then there is a natural number N such that

$$d_\varphi(f_n, f) < \frac{\varepsilon}{4C_\delta} \text{ and } d_\varphi(f_n, g_n) < \frac{\varepsilon}{4C_\delta} \text{ for } n > N. \quad \text{Then}$$

$$d_\varphi(g_n, f) \le \int_{E_1} \varphi\big(|g_n(x) - f_n(x)| + |f_n(x) - f(x)|\big) d\mu +$$

$$\int_{E_2} \varphi\big(|g_n(x) - f_n(x)| + |f_n(x) - f(x)|\big) d\mu < \varepsilon \quad \text{for } n > N,$$

where $E_1 = \{x \in E : |f_n(x) - g_n(x)| < \frac{\delta}{2} \text{ and } |f_n(x) - f(x)| < \frac{\delta}{2}\}$ and $E_2 = E \setminus E_1$.

From the above lemma and the Niemytzki Theorem (see [4]) we obtain

Theorem 4.1. *If $\varphi(+0) = 0$ and φ satisfies the condition (Δ_2) then the space $(\varphi(L), \mathcal{T}_{d_\varphi})$ is metrizable.*

5. We shall consider the problem of existence of a base for the space $(\varphi(L), \mathcal{T}_{d_\varphi})$. Before doing so, it will be convenient to prove the following

Lemma 5.1. *Let $\varphi(+0) = 0$. The conditions*
(a) $\varphi(u+0) = \varphi(u)$ *for* $u > 0$,
(b) φ *satisfies the condition* (Δ_2)
are sufficient, the condition (a) *is also necessary, and if μ is an atomless measure, then* (b) *is necessary too, in order that $A_\varphi(f,\varepsilon) \in \mathcal{T}_{d_\varphi}$ for every $f \in \varphi(L)$ and $\varepsilon > 0$.*

PROOF. *Sufficiency.* If (a) and (b) hold, then (compare [5], Theorem 3.2)

(5.1) for each $f \in \varphi(L)$ and $\varepsilon > 0$ and for each $g \in A_\varphi(f,\varepsilon)$
there is a $\delta > 0$ such that $A_\varphi(g,\delta) \subset A_\varphi(f,\varepsilon)$.

Hence $A_\varphi(f,\varepsilon) \in \mathcal{T}_{d_\varphi}$ for every $f \in \varphi(L)$ end $\varepsilon > 0$.

Necessity. If the condition (a) does not hold, or if μ is an atomless measure and the condition (b) does not hold, then (compare [5], Theorem 3.2) there are $f_0 \in \varphi(L)$ and $\varepsilon_0 > 0$ such that $A_\varphi(f_0,\varepsilon_0) \notin \mathcal{T}_{d_\varphi}$.

Applying this lemma we obtain

Theorem 5.1. *Let $\varphi(+0) = 0$. The conditions*
(a) $\varphi(u+0) = \varphi(u)$ *for* $u > 0$,
(b) φ *statisfies the condition* (Δ_2)
are sufficient, the condition (a) *is also necessary, and if μ is an atomless measure, then* (b) *is necessary too, in order that the family $\{A_\varphi(f,\varepsilon)\}_{\substack{f\in\varphi(L)\\ \varepsilon>0}}$ be an open base for the space $(\varphi(L), \mathcal{T}_{d_\varphi})$.*

By Lemma 3.1, Corollary 3.1 and Remark 3.1 one can easily prove the following

Theorem 5.2. *Let $\varphi(+0) = 0$. The condition (Δ_2) is sufficient and if μ is an atomless measure, then it is also necessary in order that the family $\{\mathrm{Int} A_\varphi(f,\varepsilon)\}_{\varepsilon>0}$ be an open base for the space $(\varphi(L), \mathcal{T}_{d_\varphi})$ at the point f for every $f \in \varphi(L)$.*

6. We shall give some remarks on convergence in the class $\varphi(L)$.

Lemma 6.1. *If a sequence $(f_n)_{n\geq 1}$, $f_n \in \varphi(L)$ for $n \geq 1$ is convergent to $f \in \varphi(L)$ in the sense of the φ-distance, then it is convergent to f in the $\mathcal{T}_{d_\varphi}$ topology.*

PROOF. Let $U \in \mathcal{T}_{d_\varphi}$ be an arbitrary set such that $f \in U$. Then $A_\varphi(f,\varepsilon) \subset U$ for some $\varepsilon > 0$. Since the sequence $(f_n)_{n\geq 1}$ is convergent to f in the sense of the φ-distance, $f_n \in A_\varphi(f,\varepsilon)$ for $n > N(\varepsilon)$. Thus $f_n \in U$for $n > N(\varepsilon)$.

Lemma 6.2. *Let $\varphi(+0) = 0$ and let φ satisfy the condition (Δ_2). If a sequence $(f_n)_{n\geq 1}$, $f_n \in \varphi(L)$ for $n \geq 1$ is convergent to $f \in \varphi(L)$ in the $\mathcal{T}_{d_\varphi}$ topology, then it is convergent to f in the sense of the φ- distance.*

PROOF. Let $\varepsilon > 0$ be an arbitrary real. Then $f_n \in \operatorname{Int} A_\varphi(f,\varepsilon)$ for $n > N$, where N is some natural number. Hence $d_\varphi(f_n, f) < \varepsilon$ for $n > N$.

The above lemmas immediately imply

Theorem 6.1. *Let $\varphi(+0) = 0$ and let φ satisfy the condition (Δ_2). The sequence $(f_n)_{n\geq 1}$, $f_n \in \varphi(L)$ for $n \geq 1$ is convergent to $f \in \varphi(L)$ in the $\mathcal{T}_{d_\varphi}$ topology if and only if it is convergent to f in the sense of the φ-distance.*

7. There are connections between $\mathcal{T}_{d_\varphi}$ and $\mathcal{T}_\varphi$. On can easily prove that $\mathcal{T}_{d_\varphi} \subset \mathcal{T}_\varphi$. The converse inclusion is described by

Lemma 7.1. *Let $\varphi(+0) = 0$ The conditions*
(a) $\varphi(u + 0) = \varphi(u)$ *for* $u > 0$,
(b) φ *satisfies the condition* (Δ_2),
are sufficient, the condition (a) *is also necessary, and if μ is an atomless measure, the condition* (b) *is necessary too, in order that $\mathcal{T}_\varphi \subset \mathcal{T}_{d_\varphi}$.*

PROOF. *Sufficiency.* Let $U \subset \mathcal{T}_\varphi$ be an arbitrary set. Then $U = \bigcup_{t\in T} \bigcap_{i=1}^{n_t} A_\varphi\left(f_i^{(t)}, \varepsilon_i^{(t)}\right)$. This implies that for any $f \in U$ there is $t_0 \in T$ such that $f \in \bigcap_{i=1}^{n_{t_0}} A_\varphi\left(f_i^{(t_0)}, \varepsilon_i^{(t_0)}\right)$. The property (5.1) implies that there is $\delta > 0$ such that $A_\varphi(f,\delta) \subset \bigcap_{i=1}^{n_{t_0}} A_\varphi\left(f_i^{(t_0)}, \varepsilon_i^{(t_0)}\right)$ and so $A(f,\delta) \subset U$. Hence $U \in \mathcal{T}_{d_\varphi}$.

Necessity. If the condition (a) does not hold or, if μ is an atomless measure and the condition (b) does not hold, then (see Lemma 5.1) there are $f_0 \in \varphi(L)$ and $\varepsilon_0 > 0$ such that $A_\varphi(f_0,\varepsilon_0) \notin \mathcal{T}_{d_\varphi}$. This implies that $\mathcal{T}_\varphi \not\subset \mathcal{T}_{d_\varphi}$.

This lemma immediately implies

Theorem 7.1. *Let $\varphi(+0) = 0$. The conditions*
(a) $\varphi(u+0) = \varphi(u)$ *for* $u > 0$,
(b) φ *satisfies the condition* (Δ_2),
are sufficient, the condition (a) *is also necessary, and if μ is an atomless measure, the condition* (b) *is necessary too, in order that $\mathcal{T}_{d_\varphi} = \mathcal{T}_\varphi$.*

From Theorems 4.1 and 7.1 we obtain

Remark 7.1. If $\varphi(u+0) = \varphi(u)$ for $u \geq 0$ and if φ satisfies the condition (Δ_2), then the space $(\varphi(L), \mathcal{T}_\varphi)$ is metrizable.

Note that this remark is a generalization of Ul'yanov Theorem (see [6]); the assumption that φ is a continuous function may be replaced by the assumption that φ is right-continuous at every point.

8. In this section we assume additionally that φ is a continuous function on $(-\infty, \infty)$. Then it is possible to define an Orlicz space L^φ (see [2]) and to introduce an F–norm on L^φ as follows:

$$\|f\| = \inf\left\{a > 0 : \int\limits_E \varphi\left(\frac{f(x)}{a}\right) d\mu \leq a\right\}.$$

By $\mathcal{T}_{L^\varphi}$ we denote the topology generated by the metric $\varrho(f,g) = \|f-g\|$. Let $K(f,\varepsilon) = \{g \in L^\varphi : \varrho(f,g) < \varepsilon\}$.

The set $\varphi(L) \subset L^\varphi$ with the induced topology $\mathcal{T}_{L^\varphi}$ we denote by $(\varphi(L), \mathcal{T}_{L^\varphi})$.

Lemma 8.1. *The topology $\mathcal{T}_{d_\varphi}$ is coarser than $\mathcal{T}_{L^\varphi}$.*

PROOF. Let $U \in \mathcal{T}_{d_\varphi}$ and let $f \in U$ be arbitrary. Then there exists $0 < \varepsilon < 1$ such that $A_\varphi(f,\varepsilon) \subset U$. We shall prove that $K(f,\varepsilon) \subset A_\varphi(f,\varepsilon)$. Let $g \in K(f,\varepsilon)$. By Theorem 1.5 in [2] it follows that $d_\varphi(f,g) \leq \|f-g\|$ and so $g \in A_\varphi(f,\varepsilon)$. Hence $K(f,\varepsilon) \subset U$.

Lemma 8.2. *Let μ be an atomless measure and let φ do not satisfy the condition (Δ_2). Then there are $f \in \varphi(L)$ and $\varepsilon > 0$ such that $K(f,\varepsilon) \notin \mathcal{T}_{d_\varphi}$.*

PROOF. There are numbers $1 < u_1 < u_2 < \ldots < u_n < \ldots$ such that

$$\varphi(u_n) > 2^n \text{ and } \varphi\left(\left(1+\frac{1}{n}\right)u_n\right) > 2^n\varphi(u_n) \quad \text{for } n \geq 1.$$

Let $E = \Big(\bigcup\limits_{n=1}^{\infty} E_n\Big) \cup \Big(E \setminus \bigcup\limits_{n=1}^{\infty} E_n\Big)$, where the sets E_n satisfy the following conditions:

$E_i \cap E_j = \emptyset$ for all $i \neq j$, $E_n \subset E$ for all $n \geq 1$ and $0 < \mu E_n = \dfrac{\mu E}{2^n \varphi(u_n)}$ for all $n \geq 1$.

Now we define a function f as follows:

$$f(x) = \begin{cases} u_n & \text{if } x \in E_n, \\ 0 & \text{if } x \in E \setminus \bigcup_{n=1}^{\infty} E_n. \end{cases}$$

Clearly $f \in \varphi(L)$. Let an $\varepsilon > 0$ be such that $\varepsilon < \min(\mu E, 1)$. There is a natural number $n_1 \geq 1$ such that $\frac{1}{\varepsilon} > 1 + \frac{1}{n_1}$. Let a $\delta > 0$ be arbitrarily chosen. Then there is a natural number $n_2 \geq 1$ such that $0 < \dfrac{\mu E}{2^{n_2}} < \delta$. Let $n_0 = \max(n_1, n_2)$ and let

$$g(x) = \begin{cases} 2u_{n_0} & \text{if } x \in E_{n_0}, \\ f(x) & \text{if } x \in E \setminus E_{n_0}. \end{cases}$$

It is clear that $g \in \varphi(L)$ and $g \in A_\varphi(f, \delta)$. Let us suppose that $g \in K(f, \varepsilon)$. This implies that $\|g - f\| < \varepsilon < 1$ and hence $d_\varphi\left(\dfrac{g}{\varepsilon}, \dfrac{f}{\varepsilon}\right) \leq \varepsilon$. On the other hand $d_\varphi\left(\dfrac{g}{\varepsilon}, \dfrac{f}{\varepsilon}\right) \geq \displaystyle\int_{E_{n_0}} \varphi\left(\left(1 + \frac{1}{n_0}\right) u_{n_0}\right) d\mu > \mu E$, a contradiction. Hence $f \in K(f, \varepsilon)$, $g \in A_\varphi(f, \delta)$ but $g \notin K(f, \varepsilon)$. That is $A_\varphi(f, \delta) \not\subset K(f, \varepsilon)$ for all $\delta > 0$. Then $K(f, \varepsilon) \notin \mathcal{T}_{d_\varphi}$.

Lemma 8.3. *If φ satisfies the condition (Δ_2), then $\mathcal{T}_{L^\varphi} \subset \mathcal{T}_{d_\varphi}$.*

Proof. Let $U \in \mathcal{T}_{L^\varphi}$ and let $f \in U$ be arbitrary. Then there exists an $\varepsilon > 0$ such that $K(f, \varepsilon) \subset U$. There is a natural number $n \geq 1$ such that $\dfrac{1}{\varepsilon} \leq 2^n$ and there is a $\delta > 0$ such that $\varphi(2^n \delta) < \dfrac{\varepsilon}{4 \mu E}$. Hence there exists a constant $C_\delta > 0$ such that $\varphi(2^n u) \leq C_\delta^n \varphi(u)$ for $u \geq \delta$. Let $0 < \gamma < \dfrac{\varepsilon}{4 C_\delta^n}$. We shall prove that $A_\varphi(f, \gamma) \subset K(f, \varepsilon)$. Let $g \in A_\varphi(f, \gamma)$. Then

$$d_\varphi\left(\frac{f}{\varepsilon}, \frac{g}{\varepsilon}\right) = \int_{E_1} \varphi\left(\frac{1}{\varepsilon}(f(x) - g(x))\right) d\mu + \int_{E_2} \varphi\left(\frac{1}{\varepsilon}(f(x) - g(x))\right) d\mu < \frac{\varepsilon}{2},$$

where $E_1 = \{x \in E : |f(x) - g(x)| < \delta\}$ and $E_2 = E \setminus E_1$.
Hence $\|f - g\| < \varepsilon$ and so $A_\varphi(f, \gamma) \subset U$. This implies that $U \in \mathcal{T}_{d_\varphi}$.

Lemmas 8.1 – 8.3 immediately imply

Theorem 8.1. *The condition* (Δ_2) *is sufficient, and if* μ *is an atomless measure, then it is also necessary in order that* $\mathcal{T}_{d_\varphi} = \mathcal{T}_{L^\varphi}$.

By Theorems 7.1 and 8.1 we obtain

Remark 8.1. If φ satisfies the condition (Δ_2), then $\mathcal{T}_{d_\varphi} = \mathcal{T}_\varphi = \mathcal{T}_{L^\varphi}$. If φ does not satisfy the condition (Δ_2) and if μ is an atomless measure then $\mathcal{T}_{d_\varphi} \subsetneq \mathcal{T}_{L^\varphi}$ and $\mathcal{T}_{d_\varphi} \subsetneq \mathcal{T}_\varphi$.

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DANUTA STACHOWIAK – GNILKA
INSTITUTE OF MATHEMATICS ADAM MICKIEWICZ UNIVERSITY
MATEJKI 48/49
60–769 POZNAŃ,
POLAND

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