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On a fixed point theorem of Greguš, Jr.

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Abstract. A constructive proof of a fixed point theorem of M. GREGUŠ,Jr. is given. A simple example is presented to show that the theorem is not a consequence of the well-known Banach fixed point theorem.

The aim of this note is to give a new proof of the fixed point theorem of M. GREGUŠ, Jr. [3]. This theorem was generalized in [1], [2] and [4]. Our proof is based on the method of nondiscrete mathematical induction proposed by V. PTÁK [5]. We construct a sequence of iterates converging to the fixed point. In [3] such a sequence was given only for uniformly convex Banach spaces.

Theorem [3, Theorem 1.1]. *Let C be a closed convex subset of a Banach space B and let $T : C \to C$ be a mapping that satisfies:*

$$(1) \qquad \|Tx - Ty\| \le a \cdot \|x - y\| + b \cdot \|Tx - x\| + c \cdot \|Ty - y\|$$

for all $x, y \in C$, where $0 < a < 1, \quad b \ge 0, \quad c \ge 0$ and $a + b + c = 1$. Then for every $x_0 \in C$, the sequence of iterates

$$x_{n+1} = (T^2 x_n + T^3 x_n)/2, \quad (n = 1, 2, 3, \dots)$$

converges to the unique fixed point x^* for T ($Tx^* = x^*$). Moreover, T is continuous at x^* and there exists $\lambda : 0 < \lambda < 1$ for which the following estimates hold:

$$\text{(a)} \qquad \|x_n - x_{n+1}\| \le (2/5)\lambda^n \|Tx_0 - x_0\|$$

$$\text{(b)} \qquad \|x_n - x^*\| \le \frac{2}{5}\frac{\lambda^n}{1-\lambda}\|Tx_0 - x_0\|$$

PROOF. As in [3] we may assume that b = c = p because (1) implies

$$\text{(1)'}\qquad \|Tx - Ty\| \leq a \cdot \|x - y\| + p \cdot \|Tx - x\| + p.\|Ty - y\|,$$

for all $x, y \in C$, where $p = (1/2)(b + c)$.

We shall use the following estimates proved in [3]:

For every $x \in C$ and every $n = 1, 2, \ldots$

$$\text{(2)}\qquad \|T^n x - T^{n-1} x\| \leq \|Tx - x\|$$

$$\text{(3)}\qquad \|T^3 x - Tx\| \leq 2(a + p)\|Tx - x\| \quad \text{and if}$$

$$\text{(4)}\qquad z = (T^2 x + T^3 x)/2$$

$$\text{(5)}\qquad \|Tz - z\| \leq \left\{1 - \frac{a(1-a)}{2(1+a)}\right\} \|Tx - x\| = \lambda \|Tx - x\|$$

Since $0 < a < 1$, $\quad 0 < \lambda < 1$ for $\lambda = 1 - \dfrac{a(1-a)}{2(1+a)}$. For any $x \in C$ and z given by (4)

$$\|x - z\| = (1/2)\|x - T^2 x + x - T^3 x\| \leq$$
$$\leq (1/2)\left\{\|x - Tx\| + \|Tx - T^2 x\| + \|x - Tx\| + \|Tx - T^2 x\| + \|T^2 x - T^3 x\|\right\}$$

therefore (2) implies

$$\text{(6)}\qquad \|x - z\| \leq (5/2)\|Tx - x\|$$

Now we are prepared to use Pták's induction theorem in the form of proposition 1.9 from [5, p.7]. Let us use the rate of convergence $w(t) = \lambda \cdot t$, let $Z(t) = \{x \in C : \|Tx - x\| \leq (2/5)t\}$ and $r_0 = (5/2)\|Tx_0 - x_0\|$. Then according to [5, Theorem 1.9] (5) and (6) imply that the sequence $x_{n+1} = (1/2)(T^2 x_n + T^3 x_n)$, $\quad n = 0, 1, 2, \ldots$ converges to an element $x^* \in C$ and the estimates (a) and (b) hold. To finish the proof we have to show that $Tx^* = x^*$, T is continuous at x^* and that there is no other fixed point of T. First observe that $\quad \|Tx_n - x_n\| \leq \lambda^n r_0$.

According to (1)'

$$\|Tx^* - x^*\| = \|Tx^* - Tx_n + Tx_n - x_n + x_n - x^*\| \leq$$

$$\leq (a+1)\|x^* - x_n\| + p \cdot \|Tx^* - x^*\| + (1+p)\|Tx_n - x_n\|.$$

$0 < p < 1$ implies

$$0 \leq (1-p)\|Tx^* - x^*\| \leq (1+a)\|x^* - x_n\| + (1+p)\|Tx_n - x_n\|.$$

Letting n tend to ∞ we obtain $Tx^* = x^*$. Suppose there is another fixed point $y^* \in C$. Then

$$\|x^* - y^*\| = \|Tx^* - Ty^*\| \leq a\|x^* - y^*\| \text{and since } 0 < a < 1$$

this shows that $x^* = y^*$.

Let $y_n \in C$ be any sequence converging to x^*, then

$$\begin{aligned}\|Ty_n - Tx^*\| &\leq a\|y_n - x^*\| + p\|Ty_n - y_n\| \leq \\ &\leq a\|y_n - x^*\| + p\|Ty_n - x^*\| + p\|x^* - y_n\|,\end{aligned}$$

It follows $\|Ty_n - Tx^*\| \leq \frac{a+p}{1-p}\|y_n - x^*\|$, i.e. Ty_n converges to $Tx^* = x^*$. This finishes the proof of the theorem.

The following simple example shows that this theorem is not a consequence of the well-known Banach fixed point theorem.

Example. Let B be any Banach space and let $b \in B$. The mapping $T : B \to B$ defined by $Tx = b - x$ is not contractive. If $y_0 \neq b/2$ then the sequence of iterates $y_{n+1} = Ty_n$ $(n = 0, 1, 2, \dots)$ is not convergent while the sequence of iterates $x_{n+1} = (1/2)(T^2x_n + T^3x_n)$ converges to the fixed point $b/2$ of T (for any $x_0 \in B$).

T satisfies the condition (1)' for arbitrary $a : 0 < a < 1$ and $p = \frac{1-a}{2}$: Indeed, for $x, y \in B$, $\|Tx - Ty\| = \|x - y\| = a\|x - y\| + \frac{1-a}{2}\|2x - 2y\| = a\|x - y\| + p\|2x - b + b - 2y\| \leq a\|x - y\| + p\|x - Tx\| + p\|Ty - y\|$.

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