Publ. Math. Debrecen
39 / 3–4 (1991), 299–304

On a functional equation characterizing inner product spaces

By C. ALSINA (Barcelona) and J. L. GARCIA ROIG (Barcelona)

Abstract. We give a characterization of inner product spaces by solving a functional equation. In doing this, a characterization of similitudes is obtained.

Let $(E, \| \ \|)$ be a real linear normed space of dimension (finite or infinite) different from 1. Since the mapping $\varrho : E \to R^+$ defined by $\varrho(x) = \frac{1}{2}\|x\|^2$ is a convex functional we can define the mappings $\varrho'_+, \varrho'_- :$ $E \times E \to R$ given by

$$\varrho'_\pm(x,y) = \lim_{t \to 0\pm} \frac{\|x+ty\|^2 - \|x\|^2}{2t}$$

The mappings $\varrho'_\pm$ play a crucial role in characterizing inner product spaces (see [2], [3], [4], [5], [6], [7], [8]). In fact, when the norm is derivable from an inner product $(E, \cdot\ )$ then $\varrho'_\pm(x,y) = x \cdot y$, i.e., $\varrho'_\pm$ reduce to the given inner product.

Our aim in this paper is to solve the functional equation:

$$(1) \qquad f\left(y - \frac{\varrho'_+(x,y)}{\|x\|^2} \cdot x\right) = f(y) - \frac{\varrho'_+(f(y),f(x))}{\|f(x)\|^2} \cdot f(x),$$

where the unknown function $f : E \to E$ is surjective, continuous and vanishes only at zero and (1) holds for all x, y in E such that $x \neq 0$. Equation (1) has a natural motivation: in inner product spaces (1) says that f transforms the height of the triangle determined by x and y into the corresponding height of the triangle generated by $f(x)$ and $f(y)$.

In order to solve (1) we quote here some elementary results concerning the functions $\varrho'_\pm$ that we will use in the sequel (for a detailed study see [2]):

(i) $\varrho'_{\pm}(x,x) = \|x\|^2$ and $|\varrho'_{\pm}(x,y)| \leq \|x\|\,\|y\|$;

(ii) $\varrho'_{+}(\alpha x,y) = \varrho'_{+}(x,\alpha y) = \alpha\varrho'_{+}(x,y), \alpha \geq 0$;

(iii) $\varrho'_{+}(\alpha x,y) = \varrho'_{+}(x,\alpha y) = \alpha\varrho'_{-}(x,y), \alpha \leq 0$;

(iv) $\varrho'_{+}(x,\alpha x + y) = \varrho'_{+}(x,y) + \alpha\|x\|^2, \alpha \geq 0$;

(v) $\varrho'_{+}(x,y) = \varrho'_{+}(y,x)$, for all x, y in E if and only if E is an inner product space.

We begin with the following:

Lemma 1. *If $f : E \to E$ is a surjective, continuous solution of (1) and $f(x) \neq 0$ whenever $x \neq 0$ then f must be a linear automorphism.*

PROOF. First we will show that f preserves the linear independence of any couple of vectors x, y in E, $x, y \neq 0$. In fact, if we had $f(y) = \lambda f(x)$ for some $\lambda \neq 0$ by (1), (i), (ii) and (iii) we immediately would obtain

$$f\left(y - \frac{\varrho'_{+}(x,y)}{\|x\|^2}\cdot x\right) = \lambda f(x) - \frac{\varrho'_{+}(\lambda f(x), f(x))}{\|f(x)\|^2}\cdot f(x) = 0,$$

and this would yield $y = \dfrac{\varrho'_{+}(x,y)}{\|x\|^2}\cdot x$ in contradiction with the independence of x and y.

Next consider u, v in E, $u, v \neq 0$, two independent vectors. The substitution $x = u$ and $y = u + v$ into (1) together with properties (1), (iv) yield:

$$\begin{aligned} f(u+v) &- \frac{\varrho'_{+}(f(u+v), f(u))}{\|f(u)\|^2}\cdot f(u) \\ &= f\left(u + v - \frac{\varrho'_{+}(u, u+v)}{\|u\|^2}\,u\right) = f\left(v - \frac{\varrho'_{+}(u,v)}{\|u\|^2}\,u\right) \\ &= f(v) - \frac{\varrho'_{+}(f(v), f(u))}{\|f(u)\|^2}\cdot f(u). \end{aligned}$$

Therefore

$$f(u+v) = f(v) + \frac{\varrho'_{+}(f(u+v), f(u)) - \varrho'_{+}(f(v), f(u))}{\|f(u)\|^2}\cdot f(u)\,, \tag{2}$$

and reversing the roles of u and v,

$$f(u+v) = f(u) + \frac{\varrho'_{+}(f(v+u), f(v)) - \varrho'_{+}(f(u), (f(v))}{\|f(v)\|^2}\cdot f(v). \tag{3}$$

By (2) and (3) and bearing in mind that $f(u)$ and $f(v)$ are independent we can conclude that f must satisfy

$$\frac{\varrho'_+(f(u+v),f(u))-\varrho'_+(f(v),f(u))}{\|f(u)\|^2}=1,$$

i.e., by (2)

$$f(u+v)=f(u)+f(v) \tag{4}$$

where (4) holds for all couples of non-zero independent vectors u and v. Moreover if $v=\lambda u$ we can choose in a plane containing u a sequence (v_n) of vectors independent of u but with $\lim_{n\to\infty} v_n=\lambda u$. By (4) and the continuity of f we obtain

$$\begin{aligned}f(u+v)=f(u+\lambda u)&=\lim_{n\to\infty}f(u+v_n)=\lim_{n\to\infty}(f(u)+f(v_n))=\\&=f(u)+f(\lambda u)=f(u)+f(v).\end{aligned}$$

Thus (4) holds for all u and v in $E\setminus\{0\}$ and obviously if either $u=0$ or $v=0$. Consequently f satisfies the classical Cauchy functional equation on E and (Aczél, 1966) f is a linear transformation. Since $f^{-1}(\{0\})=\{0\}$ and f is onto, f is an automorphism.

Remark. Observe at the end of the preceding proof that the hypothesis of f being onto is only needed in the infinite dimensional case.

Now we will prove our main result

Theorem 1. *A continuous function f from a real linear normed space $(E,\|\ \|)$ of dimension (finite or infinite) different from 1, onto itself satisfies (1) and vanishes only at zero if and only if $(E,\|\ \|)$ is an inner product space and f is a similitude.*

Proof. It is a straightforward verification to show that any similitude satisfies (1) whenever E is an inner product space. Conversely, assume that f is a solution of (1). By virtue of Lemma 1 necessarily f must be a linear automorphism and must satisfy

$$\frac{\varrho'_+(x,y)}{\|x\|^2}=\frac{\varrho'_+(f(y),f(x))}{\|f(x)\|^2}. \tag{5}$$

Consider any couple of linearly independent vectors u,v in $E\setminus\{0\}$ and any real $t\geq 0$. The substitution $x=f^{-1}(u)$ and $y=f^{-1}(u+tv)$ into

(5) implies:

$$\begin{aligned}\frac{\varrho'_+(u+tv,u)}{\|u\|^2} &= \frac{\varrho'_+\big((f\circ f^{-1})(u+tv),(f\circ f^{-1})(u)\big)}{\|(f\circ f^{-1})(u)\|^2}\\ &= \frac{\varrho'_+(f^{-1}(u),f^{-1}(u+tv))}{\|f^{-1}(u)\|^2} = \frac{\|f^{-1}(u)\|^2+\varrho'_+(f^{-1}(u),f^{-1}(tv))}{\|f^{-1}(u)\|^2}\\ &= 1+\frac{\varrho'_+(f^{-1}(u),f^{-1}(tv))}{\|f^{-1}(u)\|^2} = 1+\frac{\varrho'_+(tv,u)}{\|u\|^2} = 1+\frac{t\varrho'_+(v,u)}{\|u\|^2},\end{aligned}$$

i.e.,

$$\varrho'_+(u+tv,u) = \|u\|^2 + t\varrho'_+(v,u),$$

whence

$$\varrho'_+(v,u) = \lim_{t\to 0+}\frac{\varrho'_+(u+tv,u)-\|u\|^2}{t}. \tag{6}$$

Now we will compute the limit (6) by using (5), the definition of ϱ'_+, (ii) and (iv) and the fact that $u+tv\neq 0$:

$$\begin{aligned}\varrho'_+(v,u) &= \lim_{t\to 0+}\frac{\varrho'_+(u+tv,u)-\|u\|^2}{t}\\ &= \lim_{t\to 0+}\frac{\varrho'_+(f(u),f(u)+tf(v))\ \|u+tv\|^2-\|u\|^2\ \|f(u+tv)\|^2}{t\|f(u+tv)\|^2}\\ &= \lim_{t\to 0+}\frac{\|f(u)\|^2\ \|u+tv\|^2+t\varrho'_+(f(u),f(v))\ \|u+tv\|^2-\|u\|^2\ \|f(u+tv)\|^2}{t\|f(u+tv)\|^2}\\ &= \lim_{t\to 0+}\left\{\varrho'_+(f(u),f(v))\frac{\|u+tv\|^2}{\|f(u+tv)\|^2}+\frac{2\|f(u)\|^2}{\|f(u+tv)\|^2}\cdot\frac{\|u+tv\|^2-\|u\|^2}{2t}\right.\\ &\quad\left.-2\frac{\|u\|^2}{\|f(u+tv)\|^2}\cdot\frac{\|f(u)+tf(v)\|^2-\|f(u)\|^2}{2t}\right\}\\ &= \varrho'_+(f(u).\,f(v))\frac{\|u\|^2}{\|f(u)\|^2}+2\varrho'_+(u,v)-2\frac{\|u\|^2}{\|f(u)\|^2}\,\varrho'_+(f(u),f(v))\ \text{whence}\end{aligned}$$

$$\varrho'_+(v,u) = 2\varrho'_+(u,v)-\frac{\|u\|^2}{\|f(u)\|^2}\,\varrho'_+(f(u),(f(v)) \quad \text{and by (5)}$$

$$\varrho'_+(v,u) = 2\varrho'_+(u,v)-\frac{\|u\|^2}{\|f(u)\|^2}\cdot\frac{\|f(v)\|^2}{\|v\|^2}\,\varrho'_+(v,u)$$

therefore we obtain

$$(7) \qquad 2\varrho'_+(u,v) = \left[1 + \frac{\|u\|^2}{\|f(u)\|^2} \cdot \frac{\|f(v)\|^2}{\|v\|^2}\right] \varrho'_+(v,u).$$

By (7), $\varrho'_+(u,v) = \varrho'_+(v,u)$ if either of these values is zero. Otherwise, reversing the roles of u and v we obtain:

$$(8) \qquad 2\varrho'_+(v,u) = \left[1 + \frac{\|v\|^2}{\|f(v)\|^2} \cdot \frac{\|f(u)\|^2}{\|u\|^2}\right] \varrho'_+(u,v)$$

Combining (7) and (8) we immediately get

$$(9) \qquad \frac{\|u\|^2}{\|f(u)\|^2} \cdot \frac{\|f(v)\|^2}{\|v\|^2} = 1$$

and therefore

$$(10) \qquad \varrho'_+(u,v) = \varrho'_+(v,u)\,,$$

for independent u and v. In the case that u and v are linearly dependent using (ii) and (iii) we immediately see that (10) also holds. But, as remarked before, (10) is equivalent to the derivability of the norm from an inner product and (9) forces f to be a similitude. The theorem is proved.

Acknowledgement. The authors thank the referee GYÖRGY SZABÓ (Debrecen) for the interesting remarks made on this paper.

References

[1] J. ACZÉL, Lectures on Functional Equations and Their Applications, *Academic Press, New York*, 1966.

[2] D. AMIR, Characterizations of Inner Product Spaces, *Birkhäuser Verlag, Basel – Boston – Stuttgart*, 1986.

[3] R. C. JAMES, Orthogonality in normed linear spaces, *Duke Math. Jour.* **12** (1945), 291–301.

[4] D. LAUGWITZ, On characterizations of inner product spaces, *Proc. Amer. Math. Soc.* **50** (1975), 184–188.

[5] M. LEDUC, Caractérisation des espaces euclidiens, *Comptes Rendues Acad. Sci. Paris* **268** (1969), 943–946.

[6] P. M. MILICIC, Sur l'existence du produit scalaire, *Mat. Vestrik* **10** (1973), 3–7.

[7] R. K. RITT, A generalization of inner product, *Michigan Math. Jour.* **3** (1955), 23–26.

[8] R. A. Tapia, A characterization of inner product spaces, *Proc. Amer. Math. Soc.* **41** (1973), 569–574.

C. ALSINA and J. L. GARCIA ROIG
SEC. MATEMÀTIQUES I INFORMÀTICA
E.T.S.A.B.—U.P.C.
DIAGONAL, 649. 08028 BARCELONA
SPAIN

(Received April 13, 1989)