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Interval-filling sequences of order N and a representation of real numbers in canonical number systems

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Abstract. In this paper the authors give a characterization for interval-filling sequences of order N and using this result they show that each real number has representations in every (real) canonical number system, and give the complete analogue of a Theorem of DARÓCZY and KÁTAI for the real case.

Introduction

The concept of interval-filling sequences has been introduced by DARÓCZY, JÁRAI and KÁTAI in [1]. If (λ_n) is a strictly decreasing sequence of positive real numbers and $\sum\limits_{n=1}^{\infty} \lambda_n < +\infty$ then (λ_n) is interval-filling if for all $x \in [0, \sum\limits_{n=1}^{\infty} \lambda_n]$ there exists a sequence $(a_n) : \mathbf{N} \to \{0,1\}$ such that $x = \sum\limits_{n=1}^{\infty} a_n \lambda_n$. A characterization of interval-filling sequences can be found in [1], [2] and [3].

Let $\mathbf{R}_i$ be an integral domain (with unit element), $\alpha \in \mathbf{R}_i$ and $\mathcal{N} = \{k_1, \dots, k_n\}$ a finite subset of the rational integers $\mathbf{Z}$. $\{\alpha, \mathcal{N}\}$ is called a number system in $\mathbf{R}_i$ if every $\gamma \in \mathbf{R}_i$ has a unique representation in the form

$$(1.1) \quad \gamma = a_0 + a_1\alpha + \dots + a_k\alpha^k,\ a_i \in \mathcal{N},\ (0 \le i \le k),\ a_k \ne 0 \text{ if } k \ne 0.$$

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If $\mathcal{N} = \{0, 1, \dots, n\}$ then the number system $\{\alpha, \mathcal{N}\}$ is called a canonical number system.

This concept is a natural generalization of negative base number systems in $\mathbf{Z}$ considered by several authors. In [4] one can read a necessary and sufficient condition for the existence of canonical number systems in $\mathbf{R}_i$. In [5] PETHŐ and KOVÁCS characterized all those integral domains which have number systems and gave necessary and sufficient conditions for $\{\alpha, \mathcal{N}\}$ to be a number system in an order ϑ. In [6] KÁTAI and SZABÓ proved that if $\{\alpha, \mathcal{N}\}$ is a canonical number system in the ring of Gaussian integers, then every complex number γ can be written in the form

$$(1.2) \quad \gamma = a_k\alpha^k + \dots + a_1\alpha + a_0 + a_{-1}\alpha^{-1} + \dots, \qquad a_i \in \mathcal{N},\ (i = k, k-1, \dots).$$

This result was extended for some families of integral domains in [7], [8], [9]. In connection with this DARÓCZY and KÁTAI proved that for every non-real complex number α, $|\alpha| > 1$, there exists a set $\{0, 1, \dots, n\}$ such that every complex number γ can be represented in the form (1.2) ([10]).

In this paper we first give a characterization for interval-filling sequences of order N. Using this theorem we show that if $\{\alpha, \mathcal{N}\}$ is a canonical number system in an integral domain $\mathbf{R}_i$ such that α is real, then any real number γ can be written in the form (1.2), furthermore we prove the real analogue of the theorem of DARÓCZY and KÁTAI.

Results and proofs

In the following $\mathbf{R}_i$ will denote an integral domain of characteristic 0, $\mathbf{N}$ the set of positive integers, $\mathbf{Z}$ the ring of integers, $\mathbf{Q}$ the field of rationals and $\mathbf{R}$ the field of reals. If α is an algebraic element over $\mathbf{Q}$, $\mathbf{Z}[\alpha]$ denotes the subring of $\mathbf{Q}(\alpha)$ generated by $\mathbf{Z}$ and α.

For a given α and $\mathcal{N}$ let $\mathbf{S}(\alpha)$ be the set of all numbers γ which can be written in the form (1.2).

In order to prove our theorems we need some further information on canonical number systems.

From [4] we know the following

Lemma 1. *Let $\mathbf{R}_i$ be an integral domain of characteristic* 0. *Then there exists a canonical number system in $\mathbf{R}_i$ if and only if $\mathbf{R}_i = \mathbf{Z}[\alpha]$ for some element α which is algebraic over $\mathbf{Q}$.*

PROOF. See Theorem 1 in [4].

Lemma 2. *If $\{\alpha, \mathcal{N} = \{0, 1, \dots, n\}\}$ is a canonical number system in a $\mathbf{Z}[\beta]$ and the defining polynomial of α is $a_nx^n + \dots + a_1x + a_0$, $a_i \in \mathbf{Z}$, then $n = |a_0| - 1$.*

PROOF. See the proof of Theorem 1 in [4].

Lemma 3. *Let $\{\alpha,\mathcal{N}\}$ be a canonical number system in a $\mathbf{Z}[\beta]$, where $\beta\neq 0$ is a real algebraic element over $\mathbf{Q}$. Then $\alpha<-1$ holds.*

PROOF. i) $\mathbf{Z}[\beta]$ contains negative elements, but in the case $\alpha\geq 0$ each element $\gamma\in\mathbf{Z}[\beta]$, which has a representation in the form (1.1) is non-negative, consequently if α is the base of a canonical number system then $\alpha<0$.

ii) If $-1<\alpha<0$, then $\left|\sum_{i=0}^{\infty} a_i\alpha^i\right|<K,\ \ a_i\in\mathcal{N},\ i=0,1,\dots$.
From this we get that the set of those numbers, which have representation in the form (1.1) is bounded, but $\mathbf{Z}[\beta]$ is not bounded. This means that $\alpha\leq -1$ if $\{\alpha,\mathcal{N}\}$ is a canonical number system in $\mathbf{Z}[\beta]$.

iii) If $\alpha=-1$ then $\mathbf{Z}[\beta]=\mathbf{Z}$ and it is well-known that $\{q,\{0,1,\dots,|q|-1\}\}$ is a canonical number system in $\mathbf{Z}$ if and only if $q\leq -2$ holds. This completes the proof.

Lemma 4. *Let α be an algebraic element over $\mathbf{Q}$ with degree n. If $\{\alpha,\mathcal{N}\}$ is a canonical number system in $\mathbf{Z}[\beta]$ then $|\alpha^i|\geq 1,\ 1\leq i\leq n$, holds for each conjugate $\alpha^{(i)}$ of α.*

PROOF. See Lemma 3 in [5].

Lemma 5. *Let $\{\alpha,\mathcal{N}\}$ be a number system in $\mathbf{Z}[\beta]$, where β is an algebraic integer of degree $n\geq 1$ over $\mathbf{Q}$ and let us suppose that $|\alpha|\leq|\alpha^{(i)}|$ for every conjugate $\alpha^{(i)}$ of α over $\mathbf{Q}$. Then $\mathbf{S}(\alpha)=\mathbf{R}$ or $\mathbf{C}$ according to $\alpha\in\mathbf{R}$ or $\alpha\in\mathbf{C}\setminus\mathbf{R}$.*

PROOF. See Theorem 2 in [9].

Lemma 6. *Let α be a non-real algebraic integer of degree 3 (over $\mathbf{Q}$). If $\{\alpha,\mathcal{N}\}$ is a canonical number system in a $\mathbf{Z}[\beta]$ then $\mathbf{S}(\alpha)=\mathbf{C}$.*

PROOF. See Theorem 3 in [9].

Now we introduce a generalization of interval-filling sequences. Let N be a positive integer, let Λ_N denote the set of all real sequences (λ_n) for which $|\lambda_n|>|\lambda_{n+1}|>0$ for all $n\in\mathbf{N}$ and

$$\sum_{n=1}^{\infty}|\lambda_n|<+\infty.$$

Define the real numbers L^-, L^+ and L by

$$L^-=\sum_{n=1}^{\infty}\frac{|\lambda_n|-\lambda_n}{2},\quad L^+=\sum_{n=1}^{\infty}\frac{|\lambda_n|+\lambda_n}{2}\quad\text{and}\quad L=L^++L^-.$$

A sequence $(\lambda_n) \in \Lambda_N$ is called an interval-filling sequence of order N if for all $x \in [-NL^-, NL^+]$ there exists a sequence $(a_n) : \mathbf{N} \to \{0, 1, \ldots, N\}$ such that

$$x = \sum_{n=1}^{\infty} a_n \lambda_n.$$

It is clear that every x which has the above representation is located in $[-NL^-, NL^+]$.

If $\lambda_n > 0$ for all $n \in \mathbf{N}$ and $N = 1$ then we have the concept of interval-filling sequence introduced in [1]. In this particular case the following characterization theorem is proved in [1], [2] and [3] and in the case $N = 1$ it is proved by Z. Boros (unpublished).

Theorem 1. *Suppose that $(\lambda_n) \in \Lambda_N$. Then (λ_n) is an interval-filling sequence of order N if and only if*

$$|\lambda_n| \le N \sum_{k=n+1}^{\infty} |\lambda_k| \quad \text{for all } n \in \mathbf{N}. \tag{2.1}$$

Proof. Suppose that (λ_n) satisfies condition (2.1) and $y \in [0, NL]$. First we prove that there exists a sequence $(\varepsilon_n(y)) : \mathbf{N} \to \{0, 1, \ldots, N\}$ such that

$$y = \sum_{n=1}^{\infty} \varepsilon_n(y) |\lambda_n|. \tag{2.2}$$

Define the numbers $\varepsilon_n(y)$ inductively in the following way:

$$\varepsilon_1(y) = \begin{cases} k-1 & \text{if } (k-1)|\lambda_1| \le y < k|\lambda_1| \text{ and } k \in \{1, 2, \ldots, N\} \\ N & \text{if } N|\lambda_1| \le y \end{cases}$$

If $n > 1$ and $\varepsilon_1(y), \ldots, \varepsilon_{n-1}(y)$ have been defined then let

$$s_n(y) = \sum_{k=1}^{n-1} \varepsilon_k(y) |\lambda_k| \quad \text{and}$$

$$\varepsilon_n(y) = \begin{cases} 0 & \text{if } y < s_n(y) + |\lambda_n| \\ k-1 & \text{if } s_n(y) + (k-1)|\lambda_n| \le y < s_n(y) + k|\lambda_n| \\ & \text{and } k \in \{2, 3, \ldots, N\} \\ N & \text{if } s_n(y) + N|\lambda_n| \le y. \end{cases}$$

In what follows we show that

$$y \geq s_n(y) \quad \text{for all } n \in \mathbf{N} \tag{2.3}$$

with the convention $s_1(y) = 0$. Clearly, (2.3) holds if $n = 1$. Suppose that $n > 1$ and $y \geq s_{n-1}(y)$. Since $y - s_n(y) = y - s_{n-1}(y) - \varepsilon_{n-1}(y)|\lambda_{n-1}|$ and $\varepsilon_{n-1}(y) = k$ implies that $y - s_{n-1}(y) \geq k|\lambda_{n-1}|$ if $k \in \{0, 1, \dots, N\}$ we have (2.3) by induction. Obviously, the limit

$$s(y) = \lim_{n\to\infty} s_n(y)$$

exists and because of (2.3) we get

$$y \geq s(y) \geq s_n(y) \quad \text{for all } n \in \mathbf{N}. \tag{2.4}$$

Consider the set $\mathcal{A} = \{n \in \mathbf{N} : y < s_n(y) + N|\lambda_n|\}$. If $\mathcal{A} = \emptyset$ then $\varepsilon_n(y) = N$ for all $n \in \mathbf{N}$ thus by (2.4) $NL \geq y \geq s(y) = NL$, that is (2.2) holds in this case. If $\mathcal{A}$ is an infinite set then $y < s_n(y) + N|\lambda_n|$ for infinitely many $n \in \mathbf{N}$ therefore $y \leq s(y)$. This and (2.4) imply (2.2). The remaining case is impossible: if m is the greatest element of $\mathcal{A}$ then $\varepsilon_m(y) \leq N - 1$ and $\varepsilon_n(y) = N$ for all $n > m$. Then it follows from (2.4), according to the definition of $\varepsilon_m(y)$, that

$$s_m(y) + (\varepsilon_m(y) + 1)|\lambda_m| > y \geq s(y) = s_m(y) + \varepsilon_m(y)|\lambda_m| + N\sum_{k=m}^{\infty} |\lambda_k|$$

which contradicts (2.1). Thus (2.2) is proved.

Finally, let $x \in [-NL^-, NL^+]$ and $y = x + NL^-$ in (2.2). Define the numbers

$$\overline{\varepsilon}_n(x) = \frac{(2\varepsilon_n(y) - N)\text{sgn}\lambda_n + N}{2}$$

for all $n \in \mathbf{N}$. Then $\overline{\varepsilon}_n(x) \in \{0, 1, \dots, N\}$ and by (2.2) we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \overline{\varepsilon}_n(x)\lambda_n &= \sum_{n=1}^{\infty} \frac{(2\varepsilon_n(y) - N)|\lambda_n| + N\lambda_n}{2} = \\
&= \sum_{n=1}^{\infty} \varepsilon_n(y)|\lambda_n| - N\sum_{n=1}^{\infty} \frac{|\lambda_n| - \lambda_n}{2} = \\
&= y - NL^- = x,
\end{aligned}$$

that is (λ_n) is an interval-filling sequence of order N.

Conversely, suppose that (λ_n) is an interval-filling sequence of order N and there exists $m \in \mathbf{N}$ such that

$$|\lambda_m| > N \sum_{k=m+1}^{\infty} |\lambda_k|.$$

$$\text{If } |\lambda_m| > x > N \sum_{k=m+1}^{\infty} |\lambda_k| \quad \text{then } -NL^- \leq x - NL^- \leq NL^+,$$

therefore there exists a sequence $(a_n) : \mathbf{N} \to \{0, 1, \dots, N\}$ such that

$$x - NL^- = \sum_{n=1}^{\infty} a_n \lambda_n, \text{ that is } x = \sum_{n=1}^{\infty} b_n |\alpha_n|, \text{ where}$$
$$b_n = \frac{(2a_n - N)\text{sgn}\lambda_n + N}{2} \in \{0, 1, \dots, N\}$$

for all $n \in \mathbf{N}$. Since $x < |\lambda_m|$ we have that $b_1 = \dots = b_m = 0$ thus

$$x = \sum_{k=m+1}^{\infty} b_k |\lambda_k| \leq N \sum_{k=m+1}^{\infty} |\lambda_k|$$

which is a contradiction.

Theorem 2. *Let β be a real algebraic element over $\mathbf{Q}$ and $\{\alpha, \mathcal{N}\}$ a canonical number system in $\mathbf{Z}[\beta]$. Then every real number γ has the representation* (1.2).

PROOF. Let α be a real algebraic number over $\mathbf{Q}$ of degree n and $\{\alpha, \mathcal{N}\}$ a canonical number system in $\mathbf{Z}[\beta]$. Define the sequence (λ_n) by $\lambda_n = \alpha^{-n}$, for all $n \in \mathbf{N}$. Because of Lemma 2 the maximum element of the set $\mathcal{N}$ is $|a_0| - 1$. First we shall show that the sequence (λ_n) is an interval-filling sequence of order $|a_0| - 1$. Lemma 3 implies that $\alpha < -1$, thus

$$|\lambda_n| > |\lambda_{n+1}| > 0 \quad \text{for all } n \in \mathbf{N} \text{ and } \sum_{i=1}^{\infty} |\lambda_i| < +\infty.$$

From Theorem 1 we get that the sequence (λ_n) is an interval-filling sequence of order $|a_0| - 1$ if and only if

$$(3.1) \qquad |\lambda_n| \leq (|a_0| - 1) \sum_{i=n+1}^{\infty} |\lambda_i| \text{ holds for all } n \in \mathbf{N}.$$

From (3.1) it follows immediately by the definition of (λ_n) that

$$\frac{1}{|\alpha|^n} \le (|a_0|-1) \sum_{i=n+1}^{\infty} \frac{1}{|\alpha|^i} = \frac{|a_0|-1}{|\alpha|^{n+1}} \cdot \frac{|\alpha|}{|\alpha|-1}, \tag{3.2}$$

that is

$$|\alpha| \le |a_0| \tag{3.3}$$

is the necessary and sufficient condition for (λ_n) to be an interval-filling sequence of order $|a_0|-1$. But $|\alpha^{(i)}| \ge 1$ holds for each conjugate $\alpha^{(i)}$ of α by Lemma 4, consequently

$$|a_0| \ge |\alpha^{(1)}|\,|\alpha^{(2)}| \dots |\alpha^{(n)}| \ge |\alpha|. \tag{3.4}$$

This means that for any canonical number system $\{\alpha, \mathcal{N}\}$ under the conditions of our theorem, the sequence (λ_n) defined above forms an interval–filling sequence of order $|a_0|-1$.

Now let $\gamma \in \mathbf{R}$. The length of the interval $[-(|a_0|-1)L^-, (|a_0|-1)L^+]$ is not less then 1 therefore there exists $\gamma_0 \in \mathbf{Z} \subset \mathbf{Z}[\beta]$ such that

$$\gamma - \gamma_0 \in [-(|a_0|-1)L^-, (|a_0|-1)L^+]. \tag{3.5}$$

Since $\{\alpha, \mathcal{N}\}$ is a canonical number system in $\mathbf{Z}[\beta]$ and (λ_n) is an interval-filling sequence of order $|a_0|-1$, γ_0 and $\gamma - \gamma_0$ can be written as

$$\gamma_0 = a_M\alpha^M + \dots + a_1\alpha + a_0, \quad a_i \in \mathcal{N},\ i = 0, 1, \dots, M \tag{3.6}$$

and

$$\gamma - \gamma_0 = a_{-1}\alpha^{-1} + \dots + a_{-m}\alpha^{-m} + \dots, \qquad a_i \in \mathcal{N},\ i = -1, \dots, -m, \dots\ . \tag{3.7}$$

Because of (3.6) and (3.7) the proof is complete.

Remarks. 1 The Theorem 2 in [8] is a special case of our theorem.

2 It can be seen from the proof that, in general, a real number γ has more than one representation in the form (1.2).

Combining Theorem 2 with Lemmas 5 and 6 we obtain the following Theorem.

Theorem 3. *Let β be an algebraic integer over* $\mathbf{Q}$ *of degree at most 3 and* $\{\alpha, \mathcal{N}\}$ *be a canonical number system in* $\mathbf{Z}[\beta]$. *Then every real or complex number can be written in the form* (1.2) *according to* α *being real or a nonreal complex number.*

PROOF. If α is real then Theorem 2 gives the desired result. If α is a nonreal number of degree 2, then Lemma 5 implies the statement since all conjugates of α have the same absolute value. Finally, if α is of degree 3 and α is not real then our theorem is proved by using Lemma 6.

We note that Theorems 2 in [6], [7] and [8] are particular cases of Theorem 3.

The following theorem is similar to Theorem 5 in [10] for the real case, moreover we can give all possible sets of digits, too.

Theorem 4. *Suppose that* $\alpha \in \mathbf{R}$, $1 < |\alpha|$. *Choose* $N \in \mathbf{N}$ *such that* $N \geq |\alpha| - 1$ *and let* $\mathcal{N} = \{0, 1, \dots, N\}$. *Then every real number* γ *or non-negative real number* γ *has the representation* (1.2) *according to* $\alpha < 0$ *or* $\alpha > 0$.

PROOF. Let k be an odd natural number and $\lambda_n^{(k)} = \alpha^{k+1-n}$, $n = 1, 2, \dots$.

Then $(|\lambda_n^{(k)}|)$ is a strictly decreasing sequence and

$$|\lambda_n^{(k)}| = (|\alpha| - 1) \sum_{j=n+1}^{\infty} |\lambda_j^{(k)}| \leq N \sum_{j=n+1}^{\infty} |\lambda_j^{(k)}| \quad \text{for all } n \in \mathbf{N}.$$

Now, by Theorem 1, $(\lambda_n^{(k)})$ is an interval-filling sequence of order N. Furthermore

$$L_k^- = \sum_{n=1}^{\infty} \frac{|\lambda_n^{(k)}| - \lambda_n^{(k)}}{2} = -\alpha \frac{\alpha^{k+1}}{\alpha^2 - 1}, \quad L_k^+ = \sum_{n=1}^{\infty} \frac{|\lambda_n^{(k)}| + \lambda_n^{(k)}}{2} = \frac{\alpha^{k+1}}{\alpha^2 - 1},$$

if $\alpha < 0$ and

$$L_k^- = 0, \quad L_k^+ = \frac{\alpha^{k+1}}{\alpha - 1}, \quad \text{if } \alpha > 0.$$

Therefore $\gamma \in [-NL_k^-, NL_k^+]$ if k is large enough and the proof is complete.

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