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On a Stamate-type functional equation

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Introduction

Various authors (beginning with I. STAMATE [7]) have considered the functional equation

$$\frac{xf(y) - yf(x)}{x - y} = \varphi[\xi(x,y)], \qquad x \neq y, \tag{1}$$

and/or its diverse particular cases (cf. [2], [3], [7], [8]). Originally equation (1) was motivated by the mean value formula of POMPEIU [6], but later it began to live an independent life.

The somewhat similar functional equation, this time motivated by the classical mean value theorem of the differential calculus,

$$\frac{f(x) - f(y)}{x - y} = \varphi[\xi(x,y)], \qquad x \neq y, \tag{2}$$

and its special cases have been even more extensively studied (cf. [1], [2], [4] and the references in [1]). The most frequently considered cases of equations (1) and (2) are those, where ξ is the arithmetic mean of x and y,

$$\xi(x,y) = \frac{1}{2}(x+y), \tag{3}$$

i.e. the equations

$$\frac{xf(y) - yf(x)}{x - y} = \varphi\left(\frac{x+y}{2}\right), \qquad x \neq y, \tag{4}$$

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and

$$\frac{f(x)-f(y)}{x-y}=\varphi\left(\frac{x+y}{2}\right), \qquad x\neq y. \tag{5}$$

In fact, SH. HARUKI [4] and J. ACZÉL [1] have considered the slightly more general equation than (5) (with three unknown functions)

$$\frac{f(x)-g(y)}{x-y}=\varphi\left(\frac{x+y}{2}\right), \qquad x\neq y. \tag{6}$$

Moreover, in [1] equation (6) is treated (and solved) on an arbitrary commutative field of characteristic different from 2.

In the present paper we deal with the analogous generalization of equation (4)

$$\frac{xf(y)-yg(x)}{x-y}=\varphi\left(\frac{x+y}{2}\right), \qquad x\neq y, \tag{7}$$

on subsets of commutative fields of characteristic different from 2. Similar results can also be obtained for equation (6). They are stated without proofs at the end of the paper.

Introducing an additional unknown function g into equation (5) does not present new interesting problems; it is easily shown that equations (5) and (6) are equivalent (cf. [1], [4]). The same is also true for equations (4) and (7) provided that the domain of the equation is not too small. We will prove this for equations (1) and (8):

$$\frac{xf(y)-yg(x)}{x-y}=\varphi[\xi(x,y)], \qquad x\neq y, \tag{8}$$

under the assumption that the function ξ is symmetric:

$$\xi(x,y)=\xi(y,x). \tag{9}$$

In the sequel a *field* stands always for a commutative field. For every prime number p, the symbol $\mathbf{Z}_p$ denotes the field of integers modulo p.

1. For an arbitrary set U and an arbitrary function $\xi : U\times U\to U$ we write

$$V[U,\xi]:=\{t\in U \mid t=\xi(x,y),\ \ x,y\in U,\ \ x\neq y\}. \tag{10}$$

Theorem 1. *Let* $\mathbf{F}$ *be a field of characteristic different from 2, let* $U\subset\mathbf{F}$ *be a set such that* $U\setminus\{0\}$ *contains at least three elements and let*

$\xi : U \times U \to U$ be a symmetric function. If functions $f, g : U \to \mathbf{F}$ and $\varphi : V[U,\xi] \to \mathbf{F}$ satisfy equation (8) for all $x, y \in U,\ x \neq y$, then

$$f(x) = g(x), \qquad x \in U. \tag{11}$$

PROOF. Relations (8) and (9) imply that

$$xf(y) - yg(x) = xg(y) - yf(x), \quad x, y \in U,\ x \neq y,$$

whence

$$\frac{f(x) - g(x)}{x} = \frac{g(y) - f(y)}{y}, \quad x, y \in U\setminus\{0\},\ x \neq y. \tag{12}$$

Fix a $\overline{y} \in U\setminus\{0\}$ and put

$$c := \frac{g(\overline{y}) - f(\overline{y})}{\overline{y}}. \tag{13}$$

Thus we have by (12)

$$f(x) = g(x) + cx, \quad x \in U\setminus\{0, \overline{y}\}. \tag{14}$$

There exist $u, v \in U\setminus\{0, \overline{y}\}$, $u \neq v$. Putting in (12) $x = u$ and $y = v$ we obtain in view of (14) $c = -c$, whence $c = 0$ since the characteristic of $\mathbf{F}$ is different from 2. Thus relations (13) and (14) imply (11) for $x \in U\setminus\{0\}$. If $0 \in U$, then we put in (8) first $x = \overline{y},\ y = 0$, and then $x = 0,\ y = \overline{y}$. By (9)

$$f(0) = \varphi[\xi(\overline{y}, 0)] = \varphi[\xi(0, \overline{y})] = g(0),$$

which completes the proof.

Remark 1. If $U\setminus\{0\}$ contains less than three elements relation (11) need not hold. For instance, let $U = \{0, 1, -1\}$. The functions $f, g, \varphi : U \to U$ given by

$$\begin{cases} f(1) = 1, \\ f(0) = 0, \\ f(-1) = 0, \end{cases} \qquad \begin{cases} g(1) = 0, \\ g(0) = 0, \\ g(-1) = -1, \end{cases} \qquad \begin{cases} \varphi(1) = 0, \\ \varphi(0) = 0, \\ \varphi(-1) = 0, \end{cases}$$

satisfy equation (8) (with an arbitrary $\xi : U \times U \to U$) for all $x, y \in U,\ x \neq y$, but $f \neq g$; cf. also Theorem 3 below.

We prove yet a simple but useful lemma.

Lemma 1. *Let* $\mathbf{F}$ *be a field and suppose that sets* $W \subset U \subset \mathbf{F}$ *and a function* $\xi : U \times U \to U$ *fulfil the condition*

(i) *For every* $x \in U$ *there exists a* $y \in W \setminus \{0\}$ *such that* $y \neq x$ *and* $\xi(x, y) \in W$.
If functions $f : U \to \mathbf{F}$ *and* $\varphi : V \to \mathbf{F}$ *as well as* $f_0 : U \to \mathbf{F}$ *and* $\varphi_0 : V \to \mathbf{F}$, *where* $V = V[U, \xi]$ *is given by (10), satisfy equation (1) for all* $x, y \in U$, $x \neq y$, *and*

$$f(x) = f_0(x), \quad \varphi(t) = \varphi_0(t) \tag{15}$$

for $x \in W$, $t \in V \cap W$, *then (15) holds for all* $x \in U$, $t \in V$.

PROOF. Take an arbitrary $x \in U$ and let $y \in W \setminus \{0, x\}$ be the point whose existence is guaranteed by condition (i). Thus $t = \xi(x, y) \in V \cap W$, whence it follows by (1) and (15)

$$f(x) = \frac{1}{y}[xf(y) - (x - y)\varphi(t)] = \frac{1}{y}[xf_0(y) - (x - y)\varphi_0(t)] = f_0(x).$$

This gives the first equality in (15) for arbitrary $x \in U$. Now, according to (10), for every $t \in V$ there exist $x, y \in U$, $x \neq y$, such that $t = \xi(x, y)$. Hence we have by (1) in view of the relation (already proved) $f = f_0$

$$\begin{aligned}\varphi(t) &= \varphi[\xi(x, y)] = (x - y)^{-1}[xf(y) - yf(x)] \\ &= (x - y)^{-1}[xf_0(y) - yf_0(x)] = \varphi_0[\xi(x, y)] = \varphi_0(t).\end{aligned}$$

Thus we have obtained also the second equality in (15) for all $t \in V$.

Remark 2. Condition (i) is evidently fulfilled in many important special cases; cf. [2].

2. Theorem 1 suggests that instead of equation (7) we can consider equation (4). The latter is dealt with in the following lemma.

Lemma 2. *Let* $\mathbf{F}$ *be a field of characteristic different from 2 and let* $U \subset \mathbf{F}$ *be a set fulfilling the condition*

(ii) $\frac{1}{2}(U + U) := \{w \in \mathbf{F} \mid w = \frac{1}{2}(u + v),\ u, v \in U\} = U,\ 0 \in U,$

$U \setminus \{0\} \neq \emptyset$; *moreover, for every* $x \in U$ *there exists a* $y \in \frac{1}{2}U \setminus \{0\}$ *such that* $y \neq x$ *and* $x + y \in U \setminus \{0\}$.
If functions $f : U \to \mathbf{F}$ *and* $\varphi : V \to \mathbf{F}$, *where* $V = V[U, \xi]$ *is given by (10) with (3), satisfy equation (4) for all* $x, y \in U$, $x \neq y$, *then there exist* $\alpha, \beta \in \mathbf{F}$ *such that*

$$f(x) = \alpha x + \beta,\ x \in U,\ \varphi(t) = \beta,\ t \in V. \tag{16}$$

Conversely, for every $\alpha, \beta \in \mathbf{F}$ functions (16) actually satisfy equation (4) $(x, y \in U,\ x \neq y)$.

PROOF. The last assertion is readily checked, so assume that functions f and φ satisfy equation (4). Take arbitrary $t \in \frac{1}{2}U\setminus\{0\}$ and put in (4) $x = 2t,\ y = 0$. ($2t \neq 0$ since $\mathbf{F}$ is not of characteristic 2). We obtain $f(0) = \varphi(t)$, i.e., with $\beta := f(0)$,

$$\varphi(t) = \beta, \quad t \in \frac{1}{2}U\setminus\{0\}. \tag{17}$$

Fix a $\overline{y} \in \frac{1}{2}U\setminus\{0\} = \frac{1}{2}(U\setminus\{0\}) \neq \emptyset$. For $x \in \frac{1}{2}U,\ x \neq -\overline{y}$, we have $\frac{1}{2}(x + \overline{y}) \in \frac{1}{2}\left(\frac{1}{2}U + \frac{1}{2}U\right)\setminus\{0\} \subset \frac{1}{2}U\setminus\{0\}$, whence by (4) and (17)

$$xf(\overline{y}) - \overline{y}f(x) = (x - \overline{y})\beta, \quad x \in \frac{1}{2}U,\ x \neq \overline{y},\ x \neq -\overline{y}. \tag{18}$$

With

$$\alpha := (f(\overline{y}) - \beta)/\overline{y}, \tag{19}$$

relation (18) yields

$$f(x) = \alpha x + \beta, \quad x \in \frac{1}{2}U,\ x \neq \overline{y},\ x \neq -\overline{y}. \tag{20}$$

In fact, according to (19), formula (20) is valid also for $x = \overline{y}$. If $-\overline{y} \in \frac{1}{2}U \subset \frac{1}{2}(U + U) = U$, then by (ii) there exists a $z \in \frac{1}{2}U\setminus\{0\}$ such that $z \neq -\overline{y}$ and $z - \overline{y} \in U\setminus\{0\}$. By (20) $f(z) = \alpha z + \beta$, whence we obtain according to (17) on setting $x = z,\ y = -\overline{y}$ in (4) $f(-\overline{y}) = -\alpha\overline{y} + \beta$. Thus (20) holds also for $x = -\overline{y}$ so that

$$f(x) = \alpha x + \beta, \qquad x \in \frac{1}{2}U. \tag{21}$$

Further, if $0 \in V$, then there exists a $u \in U$ such that also $-u \in U$. Setting in (4) $x = \frac{1}{2}u \in \frac{1}{2}U,\ y = -\frac{1}{2}u \in \frac{1}{2}U$ we get by (21) $\varphi(0) = \beta$. Together with (17) this implies that

$$\varphi(t) = \beta, \qquad t \in V \cap \left(\frac{1}{2}U\right). \tag{22}$$

Relations (21) and (22) imply now (16) by virtue of Lemma 1.

Remark 3. The condition $\frac{1}{2}(U+U)=U$ (a sort of convexity; observe that it is equivalent to $\frac{1}{2}(U+U)\subset U$, since $U\subset\frac{1}{2}(U+U)$ is always true) guarantees that function (3) maps $U\times U$ into U and is quite natural when we consider equation (4) or (7). On the other hand, the condition $0\in U$ is restrictive. It would be interesting to extend Lemma 2 to the case where $0\notin U$. For $\mathbf{F}=\mathbf{R}$ (the reals) and $U=I$ (a proper real interval) this has been done in [2], but in general the problem remains open.

Lemma 3. *Let* $\mathbf{F}$ *be a field of characteristic different from 2. If a set* $U\subset\mathbf{F}$ *fulfils condition* (ii), *then* U *contains at least five elements.*

PROOF. By (ii) $0\in U$ and there exists an $x\neq 0$ belonging to U. Hence $\frac{1}{2}x=\frac{1}{2}(x+0)\in\frac{1}{2}(U+U)=U$, and we have $0\neq\frac{1}{2}x\neq x$. Similarly, $\frac{1}{4}x=\frac{1}{2}\left(\frac{1}{2}x+0\right)\in\frac{1}{2}(U+U)=U$ and $0\neq\frac{1}{4}x\neq\frac{1}{2}x$. We distinguish two cases.

I. $\frac{1}{4}x=x$. Then the characteristic of $\mathbf{F}$ is 3 and we have $\frac{1}{2}x=-x$. According to (ii) there must be in $\frac{1}{2}U=\frac{1}{2}(U+0)\subset\frac{1}{2}(U+U)=U$ a $y\neq 0$ such that $y\neq -x=\frac{1}{2}x$ and $y-x\neq 0$, i.e., $y\neq x$. Also $\frac{1}{2}y=\frac{1}{2}(y+0)$ belongs to U and is different from $0,x,y$ and $\frac{1}{2}x$. Consequently the points $0,x,y,\frac{1}{2}x,\frac{1}{2}y$ are distinct from each other and belong to U.

II. $\frac{1}{4}x\neq x$. Then we consider $\frac{1}{8}x=\frac{1}{2}\left(\frac{1}{4}x+0\right)\in\frac{1}{2}(U+U)=U$. If $\frac{1}{8}x\neq x$, then the points $0,x,\frac{1}{2}x,\frac{1}{4}x,\frac{1}{8}x$ are distinct from each other and belong to U. If, on the other hand, $\frac{1}{8}x=x$, then the characteristic of $\mathbf{F}$ is 7 so that $3x\neq 0$ and we have $\frac{3}{4}x=\frac{1}{2}\left(\frac{1}{2}x+x\right)\in U$. Consequently the points $0,x,\frac{1}{2}x,\frac{1}{4}x,\frac{3}{4}x$ are distinct from each other and belong to U.

Remark 4. As may be seen from the example of $U=\mathbf{F}=\mathbf{Z}_5$ (cf., in particular, Lemma 4 below), a set U fulfilling (ii) need not contain more than five elements.

Lemmas 2 and 3 together with Theorem 1 immediately imply the following result on equation (7).

Theorem 2. *Let* $\mathbf{F}$ *be a field of characteristic different from 2, and let* $U\subset\mathbf{F}$ *be a set fulfilling condition* (ii). *If functions* $f,g:U\to\mathbf{F}$ *and* $\varphi:V\to\mathbf{F}$, *where* $V=V[U,\xi]$ *is given by* (10) *with* (3), *satisfy equation* (7) *for all* $x,y\in U,\ x\neq y$, *then there exist* $\alpha,\beta\in\mathbf{F}$ *such that*

$$f(x)=g(x)=\alpha x+\beta,\quad x\in U,\ \varphi(t)=\beta,\ t\in V. \tag{23}$$

Conversely, for every $\alpha,\beta\in\mathbf{F}$ *functions* (23) *actually satisfy equation* (7) $(x,y\in U,\ x\neq y)$.

We prove yet

Lemma 4. *Let* $\mathbf{F}$ *be a field of characteristic different from 2. The set* $U = \mathbf{F}$ *fulfils condition* (ii) *if and only if* $\mathbf{F}$ *is not isomorphic to* $\mathbf{Z}_3$.

PROOF. If $U = \mathbf{F}$ fulfils condition (ii), then $\mathbf{F}$ cannot be isomorphic to $\mathbf{Z}_3$ as results from Lemma 3. Conversely, assume that $\mathbf{F}$ is not isomorphic to $\mathbf{Z}_3$. Evidently

$$\frac{1}{2}(\mathbf{F}+\mathbf{F}) = \mathbf{F},\ \ 0 \in \mathbf{F},\ \ \mathbf{F}\backslash\{0\} \neq \emptyset.$$

Similarly, $\frac{1}{2}\mathbf{F} = \mathbf{F}$.

Take an $x \in \mathbf{F}$. If $x = 0$, then for every $y \in \frac{1}{2}\mathbf{F}\backslash\{0\}$ we have $y \neq x$ and $x + y \in \mathbf{F}\backslash\{0\}$. If $x \neq 0$, then observe that we cannot have $\mathbf{F} = \{0, x, -x\}$ and clearly this y has all the properties specified in (ii).

Remark 5. Lemmas 2 and 4 show that Proposition 1 in [2] is valid also for a large class of fields of characteristic 3; in fact, for all such fields that are not isomorphic to $\mathbf{Z}_3$.

In view of Lemma 4 our Theorem 2 yields the general solution of equation (7) on commutative fields of characteristic different from 2 which are not isomorphic to $\mathbf{Z}_3$. The case of fields isomorphic to $\mathbf{Z}_3$ will be dealt with presently.

Observe that if a field $\mathbf{F}$ has a subfield isomorphic to $\mathbf{Z}_3$, then the characteristic of $\mathbf{F}$ is 3, and conversely, every field of characteristic 3 has a unique subfield isomorphic to $\mathbf{Z}_3$. If U is a field isomorphic to $\mathbf{Z}_3$ and $\xi : U \times U \to U$ is given by (3), then (cf. (10)) $V[U,\xi] = U$.

We will use the characteristic function $\mathcal{X}_u(x)$ defined as 1 for $x = u$ and 0 otherwise.

Theorem 3. *Let* $\mathbf{F}$ *be a field of characteristic 3 and let* $U \subset \mathbf{F}$ *be the subfield of* $\mathbf{F}$ *isomorphic to* $\mathbf{Z}_3$. *If functions* $f, g, \varphi : U \to \mathbf{F}$ *satisfy equation* (7) *for all* $x, y \in U$, $x \neq y$, *then there exist* $\alpha, \beta, \gamma, c \in \mathbf{F}$ *such that*

$$(24) \qquad \begin{cases} f(x) = [\alpha + \mathcal{X}_{-u}(x)c]x + \beta - (\gamma-\beta)\mathcal{X}_{-u}(x), \\ g(x) = [\alpha + \mathcal{X}_{u}(x)c]x + \beta - (\gamma-\beta)\mathcal{X}_{-u}(x), \qquad x \in U, \\ \varphi(x) = \beta + (\gamma-\beta)\mathcal{X}_0(x), \end{cases}$$

where u *is an arbitrarily fixed element of* $U\backslash\{0\}$. *Conversely, for every* $\alpha, \beta, \gamma, c \in \mathbf{F}$ *functions* (24) *actually satisfy equation* (7) $(x, y \in U,\ x \neq y)$.

PROOF. Fix a $u \in U\backslash\{0\}$. Thus $U = \{0, u, -u\}$. As in the proof of Theorem 1 we arrive at relation (12). Moreover, setting in (7) first $x = u,\ y = 0$ and then $x = 0,\ y = u$ we obtain

$$(25) \qquad f(0) = g(0).$$

Write

$$c := \frac{g(u) - f(u)}{u}. \tag{26}$$

The only possible values of x, y in (12) are $x = u$, $y = -u$ and $x = -u$, $y = u$. In both cases we obtain $f(u) - g(u) = f(-u) - g(-u)$, which together with (26) implies

$$g(u) = f(u) + cu, \quad g(-u) = f(-u) - c(-u). \tag{27}$$

Relations (25) and (27) can jointly be written as

$$g(x) = f(x) + [\mathcal{X}_u(x) - \mathcal{X}_{-u}(x)]cx, \quad x \in U. \tag{28}$$

We insert (28) into (7) arriving thus at the equation

$$xf(y) - yf(x) = (x-y)\varphi\left(\frac{x+y}{2}\right) + [\mathcal{X}_u(x) - \mathcal{X}_{-u}(x)]cxy, \quad x \neq y. \tag{29}$$

Putting in (29) all possible values of $x, y \in U$, $x \neq y$, we obtain six relations, of which only three are independent:

$$\varphi(u) = f(0), \;\; \varphi(-u) = f(0), \;\; f(u) + f(-u) = -\varphi(0) - cu, \tag{30}$$

while the remaining three are a consequence of (30). With

$$\alpha := [f(u) - f(0)]/u, \quad \beta := f(0), \quad \gamma := \varphi(0)$$

relations (28) and (30) yield (24). The final statement is the matter of a straightforward verification.

Remark 6. For $c = 0$ and $\gamma = \beta$ formula (24) reduces to (23).

3. If $\mathbf{F}$ is a field of characteristic 2, then function (3) and hence also equations (4) and (7) make no sense. However, if the characteristic of $\mathbf{F}$ is different from 2, equation (7) considered on $\mathbf{F}$ (or on a subfield) can equivalently be written in the form

$$\frac{xf(y) - yg(x)}{x - y} = \psi(x+y), \qquad x \neq y, \tag{31}$$

which is meaningful also for fields of characteristic 2. In the present section we are going to deal with equation (31) on fields of characteristic 2.

First observe that if U is a field of characteristic 2 and $\xi : U \times U \to U$ is the sum

$$\xi(x, y) = x + y, \tag{32}$$

then (cf. (10))

$$V[U, \xi] = U \setminus \{0\}. \tag{33}$$

In fact, if $x + y = 0$, then $x = -y = y$, which means that $0 \notin V[U, \xi]$. On the other hand, for every $t \neq 0$ in U we have $t = t + 0 \in V[U, \xi]$. Hence (33) follows.

Now we prove a theorem on equation (31).

Theorem 4. *Let* $\mathbf{F}$ *be a field of characteristic 2 and let* $U \subset \mathbf{F}$ *be a subfield of* $\mathbf{F}$ *not isomorphic to* $\mathbf{Z}_2$. *If functions* $f, g : U \to \mathbf{F}$ *and* $\psi : (U\setminus\{0\}) \to \mathbf{F}$ *satisfy equation (31) for all* $x, y \in U,\ x \neq y$, *then there exist* $\alpha, \beta \in \mathbf{F}$ *such that*

$$f(x) = g(x) = \alpha x + \beta,\ x \in U, \quad \psi(t) = \beta,\ t \in U\setminus\{0\}. \tag{34}$$

Conversely, for every $\alpha, \beta \in \mathbf{F}$ *functions* (34) *actually satisfy equation* (31) $(x, y \in U,\ x \neq y)$.

PROOF. The last assertion is clear, so assume that f, g, φ satisfy equation (31) $(x, y \in U,\ x \neq y)$. For every $t \in U\setminus\{0\}$ we obtain from (31) on setting $x = t,\ y = 0$

$$\psi(t) = f(0) =: \beta, \qquad t \in U\setminus\{0\}. \tag{35}$$

Since U is not isomorphic to $\mathbf{Z}_2$, the set $U\setminus\{0\}$ contains at least three elements. Fix a $\overline{y} \in U\setminus\{0\}$ and write

$$\alpha_1 := \frac{g(\overline{y}) - \beta}{\overline{y}}, \quad \alpha_2 := \frac{f(\overline{y}) - \beta}{\overline{y}}. \tag{36}$$

We set in (31) first $y = \overline{y}$ and next $x = \overline{y}$ (in the latter case we write then x in place of y) to get

$$f(x) = \alpha_1 x + \beta,\ g(x) = \alpha_2 x + \beta, \quad x \in U,\ x \neq \overline{y}. \tag{37}$$

Further, there are in $U\setminus\{0\}$ points $u \neq v$ such that $u \neq \overline{y} \neq v$. We put now in (31) $x = u,\ y = v$ and we obtain according to (35) and (37) (cf. also (33)) $(\alpha_1 - \alpha_2)uv = 0$. Hence

$$\alpha_1 = \alpha_2 =: \alpha. \tag{38}$$

Relations (35), (37), (36) and (38) yield (34).

Remark 7. Theorem 4 shows, in particular, that in the case where U is a subfield of $\mathbf{F}$ and $\xi : U \times U \to U$ is given by (32) Theorem 1 is true also when the characteristic of $\mathbf{F}$ is 2. There is also a perfect analogy between Theorems 2 and 4, which shows that Proposition 1 in [2] is valid also when $\mathbf{F}$ is a field of characteristic 2 not isomorphic to $\mathbf{Z}_2$ except that then relation $\psi(t) = \beta$ need not hold for $t = 0$ (ϕ in [2; Proposition 1] corresponds to our ψ).

If $U = \{0, u\}$ is isomorphic to $\mathbf{Z}_2$, then the only information we can derive from equation (31) on U is that $f(0) = g(0) = \psi(u)$. The remaining three values $f(u),\ g(u),\ \psi(0)$ may be quite arbitrary. In fact, the functions $f,\ g$ and ψ need not even be defined at the respective points; cf. also, in particular, relation (33).

4. In this section we present (mostly without proofs) some results on the functional equations (2), (5), (6),

$$\frac{f(x)-g(y)}{x-y}=\varphi[\xi(x,y)], \qquad x \neq y, \tag{39}$$

and

$$\frac{f(x)-g(y)}{x-y}=\psi(x+y), \qquad x \neq y, \tag{40}$$

analogous to those obtained in previous sections for equations (1), (4), (7), (8) and (31).

Theorem 5. *Let* $\mathbf{F}$ *be a field of characteristic different from 2, let* $U \subset \mathbf{F}$ *be a set containing at least three elements and let* $\xi : U \times U \to U$ *be a symmetric function. If functions* $f, g : U \to \mathbf{F}$ *and* $\varphi : V[U,\xi] \to \mathbf{F}$ *satisfy equation (39) for all* $x, y \in U,\ x \neq y$*, then relation (11) holds.*

THE PROOF of Theorem 5 is quite similar to that of Theorem 1; cf. also [1].

Lemma 5. *Let* $\mathbf{F}$ *be a field and suppose that sets* $W \subset U \subset \mathbf{F}$ *and a function* $\xi : U \times U \to U$ *fulfil the condition*

(iii) *For every* $x \in U$ *there exists a* $y \in W$ *such that* $y \neq x$ *and* $\xi(x,y) \in W$.
If functions $f : U \to \mathbf{F}$ *and* $\varphi : V \to \mathbf{F}$ *as well as* $f_0 : U \to \mathbf{F}$ *and* $\varphi_0 : V \to \mathbf{F}$*, where* $V = V[U,\xi]$*, satisfy equation* (2) *for all* $x, y \in U,\ x \neq y$*, and relation* (15) *holds for* $x \in W,\ t \in V \cap W$*, then* (15) *holds for all* $x \in U,\ t \in V$.

THE PROOF of Lemma 5 does not differ from that of Lemma 1.

Lemma 6. *Let* $\mathbf{F}$ *be a field of characteristic different from 2 and let* $U \subset \mathbf{F}$ *be a set fulfilling the condition*

(iv) $\frac{1}{2}(U+U)=U,\ U=2u_0-U$ *for an* $u_0 \in U,\ U\backslash\{u_0\} \neq \emptyset$.

If functions $f : U \to \mathbf{F}$ *and* $\varphi : V \to \mathbf{F}$, *where* $V = V[U,\xi]$ *is given by* (10) *with* (3)*, satisfy equation* (5) *for all* $x, y \in U,\ x \neq y$*, then there exist* $\alpha, \beta, \gamma \in \mathbf{F}$ *such that*

$$f(x)=\alpha x^2+\beta x+\gamma,\ x \in U, \quad \varphi(t)=2\alpha t+\beta,\ t \in V. \tag{41}$$

Conversely, for every $\alpha, \beta, \gamma \in \mathbf{F}$ *functions* (41) *actually satisfy equation* (5) $(x, y \in U,\ x \neq y)$.

THE PROOF of Lemma 6 is similar to that of Theorem 5 in [2], where the special case $\mathbf{F}=\mathbf{R},\ U=I$ (a proper real interval) was considered. For the case where $U=\mathbf{F}$ see [1].

The condition $U = 2u_0 - U$ means that U is symmetric with respect to u_0. (Then necessarily $u_0 \in U$, for with an $x \in U\setminus\{u_0\}$ we have $y := 2u_0 - x \in U$ and $u_0 = \frac{1}{2}(x+y) \in \frac{1}{2}(U+U) = U$). This condition is very restrictive, but sometimes we can get rid of it applying an extension procedure. E.g., let $U \subset \mathbf{F}$ be a set fulfilling $\frac{1}{2}(U+U) = U$ and containing more than one point, say, $u, v \in U$, $u \neq v$. Then also $u_0 := \frac{1}{2}(u+v) \in U$ and without loss of generality we may assume (cf. [2]) that $u_0 = 0$ so that $v = -u$. Put $U_0 := U \cap (-U)$, $V_0 := V[U_0, \xi]$ and

$$U_n := U \cap (2^n U_0), \; V_n := V[U_n, \xi], \qquad n = 1, 2, \dots ,$$

where ξ denotes function (3). Observe that U_0 is symmetric around zero and we have $U_n \subset U_{n+1}$, $n = 0, 1, 2 \dots$. Assume that, moreover,

$$U = \bigcup_{n=0}^{\infty} U_n. \tag{42}$$

Then also

$$V := V[U, \xi] = \bigcup_{n=0}^{\infty} V_n. \tag{43}$$

Assume further that

$$V_n = U_n \cap V_{n+1}, \qquad n = 0, 1, 2, \dots . \tag{44}$$

Let functions $f : U \to \mathbf{F}$ and $\varphi : V \to \mathbf{F}$ satisfy equation (5) for all $x, y \in U$, $x \neq y$. By Lemma 6 there exist $\alpha, \beta, \gamma \in \mathbf{F}$ such that (41) holds for $x \in U_0$, $t \in V_0$. For every $x \in U_{n+1}$ there exists a $y \in U_n\setminus\{x\}$ such that $\frac{1}{2}(x+y) \in U_n$. (If $x \neq 0$, then we may take $y = 0$, and if $x = 0$, then any $y \in U_n\setminus\{0\}$ will do). Using Lemma 5 and relation (44) we show by induction that (41) holds for $x \in U_n$, $t \in V_n$, $n = 0, 1, 2, \dots$. In view of (42) and (43) we obtain hence (41) in full generality.

The procedure just described works for instance when $\mathbf{F}$ is a subfield of $\mathbf{C}$ (the field of complex numbers) endowed with the topology inherited from the natural topology of the complex plane, and $U \subset \mathbf{F}$ is a non–empty open set such that $\frac{1}{2}(U+U) = U$. When U is not open, but has a non–empty interior, we obtain relation (41) for $x, t \in \operatorname{int} U$ and then we show it valid for all $x \in U$, $t \in V[U, \xi]$ (ξ is given by (3)) using Lemma 5. Note that we have $\operatorname{int} U \subset V[U, \xi]$.

Condition (iv) implies that U contains at least three elements: u_0, $x \in U\setminus\{u_0\}$ and $2u_0 - x$. (The example of $U = \mathbf{F} = \mathbf{Z}_3$ shows that U need not contain anything more). Thus the following theorem is a consequence of Lemma 6 and Theorem 5.

Theorem 6. *Let* $\mathbf{F}$ *be a field of characteristic different from 2, and let* $U \subset \mathbf{F}$ *be a set fulfilling condition* (iv). *If functions* $f, g : U \to \mathbf{F}$ *and* $\xi : V \to \mathbf{F}$, *where* $V = V[U, \xi]$ *is given by* (10) *with* (3), *satisfy equation* (6) *for all* $x, y \in U$, $x \neq y$, *then there exist* $\alpha, \beta, \gamma \in \mathbf{F}$ *such that*

$$f(x) = g(x) = \alpha x^2 + \beta x + \gamma, \ x \in U, \quad \varphi(t) = 2\alpha t + \beta, \ t \in V. \tag{45}$$

Conversely, for every $\alpha, \beta, \gamma \in \mathbf{F}$ *functions* (45) *actually satisfy equation* (6) $(x, y \in U, \ x \neq y)$.

When $\mathbf{F}$ is a field of characteristic different from 2 equation (40) on $\mathbf{F}$ (which is then equivalent to (6)) has been completely solved in [1]. The general solution of (40) $(x, y \in \mathbf{F}, \ x \neq y)$ is then

$$f(x) = g(x) = \alpha x^2 + \beta x + \gamma, \ \psi(t) = \alpha t + \beta, \qquad x, t \in \mathbf{F}. \tag{46}$$

The same conclusion can be derived also from our Theorem 6 above, because if U is a field of characteristic different from 2, then evidently U fulfils condition (iv). Consequently there is no analogue of Theorem 3 for equation (6).

On the other hand, in the case where the characteristic of $\mathbf{F}$ is 2 equation (40) can be reduced to the classical functional equation of Cauchy

$$A(x + y) = A(x) + A(y). \tag{47}$$

(Functions satisfying (47) are called *additive*). Namely, we have the following result.

Theorem 7. *Let* $\mathbf{F}$ *be a field of characteristic 2 and let* $U \subset \mathbf{F}$ *be a subfield of* $\mathbf{F}$. *If functions* $f, g : U \to \mathbf{F}$ *and* $\psi : (U \setminus \{0\}) \to \mathbf{F}$ *satisfy equation* (40) *for all* $x, y \in U$, $x \neq y$, *then there exist constants* $c, d \in \mathbf{F}$ *and an additive function* $A : U \to \mathbf{F}$ *such that*

$$\begin{aligned} &f(x) = A(x) + c, \ \ g(x) = A(x) + d, \quad x \in U, \\ &\psi(t) = [A(t) + d - c]/t, \quad t \in U \setminus \{0\}. \end{aligned} \tag{48}$$

Conversely, for every $c, d \in \mathbf{F}$ *and every additive function* $A : U \to \mathbf{F}$ *functions* (48) *actually satisfy equation* (40) $(x, y \in U, \ x \neq y)$.

PROOF. Assume that functions $f, g : U \to \mathbf{F}$ and $\psi : (U \setminus \{0\}) \to \mathbf{F}$ satisfy equation (40) for all $x, y \in U$, $x \neq y$. Since the characteristic of $\mathbf{F}$ is 2, we can always replace the sign $-$ by $+$ and vice versa.

We fix a $\overline{y} \in U$ and obtain from (40) for $x \neq y$, due to the symmetry of function (32),

$$g(x) - f(x) = g(\overline{y}) - f(\overline{y}) =: c_0, \tag{49}$$

that is,

$$g(x) = f(x) + c_0, \qquad x \in U. \tag{50}$$

Formula (50), originally obtained only for $x \in U\backslash\{\overline{y}\}$, is valid also for $x = \overline{y}$ in view of the definition (49) of c_0.

Relation (50) inserted into (40) yields the equation (we replace every $-$ by $+$)

$$\frac{f(x) + f(y)}{x + y} = \psi(x + y) + \frac{c_0}{x + y}, \qquad x \neq y. \tag{51}$$

We put $c := f(0)$ and

$$A(x) := f(x) - c, \;\; x \in U, \quad \tilde{\psi}(t) := \psi(t) + c_0 t^{-1}, \;\; t \in U\backslash\{0\}, \tag{52}$$

so that

$$A(0) = 0. \tag{53}$$

Now, on account of (52) equation (51) goes over into

$$A(x) + A(y) = (x + y)\tilde{\psi}(x + y), \qquad x \neq y, \tag{54}$$

which with $y = 0$ yields in view of (53)

$$A(x) = x\tilde{\psi}(x), \qquad x \neq 0. \tag{55}$$

From (54) and (55) we obtain (47) for $x \neq y$. For $x = y$ relation (47) is also true because of (53). Consequently the function A is additive.

Relation (48) (where $d := c + c_0$) results now from (52), (50) and (55). The converse is trivial.

The general solution of equation (47) in fields of characteristic 2 can be obtained by the standard procedure described in [5; pp. 75–85]. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be fields of a characteristic $p \neq 0$. $\mathbf{F}_1$ and $\mathbf{F}_2$ may be considered as linear spaces over the field $\mathbf{Z}_p$ and it is easy to check that the additive functions $A : \mathbf{F}_1 \to \mathbf{F}_2$ are, in fact, linear transforms (homomorphisms) from the linear space $\mathbf{F}_1$ into the linear space $\mathbf{F}_2$. Let $B \subset \mathbf{F}_1$ be a Hamel–like basis of the linear space $\mathbf{F}_1$ over $\mathbf{Z}_p$. Then every function $A_0 : B \to \mathbf{F}_2$ can be uniquely extended onto $\mathbf{F}_1$ to a solution $A : \mathbf{F}_1 \to \mathbf{F}_2$ of equation (47); and all the additive functions $A : \mathbf{F}_1 \to \mathbf{F}_2$ can be obtained in this way.

Theorem 7 implies, in particular, that an analogue of Theorem 4 is not true for equation (40). When the characteristic of $\mathbf{F}$ is 2 functions (46) ($t \neq 0$) do not yield the general solution of (40) on $\mathbf{F}$, even if $\mathbf{F}$ is not isomorphic to $\mathbf{Z}_2$. In any case the relation $f = g$ need not hold. Moreover, there exist fields of characteristic 2 with infinitely many elements. For such a field $U = \mathbf{F}$ every Hamel–like basis of $\mathbf{F}$ over $\mathbf{Z}_2$ also is infinite and consequently functions (48) may depend on an infinite number of parameters.

References

[1] J. Aczél, A mean value property of the derivative of quadratic polynomials — without mean values and derivatives, *Math. Magazine* **58** (1985), 42–45.

[2] J. Aczél and M. Kuczma, On two mean value properties and functional equations associated with them, *Aequationes Math.* **38** (1989), 216–235.

[3] B. Choczewski et M. Kuczma, Sur certaines équations fonctionnelles considérées par I. Stamate, *Mathematica (Cluj)* **4** (27) (1962), 225–233.

[4] Sh. Haruki, A property of quadratic polynomials, *Amer. Math. Monthly* **86** (1979), 577–579.

[5] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality, *Państwowe Wydawnictwo Naukowe - Uniwersytet Śląski, Warszawa - Kraków - Katowice*, 1985.

[6] D. Pompeiu, Sur une proposition analogue au théorème des acroissements finis, *Mathematica (Cluj)* **22** (1946), 143–146.

[7] I. Stamate, O proprietate a parabolei şi integrarea unei ecuaţii funcţionale, *Inst. Politehnic Cluj, Lucrari Ştiinţifice* (1959), 101–106.

[8] I. Stamate, Observatii în legătură cu o ecuaţie funcţională, *Inst. Politehnic Cluj, Lucrari Ştiinţifice* (1959), 107–110.

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