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Estimate of the rate of mean convergence of Hermite interpolating processes

By S. SZABÓ (Budapest)

The mean convergence of interpolating processes was first investigated in the famous work [1] of P. ERDŐS and P. TURÁN in 1937 on a finite interval and this was generalized for an infinite interval in 1961 by J. BALÁZS and P. TURÁN [2]. In [3] JOÓ and SZABADOS proved an L^1-convergence theorem for interpolating processes based on Laguerre nodes, with estimate for the rate of convergence and this was generalized and strenghtened in [4] by I. JOÓ. Probably, the results given in [4] are not refinable.

The aim of the present paper is to generalize the Theorem of JOÓ on the Hermite interpolating process based on Laguerre nodes in two directions: first we give a different form for the rate of convergence, and secondly, our method works for more general nodes than the Laguerrian one.

Denote by $L_n^{(\alpha)}(x) := x^{-\alpha}e^x\frac{1}{n!}[e^{-x}x^{n+\alpha}]^{(n)}$ the Laguerre polynomials $(\alpha > -1)$ and by $x_1, \dots, x_n$ the zeros of $L_{(n)}^{(\alpha)}(x)$ (in fact $x_k = x(k,n,\alpha)$; but we simplify the notation). Define by $\ell_k(x) := L_n^{(\alpha)}(x)/L_n^{(\alpha)'}(x_k)(x-x_k)$ the fundamental polynomials of the Lagrange interpolation based on the roots of $L_n^{(\alpha)}(x)$ and put

$$F_n(f,x) :=$$
$$:= \sum_{k=1}^{n}\left[f(x_k)\frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k} + f'(x_k)(x-x_k)\right]\ell_k^2(x).$$

Introduce the space $C(\lambda) := \{f \in C[0,\infty) : \lim_{x\to\infty} f(x)e^{-\lambda x} = 0\}$, and write $\omega(f,\delta) := \omega_\infty(f,\delta) := \sup_{\substack{x,y\in\mathbf{R}\\ |x-y|\le\delta}} \mid f(x) - f(y) \mid$ for the modulus of continuity of f.

We shall prove the following

Theorem. *Suppose* $\alpha > -1,\ 0 < \lambda < \mu < 1;\ f, f' \in C(\lambda)$. *Then*

$$(1)\quad \int_0^\infty x^\alpha e^{-x}|f(x) - F_n(f,x)|dx \leq$$

$$\leq C\left\{\omega\left(F^*, \frac{a}{n}\right) + \left(\frac{\lambda}{\mu}\right)^b + 2^{\frac{n}{r}} r n^\alpha e^{-6n}\right\}$$

where $r \geq 1$ *(may depend on* n *),* $a = \left(\frac{6n}{1-\mu}\right)^{1/r}$, $b = n\left(1 - r^{-1}\right)$, $F^*(x) = f'(|x|^r)e^{-\lambda|x|^r}$ *and the constant* C *doesn't depend on* n, r, f.

For the proof we need some lemmas.

Lemma 1. *Given* $0 < \lambda < \mu < 1,\ r \geq 1$*(*$r$ *may depend on* n*),* $a := (\frac{6n}{1-\mu})^{1/r},\ b = n\left(1 - \frac{1}{r}\right);\ f, f' \in C(\lambda)$*, there exists a polynomial* $p(x) \in \prod_{n+1}$ *such that*

$$\text{(a)}\quad |f'(x) - p'(x)| = \mathcal{O}\left(e^{\mu x}\right)\left\{\omega\left(F^*, \frac{a}{n}\right) + \left(\frac{\lambda}{\mu}\right)^b\right\}\quad (0 \leq x \leq a^r),$$

$$\text{(b)}\quad \int_0^\infty x^\alpha e^{-x}|f'(x) - p'(x)|dx =$$

$$= \mathcal{O}\left\{\omega\left(F^*, \frac{a}{n}\right) + \left(\frac{\lambda}{\mu}\right)^b + 2^{\frac{n}{r}} a^{r\alpha} e^{-(1-\lambda)a^r}\right\}$$

Proof.

(a) We have to modify the definition of $p(x)$ in [3], (6) so as to obtain

$$p'(x) = q(\sqrt[r]{x})\sum_{k=0}^{b}\frac{(\lambda x)^k}{k!}.$$

(b) According to [3], using the estimate

$$\int_s^\infty e^{-t}t^\beta dt = e^{-s}s^\beta\left(1 + 0(s^{-1})\right),\quad \beta \in \mathbf{R},\ s \to +\infty,$$

we obtain

$$\int_{a^r}^\infty x^\alpha e^{-x}|f'(x) - p'(x)|dx = \mathcal{O}\left(\left(\frac{2}{a^r}\right)^{\frac{n}{r}}\right)\int_{a^r}^\infty x^{\frac{n}{r}+\alpha}e^{-(1-\lambda)x}dx =$$

$$= \mathcal{O}\left(2^{\frac{n}{r}} a^{r\alpha} e^{-(1-\lambda)a^r}\right).$$

Lemma 2. *If the conditions of Lemma 1 are fulfilled, then*

$$\int_0^\infty x^\alpha e^{-x}|f(x)-p(x)|dx = \mathcal{O}\left\{\omega\left(F^*,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^b + 2^{\frac{n}{r}} r a^{r\alpha} e^{-(1-\mu)a^r}\right\}.$$

PROOF. We may suppose that $f(0)=p(0)$ and so

$$f(x)-p(x)=\int_0^x (f'(t)-p'(t))\,dt.$$

In case $0\le x\le a^r$, using Lemma 2 (a) we obtain

$$\begin{aligned}|f(x)-p(x)| &\le \int_0^x |f'(t)-p'(t)|dt \le \\ &\le \mathcal{O}\left\{\omega\left(F^*,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^b\right\}\int_0^x e^{\mu t}dt \le \\ &\le \mathcal{O}\left(e^{\mu x}\right)\left\{\omega\left(F^*,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^b\right\}.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{a^r} x^\alpha e^{-x}|f(x)-p(x)|dx &\le \mathcal{O}\left\{\omega\left(F^*,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^b\right\}\int_0^{a^r} x^\alpha e^{-(1-\mu)x}dx \le \\ &\le \mathcal{O}\left\{\omega\left(F^*,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^b\right\}.\end{aligned}$$

On the other hand we can estimate the integral

$$\int_{a^r}^\infty x^\alpha e^{-x}|f(x)-p(x)|dx$$

in the following way:

$$\int_{a^r}^\infty x^\alpha e^{-x}|f(x)-p(x)|dx \le \int_{a^r}^\infty x^\alpha e^{-x}\big(|f(x)|+|p(x)|\big)dx = I_1+I_2,$$

where, obviously

$$I_1 = \int_{a^r}^{\infty} x^\alpha e^{-x} |f(x)| dx \le C \int_{a^r}^{\infty} x^\alpha e^{-(1-\lambda)x} dx \le C a^{r\alpha} e^{-(1-\lambda)a^r}.$$

For the estimate of I_2 we have to estimate $|p(x)|$ in case $x \ge a^r$. Using

$$p(x) = \int_{a^r}^{x} p'(u) du + p(a^r)$$

and taking into account Lemma 1 (a) we obtain

$$|p'(t)| \le C\, e^{\mu t}\,, \quad 0 \le t \le a^r,$$

hence

$$|p(t)| \le c\, e^{\mu t}\,, \quad 0 \le t \le a^r,$$

consequently

$$|p(x)| \le \int_{a^r}^{x} |p'(u)| du + 0\left(e^{\mu a^r}\right).$$

Taking into account $|p'(u)| \le c\, e^{\mu u}$, $0 \le u \le a^r$ and using [6] pp. 61–62 we obtain

$$|p'(u)| \le C\, e^{\mu u} \left(\frac{2u}{a^r}\right)^{n/r}, \quad a^r \le u\,.$$

From this we get

$$|p(x)| \le c \int_{a^r}^{x} e^{\mu u} \left(\frac{2u}{a^r}\right)^{\frac{n}{r}} du + c\, e^{\mu a^r} \le$$

$$\le c\, e^{\mu x} \left(\frac{2}{a^r}\right)^{\frac{n}{r}} x^{\frac{n}{r}+1} \frac{r}{n} + c\, e^{\mu a^r}, \quad x \ge a^r,$$

so

$$I_2 = \int_{a^r}^{\infty} x^\alpha e^{-x} |p(x)| dx = \mathcal{O}\left(e^{\mu a^r}\right) \int_{a^r}^{\infty} x^\alpha e^{-x} dx +$$

$$+ \mathcal{O}\left(\left(\frac{2}{a^r}\right)^{\frac{n}{r}} \cdot \frac{r}{n}\right) \int_{a^r}^{\infty} x^{\alpha+\frac{n}{r}+1} e^{-(1-\mu)x} dx =$$

$$= \mathcal{O}\left(a^{r\alpha} e^{-(1-\mu)a^r}\right) + \mathcal{O}\left(2^{\frac{n}{r}} \frac{r}{n} a^{r\alpha+r} e^{-(1-\mu)a^r}\right),$$

hence

$$\int_{a^r}^{\infty} x^\alpha e^{-x}|f(x)-p(x)|dx \le I_1 + I_2 =$$

$$= \mathcal{O}\left\{\omega\left(F^*, \frac{a}{n}\right) + \left(\frac{\lambda}{\mu}\right)^b + a^{r\alpha}e^{-(1-\lambda)a^r} + 2^{\frac{n}{r}} r a^{r\alpha} e^{-(1-\mu)a^r}\right\} =$$

$$= \mathcal{O}\left\{\omega\left(F^*, \frac{a}{n}\right) + \left(\frac{\lambda}{\mu}\right)^b + 2^{\frac{n}{r}} r a^{r\alpha} e^{-(1-\mu)a^r}\right\},$$

and Lemma 2 is proved.

Lemma 3 ([3], [4], [5]). *If $g, g' \in C(\lambda)$, $0 < \lambda < \mu < 1$, then*

$$\int_0^\infty x^\alpha e^{-x}\left|F_n(g,x)\right| dx \le c \max_{1\le k\le n} |g(x_k)|e^{-\mu x_k} + c \max_{1\le k\le n} |g'(x_k)|e^{-\mu x_k}.$$

Now we return to the PROOF OF THE THEOREM.
Obviously

$$\int_0^\infty x^\alpha e^{-x}|f(x)-F_n(f,x)|dx \le \int_0^\infty x^\alpha e^{-x}|f(x)-p(x)|dx +$$

$$+ \int_0^\infty x^\alpha e^{-x}|F_n(f-p,x)|dx,$$

where $p(x)$ is the polynomial in Lemma 1. Applying Lemma 3 we get

$$\int_0^\infty x^\alpha e^{-x}|F_n(f-p,x)|dx \le c \max_{1\le k\le n} |f(x_k)-p(x_k)|e^{-\mu x_k} +$$

$$+ c \max_{1\le k\le n} |f'(x_k)-p'(x_k)|e^{-\mu x_k}.$$

Here $0 \le x_k \le 5n$, and applying Lemma 1 (a) we have

$$\max_{1\le k\le n} |f'(x_k)-p'(x_k)|\, e^{-\mu x_k} = \mathcal{O}\left\{\omega\left(F^*, \frac{a}{n}\right) + \left(\frac{\lambda}{\mu}\right)^b\right\}.$$

On the other hand, we have seen in the proof of Lemma 2 that

$$|f(x)-p(x)| = \mathcal{O}(e^{\mu x})\left\{\omega\left(F^*,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^b\right\},\quad 0\le x\le a^r=\frac{6n}{1-\mu},$$

so

$$\max_{1\le k\le n}|f(x_k)-p(x_k)| = \mathcal{O}\left\{\omega\left(F^*,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^b\right\},$$

hence

$$\int_0^\infty x^\alpha e^{-x}|F_n(f-p,x)|dx = \mathcal{O}\left\{\omega\left(F^*,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^b\right\}.$$

Using this last estimate and applying Lemma 2 we obtain

$$\int_0^\infty x^\alpha e^{-x}|f(x)-F_n(f,x)|dx =$$

$$= \mathcal{O}\left\{\omega\left(F^*,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^b + 2^{\frac{n}{r}}ra^{r\alpha}e^{-(1-\mu)a^r}\right\} =$$

$$= \mathcal{O}\left\{\omega\left(F^*,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^b + 2^{\frac{n}{r}}rn^{\alpha}e^{-6n}\right\}$$

and the Theorem is proved.

Remark.

1. The case $r=2$ is proved in [4].
2. Our method works also for the Jacobi, Hermite and Markov–Sonin cases.

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S. SZABÓ
1221 BUDAPEST
ANNA U. 25.
HUNGARY

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