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On ball–filling vector sequences in $\mathbf{R}^N$

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Let $N \geq 2$ be any natural number: $(\lambda_n)_1^\infty \subset \mathbf{R}^N$ fills the ball in $\mathbf{R}^N$ with center at 0 and radius $\delta > 0$ if for any $x \in \mathbf{R}^N$, $|x| = \delta$ there exists (ε_n), $\varepsilon_n = \left\{ \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right.$ such that $x = \sum \varepsilon_n \lambda_n$.
In this paper we shall prove the following

Theorem. *For any $\varepsilon > 0$ there exists $(\lambda_n) \subset \mathbf{R}^N$, $\sum_{n=1}^{\infty} |\lambda_n| = 1$ such that (λ_n) fills the ball in $\mathbf{R}^N$ with center at 0 and of radius $R_N - \varepsilon$, where*

$$R_N := \frac{1}{\sqrt{\pi}(N-1)} \cdot \frac{\Gamma(N/2)}{\Gamma\left(\frac{N-1}{2}\right)}, \tag{1}$$

but there is no such sequence $(\lambda_n) \subset \mathbf{R}^N$, $\sum |\lambda_n| = 1$, which fills the ball in $\mathbf{R}^N$ with center 0 and of radius R_N.

Remark. In the case $N = 2$ we have $R_2 = 1/\pi$ and for this case the Theorem was proved in [1] using complex functions. We give a different proof here which works for arbitrary dimension $N \geq 2$.

PROOF of the Theorem.
Let us take finitely many vectors in R^N with equal length such that $\sum |\lambda_n| = 1$ and their direction is distributed among all directions "uniformly", going as far as we can. Namely define n^{N-1} points $\lambda_{\underline{j}}$, $\underline{j} = (j_1, \dots, j_{N-1})$, $0 \leq j_1, \dots, j_{N-1} \leq n-1$ with length $|\lambda_{\underline{j}}| = \frac{1}{n^{N-1}}$ and with polar coordinates

$$\vartheta_1 = j_1 \, \frac{\pi}{n}, \dots, \vartheta_{N-2} = j_{N-2} \, \frac{\pi}{n}, \; \vartheta_{N-1} = j_{N-1} \, \frac{2\pi}{n}$$

i.e. the Cartesian coordinates of $\lambda_{\underline{j}}$ will be

$$
\begin{aligned}
x_1 &= n^{1-N}\cos\vartheta_1\\
x_2 &= n^{1-N}\sin\vartheta_1\cos\vartheta_2\\
x_3 &= n^{1-N}\sin\vartheta_1\sin\vartheta_2\cos\vartheta_3\\
&-----------------\\
x_{N-1} &= n^{1-N}\sin\vartheta_1\dots\sin\vartheta_{N-2}\cos\vartheta_{N-1}\\
x_N &= n^{1-N}\sin\vartheta_1\dots\sin\vartheta_{N-2}\sin\vartheta_{N-1}.
\end{aligned}
$$

If we take a large number of vectors i.e. n is large, then the distribution of their directions can be made more and more uniform and by taking limit, the points reachable as a subsum of the vectors can be made to be at a maximal distance R_N from the origin, where

$$R_N = \frac{1}{\omega_N}\int_0^1 x(1-x^2)^{\frac{N-3}{2}}\omega_{N-1}dx \tag{2}$$

and ω_N denotes the surface of the unit ball in $\mathbf{R}^N$. Indeed, $\omega_{N-1}(1-x^2)^{\frac{N-2}{2}}\dfrac{dx}{\sqrt{1-x^2}}$ means the surface of the unit ball between the heights x and $x+dx$; multiplying this by x we obtain the height of the sum of vectors whose direction is in this surface. So the integral R_N denotes the limit of the maximal heights of the subsums of our vectors. It is known [2] that

$$R_N = \frac{1}{\sqrt{\pi}(N-1)}\cdot\frac{\Gamma(N/2)}{\Gamma\left(\frac{N-1}{2}\right)}. \tag{3}$$

Now we shall prove that a ball of radius $R_N-\varepsilon$ can be filled. Take the first finitely many λ_n with equal length, with $\sum|\lambda_n| = 1-\frac{\varepsilon}{R_N}$ and with almost uniform distribution of the directions. If the number of vectors tends to infinity then, as the above computation shows, the subsums give a net which approximates more and more the points of the ball of radius $R_N-\varepsilon$. Take so large a number M of vectors that the subsums $\sum_1^M \varepsilon_n\lambda_n$ approximate any point of the ball of radius $R_N-\varepsilon$ with an error less than $\varepsilon/2$. Now around any point $\sum_1^M \varepsilon_n\lambda_n$ take a ball with radius $\varepsilon/2$. Repeating the above argument we can find $\lambda_{M+1},\dots,\lambda_{M'}$ such that the net $\sum_1^{M'}\varepsilon_n\lambda_n$

approximates with an error less than $\varepsilon/4$ and $\sum\limits_{M+1}^{M'} |\lambda_n| = \frac{\varepsilon}{2R_N}$. In the following step we can ensure an error $< \varepsilon/8$ furthermore $\sum\limits_{M'+1}^{M''} |\lambda_n| = \frac{\varepsilon}{4R_N}$ etc.. Repeating this process we can construct the sequence (λ_n) filling the whole ball of radius $R_N - \varepsilon$. On the other hand, we prove that an (open) ball of radius R_N cannot be filled. Take the indirect assumption. Given any direction $e \in S^{N-1}$ all points $ce, 0 \le c < R_N$ can be obtained, hence

$$\sum_{n:\langle e,\lambda_n\rangle>0} \langle e, \lambda_n\rangle \ge R_n \text{ for all } e \in S^{N-1}. \tag{4}$$

Take the integral of the left side; since for fixed n

$$\int_{\langle e,\lambda_n\rangle>0} \langle e, \lambda_n\rangle dS^{N-1} = |\lambda_n|\, \omega_N R_N$$

we get

$$\int_{S^{N-1}} \sum_{n:\langle e,\lambda_n\rangle>0} \langle e, \lambda_n\rangle dS^{N-1} = \omega_N R_N \sum |\lambda_n| = \omega_N R_N$$

which means by (4) that

$$\sum_{n:\langle e,\lambda_n\rangle>0} \langle e, \lambda_n\rangle = R_N. \tag{5}$$

Consider again the expansion of the elements ce where $c \to R_N - 0$ and $e \in S^{N-1}$. If we assume that $\langle e, \lambda_n\rangle \ne 0$ for all n then (5) implies

$$\sum_{n:\langle e,\lambda_n\rangle>0} \lambda_n = R_N e. \tag{6}$$

Let now $e' \perp e$, $e' \in S^{N-1}$ and $\delta > 0$. Excluding countably many δ's we can also ensure that

$$\sum_{n:\langle e+\delta e',\lambda_n\rangle>0} \lambda_n = R_N \frac{e + \delta e'}{\sqrt{1+\delta^2}}. \tag{7}$$

We multiply here by $\sqrt{1+\delta^2}$ and extract it from (6) to obtain

$$R_N \delta e' = -\sum_{n:\langle e,\lambda_n\rangle>0} \lambda_n + \sqrt{1+\delta^2} \sum_{n:\langle e+\delta e',\lambda_n\rangle>0} \lambda_n .$$

Taking the scalar product by e' we get

$$(8)\qquad \begin{aligned} R_N\delta = &-\sum_{\substack{n:\langle e,\lambda_n\rangle>0,\\ \langle e+\delta e',\lambda_n\rangle\le 0}} \langle \lambda_n, e'\rangle + \sum_{\substack{n:\langle e,\lambda_n\rangle\le 0,\\ \langle e+\delta e',\lambda_n\rangle> 0}} \langle \lambda_n, e'\rangle + \\ &+ (\sqrt{1+\delta^2}-1) \sum_{n:\langle e+\delta e',\lambda_n\rangle>0} \langle \lambda_n, e'\rangle. \end{aligned}$$

Here the summands of the first resp. second sum on the right are negative resp. positive, and the third member has order $O(\delta^2)$. Consequently we obtain the following statement:

($\star$) Let $e, e' \in S^{N-1}$, $e \perp e'$, $\delta > 0$ and suppose that $\langle e, \lambda_n\rangle \neq 0$, $\langle e+\delta e', \lambda_n\rangle \neq 0$ for all n. Then $\langle e, \lambda_n\rangle < 0$, $\langle e+\delta e', \lambda_n\rangle > 0$ implies $0 < \langle e', \lambda_n\rangle < R_N\delta + O(\delta^2)$.

Now let $0 < \varepsilon < 1$ be arbitrary and take $e \in S^{N-1}$ satisfying $-\varepsilon|\lambda_1| < \langle e, \lambda_1\rangle < 0$, $\langle e, \lambda_n\rangle \neq 0$ for every n. Then choose $e' \in S^{N-1}$, $e' \perp e$ such that

$$\langle e', \lambda_1\rangle > \sqrt{1-\varepsilon^2}\,|\lambda_1|.$$

Finally take $\delta > 0$ with the conditions $\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} < \delta < \frac{2\varepsilon}{\sqrt{1-\varepsilon^2}}$, $\langle e+\delta e', \lambda_n\rangle \neq 0$ for every n. Since $\langle e, \lambda_1\rangle < 0$ and $\langle e+\delta e', \lambda_1\rangle > \left(\delta\sqrt{1-\varepsilon^2} - \varepsilon\right) \quad |\lambda_1| > 0$, we have by ($\star$) that

$$\sqrt{1-\varepsilon^2}|\lambda_1| < \langle e', \lambda_1\rangle < R_N\delta + O(\delta^2)$$

and then

$$|\lambda_1| < \frac{R_N\delta + O(\delta^2)}{\sqrt{1-\varepsilon^2}} = O\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right).$$

Since $\varepsilon > 0$ can be arbitrarily small, this would imply $|\lambda_1| = 0$. The contradiction proves that a ball of radius R_N cannot indeed be filled by subsums $\sum \varepsilon_n\lambda_n$ as asserted.

References

[1] I. Joó, On the growth of the eigenfunctions of the Schrödinger operator II, *Annales Univ. Sci. Budapest. Sect. Math.* **31** (1988), 75–85.

[2] E. T. Whittaker and G. N. Watson, A course of modern analysis, *Cambridge Univ. Press* (1927).

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