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# On the difference of integer-valued additive functions

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## 1. Introduction

A classical theorem of ERDŐS [4] asserts that if a real-valued additive function satisfies $f(n+1) - f(n) \to 0$, then it must be of the form $f(n) = c \log n$. Many generalizations and analogs have been found since then. An important one is the following (still unpublished) result of E. WIRSING: if a multiplicative function satisfies $|g(n)| = 1$, $g(n+1) - g(n) \to 0$, then it must be of the form $g(n) = n^{ic}$ with some real constant $c$.

DARÓCZY and KÁTAI [3] found the following common generalization of these results: if $G$ is a locally compact, compactly generated abelian group (written additively) and a $G$–valued additive arithmetical function $f$ satisfies $f(n+1) - f(n) \to 0$, then $f$ is the restriction to the set of integers of a continuous homomorphism from the multiplicative group of positive real numbers to $G$.

DARÓCZY and KÁTAI's proof is based on an application of WIRSING's theorem to the functions $g(n) = \chi(f(n))$, where $\chi$ runs over the (continuous) characters of $G$. This approach cannot handle groups that are not separated by their characters. AJTAI, HAVAS and KOMLÓS [1] have shown that there are commutative groups with a Hausdorff topology that do not admit any nontrivial continuous character. This leaves the question of existence of nontrivial group-valued additive functions with $f(n+1) - f(n) \to 0$ open. We show that such functions do indeed exist, even for the simplest $G$, the additive group of integers.

**Theorem 1.** *There is an integer-valued completely additive function $f$, not identically 0, and a Hausdorff topology $T$ on the set of integers which makes it a topological group with the operation of addition, such that*

$$f(n+1) - f(n) \to 0 \text{ in } T.$$

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Before starting the proof we show why our function does not fall within the frames of a suitable extension of DARÓCZY and KÁTAI's theorem.

Let $G$ be a topological group. It seems reasonable to call a $G$–valued additive function *regular,* if it can be extended to a continuous homomorphism from the multiplicative group of rational numbers to $G$. (The existence of an extension to the group of positive reals depends on completeness properties of $G$.) We show that our function does not possess such an extension.

**1.1. Statement.** *Let $G$ be the group of integers with an arbitrary Hausdorff group topology. There is no nontrivial continuous homomorphism from $\mathbf{Q}^*$, the multiplicative group of positive rational numbers with the usual topology, to $G$.*

PROOF. Suppose that there is such a homomorphism $\varphi$. Take a prime $p$ such that $\varphi(p) \neq 0$, and two other primes $q$ and $r$. Write $\varphi(p) = a$, $\varphi(q) = b$, $\varphi(r) = c$; we know $a \neq 0$. Define $m = p^b q^{-a}$, $n = p^c r^{-a}$. Here $m$ contains $q$ but not $r$ while $n$ contains $r$ but not $q$ (they may and may not contain $p$), thus they are different from 1 and not powers of a common base, therefore $(\log m)/(\log n)$ is irrational. This implies that the numbers $m^u n^v$, $u, v$ integers, are dense in $\mathbf{Q}$. Since $\varphi$ vanishes on this dense set, it must be identically zero. □

An even stronger requirement would be that the function cannot be extended to an algebraical homomorphism, in other words, that it is not completely additive. This is, however, impossible.

**1.2. Statement.** *An additive function with values in an arbitrary commutative topological group that satisfies $f(n+1) - f(n) \to 0$ must be completely additive.*

PROOF. To prove complete additivity it is sufficient to show $f(p^{k+1}) = f(p^k) + f(p)$ for every prime $p$ and positive integer $k$. Now observe that the equality

$$f(p^{k+1}) - f(p^k) - f(p) = \big(f(p^k n+1) - f(p^k n)\big) - \big(f(p^{k+1} n+p) - f(p^{k+1} n)\big)$$

holds for every $n$ coprime to $p$, and the right side tends to 0 as $n \to \infty$. □

## 2. Proof of the Theorem

*2.1. Definition.* A Hausdorff topology on the set of integers is called an *arithmetical topology,* if it turns the set of integers with the operation of addition into a topological group.

*2.2. Definition.* We call a sequence $a_n$ of integers *nullpotent,* if there is an arithmetical topology in which $a_n \to 0$.

With this terminology, we need to construct a nonzero completely additive function for which the sequence $f(n+1)-f(n)$ is nullpotent. We prove slightly more.

**Theorem 2.** *There is a positive-valued function $F(n)$ such that the sequence $f(n+1)-f(n)$ is nullpotent for every completely additive function $f$ which satisfies*

$$|f(q)| > F(q) \max_{p<q} |f(p)| \tag{2.1}$$

*for all but finitely many primes $q$.*

We quote the following arithmetical description of nullpotency from RUZSA [5].

**2.3. Lemma.** *A sequence $a_n$ is nullpotent if and only if for every integer $u \neq 0$ and positive integer $k$ the equation*

$$u = e_1 a_{n_1} + e_2 a_{n_2} + \cdots + e_k a_{n_k}, \quad e_n = \pm 1 \tag{2.2}$$

*has only finitely many primitive solutions. Here a solution of (2.2) is called primitive, if none of the $2^k - 1$ nonempty subsums is 0.*

Substituting $a_n = f(n+1) - f(n)$ into (2.2), we obtain the equation

$$u = \sum e_i (f(n_i + 1) - f(n_i)). \tag{2.3}$$

If we extend $f$ to rational numbers naturally by putting $f(a/b) = f(a) - f(b)$, then (2.3) can be rewritten as

$$u = f(Q), \quad Q = \prod \left( \frac{n_i + 1}{n_i} \right)^{e_i} = \prod \left( \frac{m_i + e_i}{m_i} \right), \tag{2.4}$$

where

$$m_i = \begin{cases} n_i & \text{if } e_i = 1, \\ n_i + 1 & \text{if } e_i = -1. \end{cases} \tag{2.5}$$

For the proof the following lemma on prime factors of such products $Q$ is fundamental.

**2.4. Lemma.** *Let $m_1 \leq m_2 \leq \cdots \leq m_k$ be positive integers, $b_1, \ldots, b_k$ integers such that $|b_i| \leq A$ and $m_i + b_i > 0$ $(i = 1, \ldots, k)$. Write*

$$Q = \prod_{i=1}^{k} \left( \frac{m_i + b_i}{m_i} \right).$$

*Assume that all prime factors in the numerator and denominator of $Q$ are $\leq P$ and that*

$$\prod_{i=j}^{k}\left(\frac{m_i+b_i}{m_i}\right)\neq 1 \quad (j=1,\dots,k). \tag{2.6}$$

*Then we have $m_k \leq G(k,P,A)$, an effectively computable number depending only on $k$, $P$ and $A$.*

This lemma, which will be proved in the next section, finishes our preparation to the proof of the theorem.

PROOF OF THEOREM 2. We put $F(q)=4qG(q,q,1)$ with the function $G$ of Lemma 2.4. We have to prove that (2.3) has only finitely many primitive solutions. Let $q_0$ be such a number that (2.1) holds for $q \geq q_0$, and also that $f(p)$ is not identically 0 for $p<q_0$, in which case (2.1) also means $|f(q)|>F(q)$ $(q>q_0)$.

Without restricting the generality we may assume that the $m_i$ given by (2.5) are increasing. Consider a primitive solution and let $q$ be the largest prime that occurs in the numerator or denominator of the number $Q$ given in (2.4). From Lemma 2.4 we infer $m_k \leq G(k,q,1)$; condition (2.6) follows from the primitivity of (2.3) (in fact, we needed only the $k$ interval-subsums).

Let $K=\max(k,|u|,q_0)$. Assume first $q>K$. We have obviously

$$|f(Q)| \geq |f(q)| - r\max_{p<q}|f(p)| \geq |f(q)|(1-r/F(q)), \tag{2.7}$$

where $r$ is the total number of primes, counted with multiplicity, that occur in any of the numbers $n_i$ or $n_i+1$. Since $k<q$, we have $n_i \leq G(q,q,1)$, thus the number of prime factors in any of these numbers is at most $G(q,q,1)$ and we have $r \leq 2kG(q,q,1) < F(q)/2$. Hence (2.7) yields

$$|u| = |f(Q)| \geq |f(q)|/2 \geq F(q)/2 > q > |u|,$$

a contradiction.

Thus we have $q \leq K$, and from Lemma 2.4 we infer $m_k \leq G(k,K,1)$, a finite number of choices. □

## 3. Proof of Lemma 2.4

We use the following result of BAKER [2] in the form given by SHOREY et al. [6], p. 66 (we specialized the formulation to integers).

**3.1. Lemma.** *Let $\alpha_1, \dots, \alpha_\ell$ be positive integers, $\ell \geq 2$, $\alpha_i \leq A_i \geq 4$ for $i = 1, \dots, \ell$. Put*

$$\Omega = \prod_{i=1}^{\ell} \log A_i, \quad \Omega' = \prod_{i=1}^{\ell-1} \log A_i .$$

*Let $b_1, \dots, b_\ell$ be integers with $|b_j| \leq B \geq 4$. Then either $\alpha_1^{b_1} \dots \alpha_\ell^{b_\ell} = 1$ or*

$$\log |\alpha_1^{b_1} \dots \alpha_\ell^{b_\ell} - 1| > -\ell^{c\ell} \Omega \log \Omega' \log B \tag{3.1}$$

*with an absolute constant $c$.*

Proof of Lemma 2.4. We use Vinogradov's symbol $\ll$ where the implicit constant may depend on $k$, $A$ and $P$.

Write $Q$ in the form $Q = p_1^{t_1} \dots p_s^{t_s}$ with distinct primes $p_1, \dots, p_s$ and integers $t_i$. We have

$$|t_i| \leq 2k \log(m_k + A) \ll \log m_k$$

for all $i$. The lemma above yields

$$\log |Q - 1| \gg - \log \log m_k,$$

while a direct computation gives $|Q - 1| \ll 1/m_1$. Combining the two we obtain

$$\log m_1 \ll \log \log m_k .$$

Now we prove the inequalities

$$\log m_j \ll (\log \log m_k)^j \tag{3.2}$$

by induction on $j$. Suppose it holds for $1, \dots, j-1$. On one hand we have

$$\left| \frac{m_j + b_j}{m_j} \frac{m_{j+1} + b_{j+1}}{m_{j+1}} \dots \frac{m_k + b_k}{m_k} - 1 \right| \ll \frac{1}{m_j} . \tag{3.3}$$

On the other hand

$$\left| \frac{m_j + b_j}{m_j} \frac{m_{j+1} + b_{j+1}}{m_{j+1}} \dots \frac{m_k + b_k}{m_k} - 1 \right| = \left| Q \prod_{i=1}^{j-1} \frac{m_i}{m_i + b_i} - 1 \right| .$$

To this expression we apply our Lemma with $\ell = s + 1$, $\alpha_\ell = \prod_{i=1}^{j-1} (m_i/(m_i + b_i))$, $b_\ell = 1$. By the induction hypothesis we have

$$\log A_\ell \leq j \log(m_{j-1} + |b_{j-1}|) \ll (\log \log m_k)^{j-1} ,$$

hence

$$\log\left|\frac{m_j+b_j}{m_j}\,\frac{m_{j+1}+b_{j+1}}{m_{j+1}}\,\dots\,\frac{m_k+b_k}{m_k}-1\right| \gg$$
$$\gg -(\log\log m_k)^{j-1}\log\log m_k\,.$$

On combining this inequality with (3.3) we obtain (3.2) for $j$.
Finally, the case $j=k$ of (3.2) means

$$\log m_k \ll (\log\log m_k)^k\,,$$

which implies $m_k \ll 1$. □

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