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Structure of normal twisted group rings

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Abstract. Let $K_\lambda G$ be the twisted group ring of a group G over a commutative ring K with 1, and let λ be a factor set (2-cocycle) of G over K. Suppose $f : G \to U(K)$ is a map from G onto the group of units $U(K)$ of the ring K satisfying $f(1) = 1$. If $x = \sum_{g\in G} \alpha_g u_g \in K_\lambda G$ then we denote $\sum_{g\in G} \alpha_g f(g) u_g^{-1}$ by x^f and assume that the map $x \to x^f$ is an involution of $K_\lambda G$. In this paper we describe those groups G and commutative rings K for which $K_\lambda G$ is f-normal, i.e. $xx^f = x^f x$ for all $x \in K_\lambda G$.

1. Introduction

Let G be a group and K a commutative ring with unity. Suppose that the elements of the set

$$\Lambda = \{\lambda_{a,b} \in U(K) \mid a, b \in G\}$$

satisfy the condition

$$\lambda_{a,b}\lambda_{ab,c} = \lambda_{b,c}\lambda_{a,bc} \tag{1}$$

for all $a, b, c \in G$. Then Λ will be called a *factor system* (2-cocycle) of the group G over the ring K. The twisted group ring $K_\lambda G$ of G over the commutative ring K is an associative K-algebra with basis $\{u_g \mid g \in G\}$

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and with multiplication defined distributively by $u_g u_h = \lambda_{g,h} u_{gh}$, where $g, h \in G$ and

$$\lambda_{g,h} \in \Lambda = \{\lambda_{a,b} \in U(K) \mid a, b \in G\}.$$

Note that if $\lambda_{g,h} = 1$ for all $g, h \in G$, then $K_\lambda G \cong KG$, where KG is the group ring of the group G over the ring K.

Properties of twisted group algebras and their groups of units were studided by many authors, see, for instance, the paper by S. V. Mihovski and J. M. Dimitrova [1]. Our aim is to describe the structure of f-normal twisted group rings. This result for group rings was obtained in [2, 3].

We shall refer to two twisted group rings $K_\lambda G$ and $K_\mu G$ as being diagonally equivalent if there exists a map $\theta : G \to U(K)$ such that

$$\lambda_{a,b} = \theta(a)\theta(b)\mu_{a,b}(\theta(ab))^{-1}.$$

We say that a factor system Λ is normalized if it satisfies the condition

$$\lambda_{a,1} = \lambda_{1,b} = \lambda_{1,1} = 1$$

for all $a, b \in G$.

Hence, given $K_\mu G$ there always exists a diagonally equivalent twisted group ring $K_\lambda G$ with factor system Λ defined by $\lambda_{a,b} = \mu_{1,1}^{-1}\mu_{a,b}$ such that Λ is normalized. From now on, all the factor systems considered are supposed to be normalized.

The map ϕ from the ring $K_\lambda G$ onto $K_\lambda G$ is called *an involution*, if it satisfies the conditions

(i) $\phi(a+b) = \phi(a) + \phi(b)$; (ii) $\phi(ab) = \phi(b)\phi(a)$; (iii) $\phi^2(a) = a$ for all $a, b \in K_\lambda G$.

Let $f : G \to U(K)$ be a map from the group G onto the group of units $U(K)$ of the commutative ring K, satisfying $f(1) = 1$. For an element $x = \sum_{g\in G} \alpha_g u_g \in K_\lambda G$ we define $x^f = \sum_{g\in G} \alpha_g f(g) u_g^{-1} \in K_\lambda G$.

Let $x \to x^f$ be an involution of the twisted group ring $K_\lambda G$. The twisted group ring $K_\lambda G$ is called *f-normal* if

$$xx^f = x^f x \tag{2}$$

for all $x \in K_\lambda G$.

Recall that a p-group is called *extraspecial* (see [4], Definition III.13.1) if its centre, commutator subgroup and Frattini subgroup are equal and have order p.

Theorem. *Let $x \to x^f$ be an involution of the twisted group ring $K_\lambda G$. If the ring $K_\lambda G$ is f-normal then the group G and the ring K satisfy one of the following conditions:*

1) *G is abelian and the factor system is symmetric, i.e. $\lambda_{a,b} = \lambda_{b,a}$ for all $a, b \in G$;*
2) *G is an abelian group of exponent 2 and the factor system satisfies*

$$(\lambda_{a,b} - \lambda_{b,a})(1 + f(b)\lambda_{b,b}^{-1}) = 0 \tag{3}$$

for all $a, b \in G$;

3) *$G = H \rtimes C_2$ is a semidirect product of an abelian group H of exponent not equal to 2 and $C_2 = \langle a \mid a^2 = 1\rangle$ with $h^a = h^{-1}$ for all $h \in H$, the factor system of H is symmetric, $f(a) = -\lambda_{a,a}$ and*

$$\lambda_{a,h} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a}, \quad \lambda_{h,a} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}}; \tag{4}$$

4) *G is a hamiltonian 2-group and the factor system satisfies*
   - 4.i) *for all noncommuting $a, b \in G$*

$$\lambda_{a,b} = f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{b,a^{-1}} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}; \tag{5}$$

   - 4.ii) *$\lambda_{g,h} = \lambda_{h,g}$ for any $h \in C_G(\langle g\rangle)$ and $f(c) = \lambda_{c,c}$ for every c of order 2;*

5) *$G = \Gamma \curlyvee C_4$ is a central product of a hamiltonian 2-group Γ and a cyclic group $C_4 = \langle d \mid d^4 = 1\rangle$ with $\Gamma' = \langle d^2\rangle$. The factor system satisfies (5) and*

$$\lambda_{b,a}\lambda_{ba,d} + f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{a,b}\lambda_{ab,d^{-1}} = 0, \tag{6}$$

where $a, b \in \Gamma$, $a^4 = b^4 = 1$ and $[a, b] \neq 1$;

6) *G is either $E \times W$ or $(E \curlyvee C_4) \times W$, where E is an extraspecial 2-group, $E \curlyvee C_4$ is the central product of E and $C_4 = \langle c \mid c^4 = 1\rangle$ with $E' = \langle c^2\rangle$ and $\exp(W)|2$. The factor system satisfies:*
   - 6.i) *If $a \in G$ has order 4 then $\lambda_{a,h} = \lambda_{h,a}$ for all $h \in C_G(\langle a\rangle)$;*
   - 6.ii) *if $\langle a, b\rangle$ is a quaternion subgroup of order 8 of G then the properties (5) and (6) are satisfied for every $d \in C_G(\langle a, b\rangle)$ of order 4, and $f(v) = \lambda_{v,v}$ for all $v \in C_G(\langle a, b\rangle)$ of order 2;*

6.iii) *if* $\langle a, b \mid a^4 = b^2 = 1\rangle$ *is the dihedral subgroup of order* 8, *then* $f(b) = -\lambda_{b,b}$ *and the properties* (4) *and* (6) *are satisfied for every* $d \in C_G(\langle a, b\rangle)$ *of order* 4.

Moreover, the conditions 1)–5) *are also sufficient for* $K_\lambda G$ *to be* f-*normal. The condition* 6) *is sufficient if* K *is an integral domain of characteristic* 2.

2. Lemmas

Let C_4, Q_8 and D_8 be a cyclic group of order 4, a quaternion group of order 8 and a dihedral group of order 8, respectively. As usual, $x^y = y^{-1}xy$, $\exp(G)$ and $C_G(\langle a, b\rangle)$ denote the exponent of G and the centralizer of the subgroup $\langle a, b\rangle$ in G.

It is easy to see that $\lambda_{g,g^{-1}} = \lambda_{g^{-1},g}$ and $u_g^{-1} = \lambda_{g,g^{-1}}^{-1} u_{g^{-1}}$ hold for all $g \in G$.

Lemma 1. *The map* $x \to x^f$ *is an involution of the ring* $K_\lambda G$ *if and only if*

$$f(gh)\lambda_{g,h}^2 = f(g)f(h)$$

for all $g, h \in G$.

Proof. Let the map $x \to x^f$ be an involution of the ring $K_\lambda G$. If $g, h \in G$, then $(u_g u_h)^f = u_h^f u_g^f$. Thus

$$\begin{aligned} \lambda_{g,h} f(gh) u_{gh}^{-1} &= (\lambda_{g,h} u_{gh})^f = (u_g u_h)^f = f(g)f(h)u_h^{-1}u_g^{-1} \\ &= f(g)f(h)(\lambda_{g,h}^{-1} u_{gh})^{-1} \end{aligned}$$

and $f(gh)\lambda_{g,h}^2 = f(g)f(h)$ for all $g, h \in G$. □

Clearly, if $K_\lambda G$ is a group ring, then the map $x \to x^f$ is an involution of the group ring KG if and only if f is a homomorphism from G to $U(K)$.

Lemma 2. *If the ring* $K_\lambda G$ *is* f-*normal then the group* G *satisfies one of the conditions* 1)–6) *of Theorem 1.*

Proof. Let $K_\lambda G$ be an f-normal twisted group ring. If $a, b \in G$ and $x = u_a + u_b \in K_\lambda G$, then $x^f = f(a)u_a^{-1} + f(b)u_b^{-1}$ and by (2)

$$\begin{aligned} &f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{a^{-1},b}u_{a^{-1}b} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}u_{b^{-1}a} \\ &= f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{b,a^{-1}}u_{ba^{-1}} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}u_{ab^{-1}}. \end{aligned} \tag{7}$$

Now put $y = u_a(u_1 + u_b)$. Then $y^f = (u_1 + f(b)u_b^{-1})f(a)u_a^{-1}$ and by (2)

$$\lambda_{a,b}u_{ab} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}u_{ab^{-1}} = \lambda_{b,a}u_{ba} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}u_{b^{-1}a}. \tag{8}$$

We shall treat two cases.

I. Let $[a,b] \neq 1$ for $a, b \in G$ and $a^2 \neq 1$, $b^2 \neq 1$. Then by (8) $b^a = b^{-1}$ and by (7) $a^2 = b^2$. The factor system satisfies

$$\begin{cases} \lambda_{a,b} = f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{b,a^{-1}} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}; \\ \lambda_{b,a} = f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{a^{-1},b} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}. \end{cases} \tag{9}$$

II. Let $[a,b] \neq 1$ for $a, b \in G$ and $a^2 = 1$, $b^2 \neq 1$. Then by (8) we have $b^a = b^{-1}$ and by (7), $f(a) = -\lambda_{a,a}$. The factor system satisfies

$$\begin{cases} \lambda_{a,b} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}; \\ \lambda_{b,a} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}. \end{cases}$$

Let G be a nonabelian group and let $W = \{g \in G \mid g^2 \neq 1\}$.

First we consider the case when the elements of W commute. Then $\langle w \mid w \in W\rangle$ is an abelian subgroup and if $b \in W$ and $a \in G \setminus \langle W\rangle$ then a^2=1 and $(ab)^2 = 1$. Therefore, $b^a = b^{-1}$ for all $b \in W$. Let $c \in C_G(\langle W\rangle) \setminus \langle W\rangle$. Then $c^2 = 1$, $(cb)^2 = 1$ and $cb \notin \langle W\rangle$. But $(cb)^2 = c^2b^2 = 1$ and $b^2 = 1$, which is impossible. Therefore, $C_G(\langle W\rangle) = \langle W\rangle$ and $H = \langle W\rangle$ is a subgroup of index 2. This implies that $G = H \rtimes \langle a\rangle$ and $h^a = h^{-1}$ for all $h \in H$.

Now suppose that in W there exist elements a, b such that $[a,b] \neq 1$. Since $a^2 \neq 1$ and $b^2 \neq 1$, by (I) we have $a^2 = b^2$ and $b^a = b^{-1}$. Then $b^2 = ab^2a^{-1} = b^{-2}$ and the elements a, b are of order 4. Clearly, the subgroup $\langle a,b\rangle$ is a quaternion group of order 8. Let $c \in C_G(\langle a,b\rangle)$. If $c^2 \neq 1$ and $(ac)^2 \neq 1$ then (I) implies that $(ac)^b = (ac)^{-1}$ and $c^2 = 1$, which is impossible. Therefore, if $c \in C_G(\langle a,b\rangle)$ then either $c^2 = 1$ or $c^2 = a^2$.

Let $Q = \langle a,b\rangle$ be a quaternion subgroup of order 8 of G. Then we will prove that $G = Q \cdot C_G(Q)$. Suppose $g \in G \setminus C_G(Q)$. Pick the elements $a, b \in Q$ of order 4 such that $a^g = a^{-1}$ and $b^g = b^{-1}$. Then $(ab)^g = ab$ and $d = gab \in C_G(Q)$. It follows that $g = d(ab)^{-1}$ and $G = Q \cdot C_G(Q)$. Similary as in [3] we obtain that G satisfies the conditions 4) or 5) of the Theorem. $\square$

3. Proof of Theorem

Necessity. Let $K_\lambda G$ be f-normal. Then by Lemma 2 G satisfies one of the conditions 1)–5) of the Theorem.

First, suppose that G is abelian of exponent greater than 2 and $a, b \in G$. If $b^2 \neq 1$ then by (8) we have $\lambda_{a,b} = \lambda_{b,a}$.

Let a, b be elements of order two and assume that there exists c with $c^2 = a$. Then by (1) we have

$$\lambda_{c^2,b}\lambda_{c,c} = \lambda_{c,cb}\lambda_{c,b} \quad \text{and} \quad \lambda_{b,c^2}\lambda_{c,c} = \lambda_{bc,c}\lambda_{b,c}. \tag{10}$$

Since $c^2 \neq 1$, we have $\lambda_{c,cb} = \lambda_{bc,c}$ and $\lambda_{c,b} = \lambda_{b,c}$. Then (10) implies $\lambda_{c^2,b} = \lambda_{b,c^2}$ and $\lambda_{a,b} = \lambda_{b,a}$.

Let $a^2 = b^2 = 1$ such that neither a nor b is the square of any element of G. Then there exists c such that $(ca)^2 \neq 1$. Thus,

$$\lambda_{ca,b}\lambda_{c,a} = \lambda_{c,ab}\lambda_{a,b}, \quad \lambda_{b,ac}\lambda_{a,c} = \lambda_{ba,c}\lambda_{b,a}. \tag{11}$$

Since $\lambda_{b,ac} = \lambda_{ac,b}$ and $\lambda_{c,a} = \lambda_{a,c}$ from (11) we have $\lambda_{a,b} = \lambda_{b,a}$ for all $a, b \in G$. Therefore, if G is abelian and $G^2 \neq 1$ then the factor system is symmetric and $K_\lambda G$ is commutative.

Now, let $\exp(G) = 2$. Then by (8) $\lambda_{a,b} + f(b)\lambda_{b,b}^{-1}\lambda_{a,b} = \lambda_{b,a} + f(b)\lambda_{b,b}^{-1}\lambda_{b,a}$ for all $a, b \in G$. Therefore, $(\lambda_{a,b} - \lambda_{b,a})(1 + f(b)\lambda_{b,b}^{-1}) = 0$.

Next, let $G = H \rtimes C_2$ be a semidirect product of an abelian group H with $\exp(H) \neq 2$ and $C_2 = \langle a \mid a^2 = 1\rangle$, and with $h^a = h^{-1}$ for all $h \in H$. Clearly, $K_\lambda H$ is f-normal and the factor system of H is symmetric. Put $x = u_h + u_a$ for $h \in H$. Since $K_\lambda G$ is f-normal, we have $S_f(x) = xx^f - x^f x = 0$ and

$$\begin{aligned} &f(a)\lambda_{a,a}^{-1}\lambda_{h,a}u_{ha} + f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}}u_{ah^{-1}} \\ &\quad -f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a}u_{h^{-1}a} - f(a)\lambda_{a,a}^{-1}\lambda_{a,h}u_{ah} = 0. \end{aligned} \tag{12}$$

We will prove $u_a u_h = u_h^f u_a$ for every $h \in H$.

First, let $h^2 \neq 1$. Because $h^a = h^{-1}$, by (12) we have

$$u_a^f u_h + u_h^f u_a = 0 \tag{13}$$

and

$$\text{(14)} \qquad \begin{cases} f(a)\lambda_{a,a}^{-1}\lambda_{a,h} + f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a} = 0; \\ f(a)\lambda_{a,a}^{-1}\lambda_{h,a} + f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}} = 0. \end{cases}$$

Now, let $h^2 = 1$. Then there exists $b \in H$ with $b^2 \neq 1$ and $(hb)^2 \neq 1$. Put $x = u_a + u_h u_b$. Because $(hb)^a = (hb)^{-1}$ and $S_f(x) = xx^f - x^f x = 0$ we have

$$\text{(15)} \qquad u_a^f u_h u_b + (u_h u_b)^f u_a = 0.$$

Since $[u_h, u_b] = 1$, by (15) and (13) we have $u_a^f(u_h u_b) = u_a^f u_b u_h = -u_b^f u_a u_h$ and $u_a^f(u_h u_b) = -(u_h u_b)^f u_a = -u_b^f u_h^f u_a$. Therefore, $u_a u_h = u_h^f u_a$ for all $h \in H$ and this implies

$$\begin{cases} \lambda_{a,h} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a}; \\ \lambda_{h,a} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}}, \end{cases}$$

and, by (14), $f(a) = -\lambda_{a,a}$.

Let G be a hamiltonian 2-group. It is well known (see [5], Theorem 12.5.4) that $G = Q_8 \times W$, where Q_8 is a quaternion group and $\exp(W)|2$. If $a, b \in G$ are noncommuting elements of order 4, then $a^b = a^{-1}$ and by (8) we have 4.i) of the theorem. If $c, d \in G$ are involutions, then c and d commute with all $a \in G$ of order 4. Then $H = \langle a, d, c\rangle$ is abelian of exponent greater than 2 and $K_\lambda H$ is f-normal. By the condition 1) of the theorem, the factor system of H is symmetric, and u_a and u_b commute with u_c.

Now prove $f(c) = \lambda_{c,c}$ for all involutions $c \in G$. Choose the elements a, b of order 4 such that $b^a = b^{-1}$. Put $x = u_c u_a + u_b$. Since $\lambda_{a,c} = \lambda_{c,a}$ and $\lambda_{b,c} = \lambda_{c,b}$ by (2), for x we obtain

$$\begin{aligned} S_f(x) = (&f(b)u_a u_b^{-1} + f(a)f(c)\lambda_{c,c}^{-1}u_b u_a^{-1} \\ &- f(b)u_b^{-1}u_a - f(a)f(c)\lambda_{c,c}^{-1}u_a^{-1}u_b)u_c = 0 \end{aligned}$$

and $f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}} = f(c)f(a)\lambda_{c,c}^{-1}\lambda_{a,a^{-1}}^{-1}\lambda_{a^{-1},b}$. From this property and (9) we deduce $f(c) = \lambda_{c,c}$.

Now, suppose that either $G = E \times W$ or $G = (E \curlyvee C_4) \times W$, where E is an extraspecial 2-group, $\exp(W)|2$ and $E \curlyvee C_4$ is the central product of E and $C_4 = \langle c\rangle$ with $E' = \langle c^2\rangle$.

Let a be an element of order 4 and $h \in C_G(\langle a\rangle)$. Then by the condition 1) of the theorem $\lambda_{a,h} = \lambda_{h,a}$.

Let $\langle a, b \mid a, b \in G\rangle$ be the quaternion subgroup of order 8. Then by 4) we obtain (5).

Now, let $G = \langle a, b\rangle \mathrm{Y} \langle d \mid d^4 = 1\rangle$ be a subgroup of G and $d^2 = a^2$. Then $a^b = a^{-1}$, and $\langle a, d\rangle$ and $\langle b, d\rangle$ are abelian subgroups of exponent not equal to 2 and by the condition 1) of the theorem, $\lambda_{a,d} = \lambda_{d,a}$ and $\lambda_{b,d} = \lambda_{d,b}$. Put $x = u_b + u_a u_d$. Since $K_\lambda G$ is f-normal, we obtain

$$\begin{aligned}
&f(b)\lambda^{-1}_{b,b^{-1}}\lambda_{a,b^{-1}}u_{ab^{-1}}u_d + f(d)f(a)\lambda^{-1}_{a,a^{-1}}\lambda^{-1}_{d,d^{-1}}\lambda_{b,a^{-1}}u_{ba^{-1}}u_{d^{-1}}\\
&= f(d)f(a)\lambda^{-1}_{a,a^{-1}}\lambda^{-1}_{d,d^{-1}}\lambda_{a^{-1},b}u_{a^{-1}b}u_{d^{-1}} + f(b)\lambda^{-1}_{b,b^{-1}}\lambda_{b^{-1},a}u_{b^{-1}a}u_d
\end{aligned}$$

and by (5)

$$\begin{aligned}
&\lambda_{b,a}\lambda_{ab^{-1},d}u_{ab^{-1}d} + f(d)\lambda^{-1}_{d,d^{-1}}\lambda_{a,b}\lambda_{ba^{-1},d^{-1}}u_{ba^{-1}d^{-1}}\\
&= f(d)\lambda^{-1}_{d,d^{-1}}\lambda_{b,a}\lambda_{a^{-1}b,d^{-1}}u_{a^{-1}bd^{-1}} + \lambda_{a,b}\lambda_{b^{-1}a,d}u_{b^{-1}ad}.
\end{aligned}$$

Since $d^2 \in G'$ and $a^2 = b^2$, we have $a^{-1}bd^{-1} = abd$, $ab^{-1}d = ba^{-1}d^{-1}$ and

$$\lambda_{b,a}\lambda_{ba,d} + f(d)\lambda^{-1}_{d,d^{-1}}\lambda_{a,b}\lambda_{ab,d^{-1}} = 0.$$

Therefore, we proved 6.i).

If $\langle a, b \mid a^4 = b^2 = 1\rangle$ is the dihedral subgroup of order 8 of G, then by 3) of the theorem we have (4) and $f(b) = -\lambda_{b,b}$.

Let $L = D_8 \mathrm{Y} C_4 = \langle a, b \mid a^4 = b^2 = 1\rangle \mathrm{Y} \langle c\rangle$. Then any $x \in K_\lambda L$ can be written as $x = x_0 + x_1 u_c$, where $x_0, x_1 \in K_\lambda D_8$. Since $K_\lambda G$ is f-normal, $K_\lambda L$ is f-normal, too, and $(x_0^f x_1 - x_1 x_0^f)u_c = (x_0 x_1^f - x_1^f x_0)u_c^f$. By the f-normality of $K_\lambda D_8$ $(x_0 + x_1)(x_0 + x_1)^f = (x_0 + x_1)^f(x_0 + x_1)$ and we have

$$(x_0^f x_1 - x_1 x_0^f)u_c - (x_0 x_1^f - x_1^f x_0)u_c^f = (x_0^f x_1 - x_1 x_0^f)(u_c - u_c^f).$$

If $x_0^f x_1 - x_1 x_0^f$ can be written as a sum of elements of form $u_a^f u_b - u_b u_a^f$ then

$$\begin{aligned}
(x_0^f x_1 - x_1 x_0^f)(u_c - u_c^f) &= (\lambda_{b,a}\lambda_{ba,c} + f(c)\lambda^{-1}_{c,c^{-1}}\lambda_{a,b}\lambda_{ab,c^{-1}})u_{bac}\\
&\quad -(\lambda_{a,b}\lambda_{ab,c} + f(c)\lambda^{-1}_{c,c^{-1}}\lambda_{b,a}\lambda_{ba,c^{-1}})u_{abc} = 0
\end{aligned}$$

and we have (6).

Sufficiency. We wish to prove that $S_f(x) = xx^f - x^f x$ is equal to 0 for all $x \in KG$. Let $x = \sum_{g\in G} \alpha_g u_g \in K_\lambda G$. It is easy to see that $S_f(x)$ is a sum of elements of the form

$$S_f(g,h) = \alpha_g\alpha_h(f(h)\lambda^{-1}_{h,h^{-1}}\lambda_{g,h^{-1}}u_{gh^{-1}} + f(g)\lambda^{-1}_{g,g^{-1}}\lambda_{h,g^{-1}}u_{hg^{-1}} - f(h)\lambda^{-1}_{h,h^{-1}}\lambda_{h^{-1},g}u_{h^{-1}g} - f(g)\lambda^{-1}_{g,g^{-1}}\lambda_{g^{-1},h}u_{g^{-1}h}).$$

First, let G be abelian of exponent greater than 2, and assume that the factor system of G is symmetric. Then $K_\lambda G$ is commutative, and therefore, f-normal.

Next, suppose that G is of exponent 2 and the factor system satisfies $(\lambda_{g,h} - \lambda_{h,g})(1 + f(h)\lambda^{-1}_{h,h}) = 0$ for all $g,h \in G$.
This implies $(\lambda_{g,h} - \lambda_{b,h})(f(g)\lambda^{-1}_{g,g} - f(h)\lambda^{-1}_{h,h}) = 0$ for all $g,h \in G$. Then

$$S_f(g,h) = \alpha_g\alpha_h(f(h)\lambda^{-1}_{h,h}\lambda_{g,h}u_{gh} + f(g)\lambda^{-1}_{g,g}\lambda_{h,g}u_{hg} - f(h)\lambda^{-1}_{h,h}\lambda_{h,g}u_{hg} - f(g)\lambda^{-1}_{g,g}\lambda_{g,h}u_{gh}) = \alpha_g\alpha_h(f(h)\lambda^{-1}_{h,h} - f(g)\lambda^{-1}_{g,g})(\lambda_{g,h} - \lambda_{h,g})u_{gh} = 0$$

and $S_f(x) = 0$, thus, $K_\lambda G$ is f-normal.

Now, let $G = H \rtimes C_2$, where H is an abelian group of exponent not equal to 2 and $C_2 = \langle a\rangle$ with $h^a = h^{-1}$ for all $h \in H$. Using the properties of the factor system we obtain

$$f(a)u_a^{-1}u_h = -f(h)u_h^{-1}u_a, \qquad f(a)u_hu_a^{-1} = -f(h)u_au_h^{-1}, \qquad u_a^f y = -y^f u_a, \qquad yu_a^f = -u_a y^f \tag{16}$$

for any $h \in H$ and $y \in K_\lambda H$. If $x = x_1 + x_2u_a \in K_\lambda G$ where $x_1, x_2 \in K_\lambda H$, then $x^f = x_1^f + f(a)u_a^{-1}x_2^f$ and

$$xx^f = x_1x_1^f + f(a)x_1u_a^{-1}x_2^f + x_2u_ax_1^f + f(a)x_2x_2^f.$$

Because in $K_\lambda H$ the factor system is symmetric and $K_\lambda H$ is commutative, by (16) we have

$$xx^f = x_1x_1^f + (x_2x_1 - x_1x_2)u_a + f(a)x_2x_2^f = x_1x_1^f + f(a)x_2x_2^f.$$

Similarly, $x^f x = x_1^f x_1 + f(a)x_2^f x_2$ and we conclude that $S_f(x) = 0$ and $K_\lambda G$ is f-normal.

Next, let G be a hamiltonian 2-group. Then $G = Q_8 \times W$, where $Q_8 = \langle a, b\rangle$ is a quaternion group and $\exp(W)|2$. Suppose that the conditions 4.i)–4.ii) of the theorem are satisfied. If $H = \langle a^2, W\rangle$ then any element $x \in K_\lambda G$ can be written as

$$x = x_0 + x_1u_a + x_2u_b + x_3u_{ab},$$

where $x_i \in K_\lambda H$, $(i = 0, \dots, 3)$. Since $\langle a\rangle \times H$ and $\langle b\rangle \times H$ are abelian groups of exponent 4, by the condition 1) of the theorem the elements x_0, x_1, x_2, x_3 commute with u_a, u_b and u_{ab}. Since $K_\lambda H$ is f-normal, we have $x_ix_j^f - x_i^fx_j = x_j^fx_i - x_jx_i^f$. Using these properties we obtain

$$\begin{aligned} S_f(x) = {} & (x_1x_2^f - x_1^fx_2)(\lambda_{b,a}u_{ba} - \lambda_{a,b}u_{ab}) \\ & + (x_1x_3^f - x_1^fx_3)(\lambda_{ab,a}u_b - \lambda_{a,ab}u_{b^3}) \\ & + (x_2x_3^f - x_2^fx_3)(\lambda_{ab,b}u_{a^3} - \lambda_{b,ab}u_a). \end{aligned}$$

Clearly, the element $x_ix_j^f - x_i^fx_j$ can be written as a sum of elements of form

$$S_f(c, d) = \gamma_{c,d}(f(d)u_cu_d^{-1} - f(c)u_c^{-1}u_d),$$

where $c, d \in H$. Since H is an elementary 2-subgroup, by the condition 4.ii) $f(d) = \lambda_{d,d}$, $f(c) = \lambda_{c,c}$, and we obtain

$$S_f(c, d) = \gamma_{c,d}(f(d)\lambda_{d,d}^{-1}\lambda_{c,d}u_{cd} - f(c)\lambda_{c,c}^{-1}\lambda_{c,d}u_{cd}) = 0.$$

Therefore, $S_f(x) = 0$ and $K_\lambda G$ is f-normal.

Next, let $G = H \times W$, where H is an extraspecial 2-group and $\exp(W)|2$. Since G is a locally finite group, it suffices to establish the f-normality of all finite subgroups H of G. Let G be a finite group and $G = H \times W$, where H is a finite extraspecial 2-group and $\exp(W)|2$. We know (see [4], Theorem III.13.8) that H is a central product of n copies of dihedral groups of order 8 or a central product of a quaternion group of order 8 and $n-1$ copies of dihedral groups of order 8. We can write $H_n = H$. Then $G = H_n \times W$ and by induction on n we prove the f-normality of $K_\lambda G$.

If $n = 1$ then either $H_1 = Q_8$ or $H_1 = D_8$ or $H_1 = Q_8 \curlyvee C_4$. In the first and second cases the f-normality $K_\lambda G$ is implied by the conditions 3) or 4) of the theorem.

Let $G = Q_8 \curlyvee C_4$. Then any element $x \in K_\lambda G$ can be written as $x = x_0 + x_1 u_c$, where $x_i \in K_\lambda Q_8$, $c \in C_4$ and $c^2 \in Q_8$. From the f-normality of $K_\lambda Q_8$ we obtain $x_0^f x_1 - x_1 x_0^f = x_1^f x_0 - x_0 x_1^f$ and $S_f(x) = (x_0^f x_1 - x_1 x_0^f)(u_c - u_c^f)$. The element $x_0^f x_1 - x_1 x_0^f$ can be written as a sum of elements of form $\alpha(u_a^f u_b - u_b u_a^f)$, where $\alpha \in K$, $a, b \in Q_8$. We will prove $S_f(a,b) = (u_a^f u_b - u_b u_a^f)(u_c - u_c^f) = 0$ for all $a, b \in Q_8$.

If $a, b \in Q_8$ does not generate Q_8 then $u_a u_b = u_b u_a$ and $S_f(a,b) = 0$. Let $\langle a, b\rangle = Q_8$. Then by (5)

$$\begin{aligned} S_f(a,b) &= (\lambda_{b,a} u_{ba} - \lambda_{a,b} u_{ab})(u_c - u_c^f) \\ &= (\lambda_{b,a}\lambda_{ba,c} + f(c)\lambda_{c,c^{-1}}^{-1}\lambda_{a,b}\lambda_{ab,c^{-1}})u_{bac} \\ &\quad + (\lambda_{a,b}\lambda_{ab,c} + f(c)\lambda_{c,c^{-1}}^{-1}\lambda_{b,a}\lambda_{ba,c^{-1}})u_{abc} \end{aligned}$$

and from (6) $S_f(a,b) = 0$.

It is easy to see $D_8 \curlyvee D_8 \cong Q_8 \curlyvee Q_8$, and H_n $(n > 1)$ can be written as $Q_8 \curlyvee H_{n-1}$.

Let $Q_8 = \langle a, b\rangle$ and $L = W \times H_{n-1}$. Then any element $x \in K_\lambda G$ can be written as

$$x = x_0 + x_1 u_a + x_2 u_b + x_3 u_a u_b,$$

where $x_i \in K_\lambda L$. By 6.i) the x_i commute with u_a and u_b. Since $\langle a, b\rangle$ is a quaternion group of order 8, by the condition 6.ii) of the theorem we have $u_a u_b = u_b^f u_a = u_b u_a^f$. Hence,

$$\begin{aligned} S_f(x) = (x_0x_1^f - x_1^f x_0)u_a^f &+ (x_0x_2^f - x_2^f x_0)u_b^f + (x_0x_3^f - x_3^f x_0)u_b^f u_a^f \\ &+ (x_1x_0^f - x_0^f x_1)u_a + (x_1x_2^f - x_1^f x_2)u_a u_b^f + (x_1x_3^f - x_1^f x_3)u_b f(a) \\ &+ (x_2x_0^f - x_0^f x_2)u_b + (x_2x_1^f - x_2^f x_1)u_a u_b + (x_2x_3^f - x_2^f x_3)u_a^f f(b) \\ &+ (x_3x_0^f - x_0^f x_3)u_a u_b + (x_3x_1^f - x_3^f x_1)u_a u_{ab} \\ &+ (x_3x_2^f - x_3^f x_2)u_a f(b). \end{aligned} \tag{17}$$

Since by induction $K_\lambda L$ is f-normal, $(x_i + x_j)(x_i + x_j)^f = (x_i + x_j)^f(x_i + x_j)$ implies $x_i x_j^f - x_i^f x_j = x_j^f x_i - x_j x_i^f$ and $x_i x_j^f - x_j^f x_i =$

$x_i^f x_j - x_j x_i^f$. Therefore, by (17)

$$
\begin{aligned}
S_f(x) = {} & (x_0 x_1^f - x_1^f x_0)(u_a^f - u_a) + (x_0 x_2^f - x_2^f x_0)(u_b^f - u_b) \\
& + (x_0 x_3^f - x_3^f x_0)(u_a^f - u_a)u_b + (x_1 x_2^f - x_1^f x_2)u_a(u_b^f - u_b) \\
& + (x_1 x_3^f - x_1^f x_3)u_a(u_b - u_b^f)f(a) + (x_2 x_3^f - x_2^f x_3)(u_a^f - u_a)f(b).
\end{aligned}
$$

Clearly, the element $x_i x_j^f - x_j^f x_i$ can be written as a sum of elements of form $S_f(c,d) = \gamma_{c,d}(u_c u_d^f - u_d^f u_c)$, where $c, d \in L$, $\gamma_{c,d} \in K$. We will prove $S_f(c,d,a) = (u_c u_d^f - u_d^f u_c)(u_a - u_a^f) = 0$ for any $c, d \in L$.

We consider the following cases:

Case 1). Let $[c,d] = 1$. Then $L = \langle c, d, a\rangle$ is abelian with $\exp(L) \neq 2$, and by 6.i) the factor system is symmetric and $S_f(c,d,a) = 0$.

Case 2). Let $\langle c, d\rangle = Q_8$. Then by 6.ii) (5) holds and $(u_c u_d^f - u_d^f u_c)(u_a - u_a^f) = (\lambda_{d,c}\lambda_{dc,a} + f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{c,d}\lambda_{cd,a^{-1}})u_{dca} - (\lambda_{c,d}\lambda_{cd,a} + f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{d,c}\lambda_{a^{-1},dc})u_{cda}$. Now by 6.ii) the property (6) is satisfied and we conclude $S_f(c,d,a) = 0$.

Case 3). Let $\langle c, d\rangle = D_8$ and $c^4 = d^2 = 1$. Then by 6.iii) $f(d) = -\lambda_{d,d}$ and by (4)

$$
\begin{aligned}
(u_c u_d^f - u_d^f u_c)(u_a - u_a^f) &= (\lambda_{d,c}u_{dc} - \lambda_{c,d}u_{cd})(u_a - u_a^f) \\
&= (\lambda_{c,d}\lambda_{cd,a} + f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{dc,a^{-1}}\lambda_{d,c})u_{cda} \\
&\quad + (\lambda_{d,c}\lambda_{dc,a} + f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{cd,a^{-1}}\lambda_{c,d})u_{dca}.
\end{aligned}
$$

Now by 6.ii) we have (6) and we conclude $S_f(c,d,a) = 0$.

Case 4). Let $\langle c, d\rangle = D_8$ and $d^4 = c^2 = 1$. Then by (4)

$$
\begin{aligned}
u_c u_d^f - u_d^f u_c &= f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{c,d^{-1}}u_{dc} - f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{d^{-1},c}u_{cd} \\
&= \lambda_{d,c}u_{dc} - \lambda_{c,d}u_{cd}.
\end{aligned}
$$

Similarly to the case 3) we have $S_f(c,d,a) = 0$.

Case 5). Let $\langle c, d\rangle = D_8$ and $d^2 = c^2 = 1$. Then by 6.iii) $f(d) = -\lambda_{d,d}$. In $\langle c, d\rangle$ we choose a new generator system $\{a_1, b_1 \mid a_1^4 = b_1^2 =$

$1, a_1^{b_1} = a_1^{-1}\}$ such that $c = b_1$ and $d = a_1^i b_1$, where $i = 1$ or 3. Then $a^2 = a_1^2$ and

$$\begin{aligned}(u_c u_d^f - u_d^f u_c)(u_a - u_a^f) &= (u_d u_c - u_c u_d)(u_a - u_a^f)\\ &= \lambda^{-1}_{a_1^i, b_1}(u_{a_1^i} u_{b_1} - u_{b_1} u_{a_1^i})(u_a - u_a^f) u_{b_1}.\end{aligned}$$

As in the Case 3) it is easy to see $(u_{a_1^i} u_{b_1} - u_{b_1} u_{a_1^i})(u_a - u_a^f) = 0$ and $S_f(c, d, a) = 0$.

Analogously, the element $x_i x_j^f - x_i^f x_j$ can be written as a sum of elements of form $\gamma_{c,d}(u_c u_d^f - u_c^f u_d)$, where $c, d \in L$. Let us prove that if $c, d \in L$, then $S_f(c, d, a) = (u_c u_d^f - u_c^f u_d)(u_a - u_a^f) = 0$.

Let $z \in L$, $a \in Q_8$ be commuting elements of order 4 with $z^2 = a^2$. First, we will prove that K is of characteristic 2, then $(u_z + u_z^f)(u_a + u_a^f) = 0$.

Indeed,

$$\begin{aligned}(u_z + u_z^f)(u_a + u_a^f) = (\lambda_{z,a} + f(z)\lambda^{-1}_{z,z^{-1}} f(a)\lambda^{-1}_{a,a^{-1}}\lambda_{z^{-1},a^{-1}})u_{za}\\ +(f(a)\lambda^{-1}_{a,a^{-1}}\lambda_{z,a^{-1}} + f(z)\lambda^{-1}_{z,z^{-1}}\lambda_{z^{-1},a})u_{za^3}.\end{aligned}$$

First let za be a noncentral element of order 2. Then by 6.iii) $f(za) = \lambda_{za,za}$. Since $((u_z u_a)u_a)u_{a^3} = u_z(u_a(u_a u_{a^3}))$ we conclude that

$$\lambda_{z,a}\lambda_{za,a}\lambda_{za^2,a^3} = \lambda_{z,a}\lambda_{a,1}\lambda_{a,a^{-1}}$$

and $\lambda^{-1}_{a,a^{-1}} = \lambda^{-1}_{z^3,a^3}\lambda^{-1}_{za,a}$. Clearly, $f(z)f(a) = f(za)\lambda^2_{z,a} = \lambda_{za,za}\lambda^2_{z,a}$ and

$$\begin{aligned}&\lambda_{z,a} + f(z)\lambda^{-1}_{z,z^{-1}} f(a)\lambda^{-1}_{a,a^{-1}}\lambda_{z^{-1},a^{-1}}\\ &\qquad= \lambda_{z,a}(1 + (\lambda_{za,az}\lambda_{a,z})\lambda^{-1}_{z,z^{-1}}\lambda^{-1}_{a,a^{-1}}\lambda_{z^{-1},a^{-1}})\\ &\qquad= \lambda_{z,a}(1 + \lambda_{z,za^2}\lambda_{a,az}\lambda^{-1}_{a,a^{-1}}\lambda^{-1}_{z,za^2}\lambda_{z^{-1},a^{-1}}\\ &\qquad= \lambda_{z,a}(1 + (\lambda_{a,az}\lambda_{zaa,a^{-1}})\lambda^{-1}_{a,a^{-1}})\\ &\qquad= \lambda_{z,a}(1 + \lambda_{za,aa^{-1}}\lambda_{a,a^{-1}}\lambda^{-1}_{a,a^{-1}}) = 2\lambda_{z,a} = 0.\end{aligned} \tag{18}$$

By (1) we have

$$\begin{aligned}(\lambda_{z,a^{-1}}\lambda_{za^{-1},za^{-1}})\lambda_{z^{-1},a} = \lambda_{z,a^{-1}za^{-1}}\lambda_{a^{-1},z^{-1}a}\lambda_{z^{-1},a}\\ = \lambda_{z,z^{-1}}(\lambda_{a^{-1},az^{-1}}\lambda_{a,z^{-1}}) = \lambda_{z^{-1},z}\lambda_{aa^{-1},z^{-1}}\lambda_{a,a^{-1}} = \lambda_{z^{-1},z}\lambda_{a,a^{-1}}\end{aligned}$$

and since az^{-1} has order 2, $f(az^{-1})=\lambda_{az^{-1},az^{-1}}$, and we obtain

$$\begin{aligned}
&f(a^{-1})^{-1}f(a^{-1})(f(a)\lambda^{-1}_{a,a^{-1}}\lambda_{z,a^{-1}}+f(z)\lambda^{-1}_{z,z^{-1}}\lambda_{z^{-1},a})\\
&\quad=f(a^{-1})^{-1}(\lambda^{2}_{a,a^{-1}}\lambda^{-1}_{a,a^{-1}}\lambda_{z,a^{-1}}+f(az^{-1})\lambda^{2}_{a^{-1},z}\lambda^{-1}_{z,z^{-1}}\lambda_{z^{-1},a})\\
&\quad=f(a^{-1})^{-1}(\lambda_{a,a^{-1}}\lambda_{z,a^{-1}}+\lambda_{z,a^{-1}}(\lambda_{z,a^{-1}}\lambda_{az^{-1},az^{-1}}\lambda_{z^{-1},a})\lambda^{-1}_{z,z^{-1}})\\
&\quad=f(a^{-1})^{-1}(\lambda_{z,a^{-1}}(\lambda_{a^{-1},a}-\lambda_{z,z^{-1}}\lambda_{a^{-1},a}\lambda^{-1}_{z^{-1},z}\lambda_{z^{-1},a})\\
&\quad=2f(a^{-1})^{-1}\lambda_{z,a^{-1}}\lambda_{a,a^{-1}}=0.
\end{aligned}\tag{19}$$

Clearly, if $[c,d]=1$ then $S_f(c,d,a)$ can be written as

$$\begin{aligned}
S_f(c,d,a)&=(u_cu^f_d+(u^f_du_c)^f)(u_a-u^f_a)\\
&=f(d)\lambda_{d,d^{-1}}\lambda_{c,d^{-1}}(u_{cd^{-1}}-u^f_{cd^{-1}})(u_a-u^f_a).
\end{aligned}\tag{20}$$

Similarly, the element $x_ix^f_j-x^f_ix_j$ can be written as a sum of elements of form $\gamma_{c,d}(u_cu^f_d-u^f_cu_d)$, where $c,d\in L$. Now let us prove $S_f(c,d,a)=(u_cu^f_d-u^f_cu_d)(u_a-u^f_a)=0$, where $c,d\in L$.

We consider the following cases:

Case 1). Let $[c,d]=1$, $c^2=d^2=1$ and $c,d\notin\zeta(G)$. Then $S=\langle c,d,a\rangle$ is abelian of exponent greater that 2 and by 6.i) the factor system of S is symmetric. We know that in L every element of order 2 is either central or coincides with a noncentral element of some dihedral subgroup of order 8. Since $c,d\notin\zeta(G)$, we have $f(c)=\lambda_{c,c}$ and $f(d)=\lambda_{d,d}$ and

$$S_f(c,d,a)=\lambda_{c,d}(f(d)\lambda^{-1}_{d,d}-f(c)\lambda^{-1}_{c,c})u_{cd}(u_a-u^f_a)=0.$$

Case 2). Let $[c,d]=1$, $c^2=d^2=1$ and $c,d\in\zeta(G)$. Then $c=d=a^2$ and $S_f(c,d,a)=0$.

Case 3). Let $[c,d]=1$, $c^2=d^2=1$ and $c\in\zeta(G)$, $d\notin\zeta(G)$. Then $f(d)=\lambda^{-1}_{d,d}$, $c=a^2$ and

$$\begin{aligned}
S_f(c,d,a)&=-u_d(u_{a^2}+u^f_{a^2})(u_a-u^f_a)\\
&=-u_d(\lambda_{a,a^2}u_{a^{-1}}-f(a)\lambda^{-1}_{a,a^{-1}}u_a)(1+f(a^2)\lambda^{-1}_{a^2,a^2}).
\end{aligned}$$

Since K is an integral domain of characteristic 2 and $f^2(a^2)=\lambda^2_{a^2,a^2}f(a^4)=\lambda^2_{a^2,a^2}$, we conclude $f(a^2)=\pm\lambda_{a^2,a^2}$ and $S_f(c,d,a)=0$.

Case 4). Let $[c,d]=1$, $d^2=1$ and suppose that c has order 4. Then dc has order 4 and by (20) $S_f(c,d,a)=0$.

Case 5). Let $[c,d]=1$ with c,d of order 4. Then $d^2=c^2=a^2$,

$$S_f(c,d,a)=(f(d)\lambda^{-1}_{d,d^{-1}}\lambda_{c,d^{-1}}+f(c)\lambda^{-1}_{c,c^{-1}}\lambda_{c^{-1},d})u_{cd^{-1}}(u_a-u^f_a),$$

and by (19) we have $S_f(c,d,a)=0$.

Case 6). Let $\langle c,d\rangle$ be a quaternion group of order 8. Then by 6.ii) (5) holds and

$$\begin{aligned}u_cu^f_d-u^f_cu_d&=(f(d)\lambda^{-1}_{d,d^{-1}}\lambda_{c,d^{-1}}-f(c)\lambda^{-1}_{c,c^{-1}}\lambda_{c^{-1},d})u_{c^{-1}d}\\&=(\lambda_{d,c}-\lambda_{d,c})u_{c^{-1}d}=0.\end{aligned}$$

Case 7). Let $\langle c,d\rangle$ be a dihedral group of order 8. If $c^2\neq 1$ then $f(d)=\lambda_{d,d}$ and

$$\begin{aligned}S_f(c,d,a)&=(\lambda_{c,d}u_{cd}+f(c)\lambda^{-1}_{c,c^{-1}}\lambda_{c^{-1},d}u_{dc})(u_a-u^f_a)\\&=(\lambda_{c,d}u_{cd}+\lambda_{d,c}u_{dc})(u_a-u^f_a)=(\lambda_{c,d}\lambda_{cd,a}+f(a)\lambda_{a,a^{-1}}\lambda_{d,c}\lambda_{dc})u_{acd}\\&\quad-(\lambda_{d,c}\lambda_{dc,a}+f(a)\lambda_{a,a^{-1}}\lambda_{c,d}\lambda_{cd,a^{-1}})u_{adc}.\end{aligned}$$

By (6) we obtain $S_f(c,d,a)=0$.

Case 8). Let $\langle c,d\rangle$ be a dihedral group of order 8 and $c^2=d^2=1$. Then $f(d)=\lambda_{d,d}$, $f(c)=\lambda_{c,c}$ and $S_f(c,d,a)=2u_cu_d(u_a-u^f_a)=0$. □

References

[1] S. V. Mihovski and J. M. Dimitrova, Units, isomorphisms and automorphisms of crossed products of UP-groups, *Comm. in Algebra* **24** (7) (1996), 2473–2499.

[2] S. D. Berman, On the equation $x^m=1$ in an integral group ring, *Ukrain. Mat. Zh.* **7** (1955), 253–261.

[3] A. A. Bovdi, P. M. Gudivok and M. S. Semirot, Normal group rings, *Ukrain. Mat. Zh.* **37** (1985), 3–8.

[4] B. Huppert, Endliche Gruppen I, *Springer-Verlag, Berlin–Heidelberg–New York*, 1967, 410.

[5] M. Hall, Group theory, *The Macmillian Company, New York*, 1959, 468.

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