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Retrieving a topological space from its lattice of open sets

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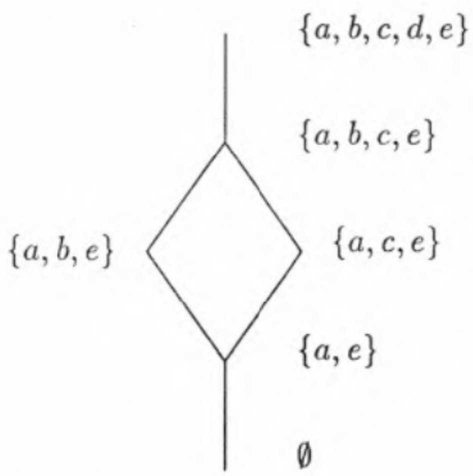
It can be readily verified that the subsets:

$$(1) \qquad \emptyset, \quad \{a, e\}, \quad \{a, b, e\}, \quad \{a, c, e\} \quad \{a, b, c, e\} \quad \{a, b, c, d, e\}$$

of the set $\{a, b, c, d, e\}$ define a topology T on that set.

If we view the topology T, given by (1), as a partial order with respects to the set theoretic inclusion $(\subseteq)$, we can represent T by the following diagram:

(2)



As expected, $(T, \subseteq)$ is a partially ordered set (*poset*, for short) which is:

(3) *a distributive lattice*

such that:

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(4) *the greatest lower bound* (glb) *of any two elements of* T *is their set theoretic intersection*

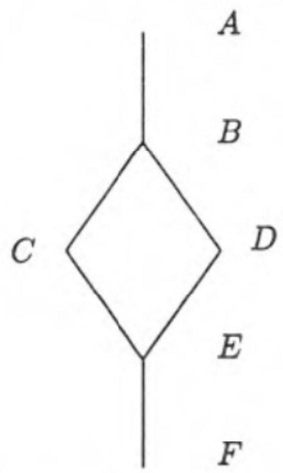
and

(5) *the least upper bound* (lub) *of any two elements of* T *is their set theoretic union.*

It is natural to call any finite partially ordered set which is isomorphic to a poset of sets satisfying (3), (4), (5) a *finite topology poset.* We shall refer to the latter simply as a *topology poset* when no confusion is likely to arise.

We may further simplify Diagram (2) by replacing the explicit appearance of sets in it with capital letters, obtaining, say:

(6)



We see that, by our above definition, Diagram (6) represents a topology poset.

Comparing Diagrams (2) and (6), we notice that by replacing $\{a, b, c, d, e\}$ with A and $\{a, b, c, e\}$ with B, etc., we have lost some information. Namely, what sets (explicitly given in terms of their elements) could $A, B, C, \ldots,$ represent? In this connection the following two questions naturally arise:

(i) When does a diagram of capital letters represent a topology poset?

(ii) Given a diagram of capital letters representing a topology poset, can we retrieve the open sets (by explicitly giving their elements) that these letters could represent?

Question (i), answered by Theorem 1 below, can also be dealt with by applying Theorem 3.1 of [4] to the finite case. We give below a direct and independent proof of Theorem 1.

Question (ii) is answered in the affirmative by us in Theorems 2 and 3 below.

Theorem 1. *A finite poset $(P, \leq)$ is a topology poset iff $(P, \leq)$ is a distributive lattice.*

PROOF. If $(P, \leq)$ is a topology poset, by (3) it is a distributive lattice.

To prove the converse, we first recall that an element x of a poset $(P, \leq)$ is called *join irreducible* [2, p.65] iff x is not the minimum of P and x is not the lub of two elements $< x$. Now let $J(P)$ be the set of all join irreducible elements of P. We will define a topology T on the set $J(P)$ and show that $(P, \leq)$ is isomorphic to $(T, \subseteq)$.

Consider the mapping f from P into the power set $\mathcal{P}(J(P))$ of $(J(P))$, where f is given by

$$f(x) = \{y \mid y \in J(P) \text{ and } y \leq x\}\,. \tag{7}$$

We show that $T = \{f(x) \mid x \in P\}$ is a topology on $J(P)$.

As P is a finite lattice it has a minimum 0 and a maximum 1. Clearly $f(0) = \emptyset$ and $f(1) = J(P)$. Thus,

$$\emptyset \in T \quad \text{and} \quad J(P) \in T\,. \tag{8}$$

To show that T is closed under intersections, let $f(x) \in T$ and $f(y) \in T$. We show

$$\big(f(x) \cap f(y)\big) = f(\mathrm{glb}\{x, y\})\,. \tag{9}$$

By the definition (7) of f we have

$$\begin{aligned} z \in \big(f(x) \cap f(y)\big) \quad &\text{iff} \;\; z \in J(P) \text{ and } \; z \leq x, \text{ and } z \leq y \\ &\text{iff} \;\; z \in J(P) \;\; \text{and} \;\; z \leq \mathrm{glb}\{x, y\}\,, \end{aligned}$$

the latter because P is a lattice. Hence $z \in \big(f(x) \cap f(y)\big)$ iff $z \in f(\mathrm{glb}\{x, y\})$, which establishes (9).

By a dual argument it is easily shown that

$$\big(f(x) \cup f(y)\big) = f(\mathrm{lub}\{x, y\}) \tag{10}$$

which shows that T is closed under unions.

From (8), (9), (10), it follows that T is a topology on the set $J(P)$.

We now show that f is an order-isomorphism from $(P, \leq)$ onto $(T, \subseteq)$.

By its definition f is onto. We claim that f is also $1-1$. To prove this we first show that $x = \mathrm{lub}\, f(x)$ for every $x \in P$. Assume not, and since P is finite, let $a \in P$ be minimal with respects to the property that $a \neq \mathrm{lub}\, f(a)$, which clearly implies that a cannot itself be join irreducible. Thus we must have $a = \mathrm{lub}\{b, c\}$ for some $b < a$ and $c < a$. By the minimality of a, we have $b = \mathrm{lub}\, f(b)$ and $c = \mathrm{lub}\, f(c)$. We therefore conclude that $a = \text{lub}\,(f(b) \cup f(c))$. But then as a is an upper bound of $f(a)$, and $(f(b) \cup f(c)) \subseteq f(a)$, we must have $a = \text{lub}\, f(a)$, which is a contradiction. Hence $x = \text{lub}\, f(x)$ for every $x \in P$, as claimed. It follows therefore that if $x \neq y$ then $\text{lub}\, f(x) \neq \text{lub}\, f(y)$. However, both lubs exist in P as P is a lattice. Consequently, $f(x) \neq f(y)$ and f is $1-1$.

We now show that both f and its inverse are order-preserving. If $x \leq y$ we have by (7) that $f(x) \subseteq f(y)$. On the other hand, if $f(x) \subseteq f(y)$, then as $x = \text{lub}\, f(x)$ and $y = \text{lub}\, f(y)$, we see that $x \leq y$.

It follows that f is an isomorphism from $(P, \leq)$ onto the lattice of open sets $(T, \subseteq)$, and therefore $(P, \leq)$ is a topology poset.

Thus Theorem 1 is proved.

Let us apply the procedure indicated by the proof of Theorem 1 to retrieve a space corresponding to the distributive lattice which is represented by Diagram (6), and which we denote by $(L, \leq)$. To this end we observe that the join irreducible elements of L are A, C, D, E, and therefore

$$J(L) = \{A, C, D, E\} \tag{11}$$

From (7) it follows that the open sets of a topology on $J(L)$ are given by $f(F) = \emptyset,\ f(E) = \{E\}, \ldots,\ f(A) = \{A, C, D, E\}$, i.e. by:

$$\emptyset,\ \{E\},\ \{D, E\},\ \{C, E\},\ \{C, D, E\},\ \{A, C, D, E\} \tag{12}$$

As expected in view of Theorem 1, we see that the lattice of the open sets in (12) is isomorphic to $(L, \leq)$, given by Diagram (6).

Remark 1. We observe that the lattices of the open sets of the topologies given by (1) and (12) are both isomorphic to (6), and hence to each other. However, the topologies (1) and (12) are not homeomorphic. First of all, the spaces $\{a, b, c, d, e\}$ and $\{A, C, D, E\}$ have different cardinalities. Secondly, the topology (12) is T_0, while (1) is not.

Since our method of retrieval applied to Diagram (6) yielded the T_0 topology given by (12), we may suspect that our method (based on (7)) will always yield a unique (up to homeomorphism) T_0 topology when applied to a given finite distributive lattice. This is indéed the case, as is shown by:

Theorem 2. *A finite poset $(P, \leq)$ is isomorphic to the lattice of open sets of a unique (up to homeomorphism) T_0 topological space iff P is a distributive lattice.*

PROOF. If P is isomorphic to the lattice of open sets of some topological space, then P is a topology poset, and hence by (3) we see that P is a distributive lattice.

To prove the converse, in view of Theorem 1, it remains to show that $(J(P), T)$ is a T_0 topological space. To this end let x and y be distinct elements of $J(P)$. Since x is join irreducible, $x \in f(x)$. Similarly, $y \in f(y)$.

If $x < y$ then by (7) we see that $y \notin f(x)$, and if $y < x$ then $x \notin f(y)$. Otherwise $x \notin f(y)$, and $y \notin f(x)$, as x and y are not comparable.

Hence in every case T has an open set containing one of x and y but not the other. Therefore T is a T_0 topology on the set $J(P)$. The uniqueness of T follows from Theorem 2.1 of [4] and Corollary 5.1 of [1].

Thus, Theorem 2 is proved.

Remark 2. If $(P, \leq)$ is a finite distributive lattice then P cannot be isomorphic to the lattice of the open sets of any infinite (or sufficiently large finite) T_0 topological space. This follows from Theorem 2, since $J(P) \subseteq P$ and hence $J(P)$ is finite. The same can be shown independently by a combinatorial argument.

Remark 3. The lattice of open sets of any topological space is a distributive lattice. However, the example given in Remark 1, together with Theorem 2, lead us to cocnlude that if a finite topological space is not T_0, this fact cannot be detected from a diagram of capital letters (such as (6)) representing its lattice of open sets.

Remark 3 may lead us to wonder, if a finite topological space is not only T_0 but T_1 as well. Can this fact be detected from a diagram of capital letters representing its lattice of open sets? The answer is yes, as follows from Theorem 3 below (as every discrete topological space is T_1).

We first recall that an element x of a poset P is called an *atom* iff x is a nonminimum minimal element of P. We further recall that a lattice $(P, \leq)$ is called *atomic* iff the only join irreducible elements of P are atoms.

Theorem 3. *A finite poset $(P, \leq)$ is isomorphic to the lattice of open sets of a unique (up to homeomorphism) discrete topological space iff P is a distributive atomic lattice.*

PROOF. If $(P, \leq)$ is isomorphic to the lattice of open sets $(T, \subseteq)$ of the discrete topological space (X, T), then (X, T) is T_0, and hence by Theorem 2 it follows that P is a distributive lattice. Since (X, T) is discrete, the only join irreducible elements of the lattice $(T, \subseteq)$ are the singleton subsets of X. Hence $(T, \subseteq)$ is atomic, and so is $(P, \leq)$ by the assumed isomorphism.

Conversely, let $(P, \leq)$ be a finite distributive atomic lattice. From the proof of Theorem 2, we have that $T = \{f(x) \mid x \in P\}$ defines a T_0 topology on $J(P)$, where f is given by (7). As P is atomic, the elements of $J(P)$ are precisely the atoms of P. Hence $f(x) = \{x\}$ is open for every $x \in P$, and therefore (X, T) is a discrete topological space. Finally, from the proof of Theorem 2 it follows that $(P, \leq)$ is isomorphic to $(T, \subseteq)$, and that (X, T) is unique up to homeomorphism.

Thus, Theorem 3 is proved.

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