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Solving convolution equations in S'_+ by numerical method

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Abstract. By using expansions of elements from S'_+ into Laguerre series we investigate the convolution equations in this space. We give examples of series expansions and present a numerical method for solving convolution equations. Also, we consider the convolution equations in LG'_e.

0. Introduction

Convolution equations in S'_+ include as special cases a lot of types of differential and integrodifferential equations. This space is a convolution algebra and a natural frame for the extension and the use of the Laplace transformation.

In the first part of the paper we give the structural properties of the basic spaces and their duals S'_+ and LG'_e from the point of view of Laguerre expansions of their elements. Note that the coefficients of $f \in S'_+ \equiv LG'$, respectively of $f \in LG'_e$, expanded into Laguerre series $f = \sum a_n l_n$ satisfy $\sum |a_n|^2 n^{-2k} < \infty$ for some $k > 0$, respectively $\sum |a_n|^2 k^{-2n} < \infty$ for some $k > 0$. By using expansions of elements from $S'_+(LG'_e)$ into Laguerre series we investigate the convolution in it and, in the second part of the paper, the convolution equation $f * g = h$, where $f \in S'_+(LG'_e)$ and $h \in S'_+(LG'_e)$ are known. We give examples of series expansions and a numerical method of finding coefficients in the expansion of g.

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We prove that if $f \in LG'_e$, then the convolution equation is solvable in LG'_e for all h from LG'_e iff $a_0 \neq 0$, where a_0 is the first coefficient in Laguerre's expansion of f. Finally, we give some comments on the error estimate.

1. Basic spaces

The space of smooth rapidly decreasing functions S is defined as the space of all smooth functions φ defined on the real line $\mathbf{R}$ ($\varphi \in C^\infty(\mathbf{R})$) for which all the norms

$$\|\varphi\|_k = \sup\{(1+|x|^k)|\varphi^{(i)}(x)|; \quad x \in \mathbf{R},\ i = 0, \dots k\}, \quad k \in \mathbf{N}_0\,,$$

are finite. ($\mathbf{N}$ is the set of naturals, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$). Its dual space is the well-known Schwartz's space of tempered distributions S'. Let us recall (see [6], for example) that

$$S = \operatorname*{proj\,lim}_{k\to\infty}(S_k\,, \|\ \|_k)$$

where $S_k = \{\varphi \in C^k(\mathbf{R});\ (1+|x|)^k|\varphi^{(i)}(x)| \to 0, \quad |x| \to \infty,\ i = 0, \dots k\}$, and $C^k(\mathbf{R})$ is the space of functions with continuous derivatives on $\mathbf{R}$ of order $\leq k$. In fact, S_k is the completion of S under the norm $\|\ \|_k$.

We have ([6]) $S' = \operatorname*{ind\,lim}_{k\to\infty} S'_k$ (in the topological sense) where S'_k is the dual of S_k, $k \in \mathbf{N}_0$, endowed with the dual norm $\|\ \|'_k$. The following three conditions for a sequence f_n from S' are equivalent $(n \to \infty)$:

(i) $f_n \to 0$ in the sense of the weak topology;

(ii) $f_n \to 0$ in the sense of the strong topology;

(iii) there exists $k \in \mathbf{N}$ such that $f_n \in S'_k$, $n \in \mathbf{N}$ and $f_n \to 0$ in the sense of the norm in S'_k.

It is well-known that S' is an A'–type space, A'–type spaces were introduced and studied in ([8], Ch. 9). The A'–type spaces whose elements have unique orthonormal expansions into Laquerre series were studied by Zemanian [8], Zayed [7], Duran [9] and Pilipović [5]. Let us recall the basic facts concerning these spaces. Denote by $\{l_n\}$, $n \in \mathbf{N}_0$, the Laguerre orthonormal base of the space $L^2(\mathbf{R}_+)$, ($\mathbf{R}_+ = (0,\infty)$, $\overline{\mathbf{R}}_+ = [0,\infty)$) whose elements are defined on $\mathbf{R}_+$ by $l_n(t) = e^{-t/2}L_n(t)$, where

$$L_n(t) = \sum_{m=0}^{n} \binom{n}{n-m} \frac{(-t)^m}{m!}, \quad n \in \mathbf{N}_0\,.$$

We denote by $\mathcal{R}$ a differential operator of the form $\mathcal{R} = e^{t/2}Dte^{-t}De^{t/2}$ $(D = d/dt)$; $\mathcal{R}^{k+1} = \mathcal{R}(\mathcal{R}^k)$, $k \in \mathbf{N}_0$, $\mathcal{R}^0$ is the identity operator.

The space LG is defined as the space of all $\varphi \in C^\infty(\mathbf{R}_+)$ for which all the norms

$$|||\varphi|||_k = |||\mathcal{R}^k\varphi|||_0 = \left(\int_0^\infty |\mathcal{R}^k\varphi(t)|^2 dt\right)^{1/2}, \quad k \in \mathbf{N},$$

are finite, and

$$\langle \mathcal{R}^k\varphi, l_n\rangle = \langle \varphi, \mathcal{R}^k l_n\rangle = (-n)^k\langle \varphi, l_n\rangle, \quad k \in \mathbf{N}_0, \quad n \in N_0,$$

where

$$\langle \varphi, \psi\rangle = (\varphi, \bar{\psi}) = \int_0^\infty \varphi(t)\psi(t)dt, \quad \varphi, \psi \in L^2(\mathbf{R}_+).$$

LG is the space of all $\varphi \in C^\infty(\overline{\mathbf{R}}_+)$ for which all the norms

$$\sup\{t^k|\varphi^{(j)}(t)|;\ t \in [0,\infty),\ j = 0,\dots k\}, \quad k \in \mathbf{N}_0,$$

are finite ([7]) and the dual space LG' is in fact S'_+ — the space of tempered distributions supported by $\overline{\mathbf{R}}_+$ ([4]).

Let $L_k,\ k \in \mathbf{R}$, $\quad (Le_k,\ k \geq 0)$ be the space of all the formal series

$$\varphi = \sum_{n=0}^\infty a_n l_n \ \text{ such that } \ |\varphi|_k = \left(|a_0|^2 + \sum_{n=1}^\infty |a_n|^2 n^{2k}\right)^{1/2} < \infty\,,$$

$$\left(\varphi = \sum_{n=0}^\infty a_n l_n \ \text{ such that } \ |\varphi|_{e,k} = \left(|a_0|^2 + \sum_{n=1}^\infty |a_n|^2 k^{2n}\right)^{1/2} < \infty\right).$$

We know that ([5])

(a) The L_k are B–spaces, $k \in \mathbf{R}$;

(b) The inclusion mappings $L_k \to L_\ell,\ k > \ell$ are compact;

(c) $LG = \operatorname{proj}\lim\limits_{k\to\infty} L_k$; $\quad LG' = S'_+ = \operatorname{ind}\lim\limits_{k\to\infty} L'_k$

where the L'_k, are the duals of $L_k,\ k \in \mathbf{R}$, endowed with the dual norms;

(d) $L'_k = \left\{ \sum\limits_{n=0}^\infty b_n l_n;\ \left(\sum\limits_{n=1}^\infty |b_n|^2 n^{-2k} + |b_0|^2\right)^{1/2} < \infty\right\} = L_{-k},\ \ k \in \mathbf{R}.$

Clearly, $Le_k \hookrightarrow L_k \hookrightarrow L^2$ for $k > 0$, where $A \hookrightarrow B$ means that A is a dense subspace of B and that the inclusion mapping is continuous.

$Le_k,\ k>0$, are B–spaces and the inclusion mappings $Le_k \to Le_\ell$, $k>\ell$, are compact.

Let $LG_e = \operatorname{proj}\lim_{k\to\infty} Le_k$. We have $LG'_e = \operatorname{ind}\lim_{k\to\infty} Le'_k$ and $(k>0)$

$$Le'_k = \left\{ \sum_{n=0}^{\infty} b_n l_n;\ \left(\sum_{n=0}^{\infty} |b_n|^2 k^{-2n} + |b_0|^2 \right)^{1/2} < \infty \right\} = Le_{1/k}\,.$$

The space LG'_e has been introduced in [3], where we studied spaces $\exp(\mathcal{A}')$ in general. From [3] we have

$f \in LG'_e \iff f = \sum\limits_{n=0}^{\infty} \frac{k^n}{n!} \mathcal{R}^n F_n$ for some $k>0$ and some sequence F_n from $L^2(\mathbf{R}_+)$ for which $\sum\limits_{n=0}^{\infty} |||F_n|||_0 < \infty$ holds.

The weak and the strong convergence in $LG'(LG'_e)$ are equivalent and $f_n \to f$ in $LG'(LG'_e)$, where

$$f_n = \sum_{m=0}^{\infty} b_m^{(n)} l_m, \quad f = \sum_{m=0}^{\infty} b_m l_m,$$

iff for some $k>0$

$$\sum_{m=0}^{\infty} \left| b_m^{(n)} - b_m \right|^2 m^{-2k} \to 0,\ \left(\sum_{m=0}^{\infty} \left| b_m^{(n)} - b_m \right|^2 k^{-2m} \to 0 \right),\quad n \to \infty\,.$$

Note that if we consider f_n and f as elements from S' then $f_n \to f$ in LG' iff $f_n \to f$ in S', $n \to \infty$.

2. The convolution and the Laplace transformation

These two notions are well-known for the space S'_+ which is a convolution algebra and for which we have

$$\mathcal{L} : S'_+ \to H(\mathbf{R}_+) \tag{1}$$

where $\mathcal{L}$ is the Laplace transformation defined by

$$(\mathcal{L}f)(s) = F(s) = \langle f(t),\ \mathcal{H}(t)e^{-st}\rangle,\quad s \in \mathbf{R} + i\mathbf{R}_+\ ,$$

where $\mathcal{H} \in C^\infty,\ \mathcal{H} = 1$ on $(-\varepsilon,\infty),\ \mathcal{H} = 0$ on $(-\infty,-2\varepsilon),\ \ \varepsilon > 0$ and $H(\mathbf{R}_+)$ is a space of holomorphic functions in $\mathbf{R}+i\mathbf{R}_+$ which satisfies the suitable growth conditions; in fact the mapping (1) is a bijection. Note that this definition does not depend on ε. We shall not repeat all the facts concerning the Laplace transformation which is deeply analyzed, for example, in [6]. In section 3. we shall recall and use some results for the convolution algebra S'_+ from [6].

Since any f from S'_+ is of the form $f = D^m F$, where $m \in \mathbf{N}_0$ and F is a continuous function on $\mathbf{R}$ bounded by a polynomial with $\operatorname{supp} F \subset [0, \infty)$, (D is the distributional derivative), the convolution of $f, g \in S'$ is

$$f * g = D^{m+r} \left(\int_0^x F(t)G(x-t)dt \right),$$

where f is of the given form and $g = D^r G,\ r \in \mathbf{N}_0$, while G is a continuous function with $\operatorname{supp} G \subset [0, \infty)$ and bounded by a polynomial.

Proposition 1. *Let f_n and g_n be sequences from S'_+ which converge in S' to f and g from S'_+. Then*

$$f_n * g_n \to f * g, \quad n \to \infty, \text{ in } S'.$$

Proof. This assertion follows directly from the topological properties of S'_+. However, we shall give here an elementary proof. As we noted in the introduction, there exists $k \in \mathbf{N}$ such that

$$f_n \to f, \quad g_n \to g \ \text{ in } \ S'_k.$$

Observe the sequence f_n. Let $x > 0$. For sufficiently large m the function

$$t \to \mathcal{H}(t)\frac{1}{m!}(x-t)_+^{m-1} \text{ is from } S_k, \text{where}$$

$$t \to \frac{1}{m!}(x-t)_+^{m-1} = \begin{cases} \frac{1}{m!}(x-t)^{m-1} & , \ -2 \le t \le x \\ 0 & , \ t > x \end{cases}$$

and $\mathcal{H}(t) \in C^\infty,\ \mathcal{H}(t) = 1$ for $t > -\frac{1}{2},\ \mathcal{H}(t) = 0$ for $t < -1$. We have

$$\langle f_n(t), \mathcal{H}(t)\frac{1}{m!}(x-t)_+^{m-1}\rangle \to \langle f(t), \mathcal{H}(t)\frac{1}{m!}(x-t)_+^{m-1}\rangle.$$

If we put

$$F_n(x) = \begin{cases} \langle f_n(t), \mathcal{H}(t)\frac{1}{m!}(x-t)_+^{m-1}\rangle & , \ x > 0 \\ 0 & , \ x \le 0, \end{cases}$$

$$F(x) = \begin{cases} \langle f(t), \mathcal{H}(t)\frac{1}{m!}(x-t)_+^{m-1}\rangle & , \ x > 0 \\ 0 & , \ x \le 0, \end{cases}$$

we have (from the boundedness of the sequence f_n in S'_k) that for every $n \in \mathbf{N}$, there is $C > 0$ such that

$$\max\left\{|F(x)|\,,\,|F_n(x)|\right\} \leq$$
$$\leq C \sup\left\{\left|(\mathcal{H}(t)\frac{1}{m!}(x-t)_+^{m-1})^{(\alpha)}\right|;\ -2 \leq t \leq x,\ \alpha = 0, \dots k\right\}$$

i.e. for suitable $C_1 > 0$

$$\max\left\{|F(x)|,\ |F_n(x)|\right\} \leq C_1 x^{m-1}, \quad x > 0,\ n \in \mathbf{N}_0.$$

This implies that $F_n,\ n \in \mathbf{N}$, and F are continuous functions supported by $[0,\infty)$ for which we have

$$F_n^{(m)}(x) = f_n(x), \quad F^{(m)}(x) = f(x), \quad F_n(x) \to F(x), \quad x \in \mathbf{R}$$
$$\text{and} \quad \frac{F_n(x)}{(1+|x|)^{m-1}}, \quad \frac{F(x)}{(1+|x|)^{m-1}} < C_1, \quad x \in \mathbf{R}.$$

Similarly, we have for $g_n,\ n \in \mathbf{N}$, and g and some $\overline{m} \in \mathbf{N}_0$ that

$$G_n^{(\overline{m})}(x) = g_n(x), \quad G^{(\overline{m})}(x) = g(x),$$

$$G_n(x) \to G(x), \quad \frac{G_n(x)}{(1+|x|)^{\overline{m}-1}}, \frac{G(x)}{(1+|x|)^{\overline{m}-1}} < \tilde{C}_1, \quad x \in \mathbf{R},$$

where G_n and G have properties as F_n and F.

So from

$$(f_n * g_n)(x) = \left(\int_0^x F_n(t)G_n(x-t)dt\right)^{(m+\overline{m})},$$

$$(f * g)(x) = \left(\int_0^x F(t)G(x-t)dt\right)^{(m+\overline{m})}, \quad x \in \mathbf{R}\,.$$

By using the Lebesgue theorem, we get

$$\int_0^x F_n(t)G_n(x-t)dt \longrightarrow \int_0^x F(t)G(x-t)dt, \quad n \to \infty,\ \text{in}\ S'\,,$$

and this implies the assertion of the theorem. □

Proposition 2. *Let* $f = \sum\limits_{m=0}^{\infty} b_m l_m,\ g = \sum\limits_{m=0}^{\infty} c_m l_m$ *be from* S'_k *for some* $k \in \mathbf{R}$. *Then* $f * g \in S'_{2k+r},\ r > \frac{3}{2}$, *and*

$$\text{(2)}\quad f * g = \sum_{m=0}^{\infty} \left(\sum_{p+q=m} b_p c_q - \sum_{p+q=m-1} b_p c_q \right) l_m \quad (\textit{As usual}, \sum_{p+q=-1} = 0).$$

Proof. Let us put $f_n = \sum\limits_{m=0}^{\infty} b_m^{(n)} l_m,\ g_n = \sum\limits_{m=0}^{\infty} c_m^{(n)} l_m$, where $b_m^{(n)} = b_m,\ c_m^{(n)} = c_m$ for $m \le n$ and $b_m^{(n)} = c_m^{(n)} = 0$ for $m > n,\ n \in \mathbf{N}$. From Proposition 1. we have

$$\text{(3)}\qquad \left(\sum_{m=0}^{\infty} b_m^{(n)} l_m \right) * \left(\sum_{m=0}^{\infty} c_m^{(n)} l_m \right) \longrightarrow f * g, \quad n \to \infty \text{ in } S'.$$

Since $l_p * l_q = l_{p+q} - l_{p+q+1}$ (see [2, p.191 (31)]), we have that the left side of (3) converges to

$$\sum_{m=0}^{\infty} \left(\sum_{p+q=m} b_p^{(n)} c_q^{(n)} - \sum_{p+q=m-1} b_p^{(n)} c_q^{(n)} \right) l_m \text{ in } S',\ n \to \infty .$$

Since $|b_m|$, $|c_m| \le C m^k,\ m \in \mathrm{N}_0$, for some C, we have that for suitable C_1

$$\left| \sum_{p+q=m} b_p c_q \right| \le C_1 m^{2k+1}, \quad m \in \mathbf{N} .$$

So, we have that $f * g$ is of the form (2) and it belongs to $S'_{2k+r},\ r > \frac{3}{2}$. □

We define the convolution of f and g from LG'_e by (2). We have

Proposition 3. LG'_e *is a convolution algebra. Moreover, if*

$$f = \sum_{m=0}^{\infty} b_m l_m, \quad g = \sum_{m=0}^{\infty} c_m l_m \quad \textit{are from } \ Le'_k ,$$

then $f * g \in Le'_s$ *for any* $s > k$.

Proof. Since $|b_m|$, $|c_m| \le C k^m, \quad m \in N_0$, we have, for any $k_1 > k$ and C_1 which depends on k and k_1,

$$\left| \sum_{p+q=m} b_p c_q \right| \le C^2 (m+1) k^m \le C_1 k_1^m .$$

This implies the assertion. $\square$

For the Laplace transform of an $f = \sum\limits_{m=0}^{\infty} b_m l_m \in L'_k$ we have

$$(\mathcal{L}f)(s) = \sum_{n=0}^{\infty} b_n \frac{(s-1/2)^n}{(s+1/2)^{n+1}}, \quad s \in \mathbf{R}+i\mathbf{R}_+ .$$

Now, by using the ordinary multiplication of series and (2) we get at once the well-known formula

$$\mathcal{L}(f * g)(s) = (\mathcal{L}f)(s)(\mathcal{L}g)(s), \quad s \in \mathbf{R}+i\mathbf{R}_+ .$$

In the sequel of this part we shall give several explicite expansions for elements from S'_+.

Let us first remark that the derivative of an $f \in S'_+$, considered in this paper as an element from $(LG)'$ is the same as the derivative of f considered as an element from S'. From [2. p. 189 (15), p. 192 (38)] we have

$$l'_n = -\sum_{n=0}^{n-1} l_m - \frac{1}{2} l_n, \quad \left(\sum_{0}^{-1} = 0\right), \quad n \in N_0 ,$$

which leads to the following assertion:

$$\text{(5)} \qquad \text{if} \quad f = \sum_{n=0}^{\infty} b_n l_n \in S'_+, \quad \text{then} \quad f' = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n-1} b_m + \frac{1}{2} b_n \right) l_n .$$

Examples.

1° Since $\int\limits_0^{\infty} l_n(t)dt = 2(-1)^n,\ n \in \mathbf{N}_0$, ([2, p. 191 (32)]), for Heaviside's function we have

$$H(x) = 2\sum_{n=0}^{\infty} (-1)^n l_n(x) . \tag{6}$$

This is an element from $S'_r,\ r > 1/2$.

2° For $s \in \mathbf{C}$, $\operatorname{Re} s > 0$, we have

$$H(x)e^{-sx} = \sum_{n=0}^{\infty} \frac{(s-1/2)^n}{(s+1/2)^{n+1}} l_n(x) . \tag{7}$$

Note, $\left|\dfrac{s-1/2}{s+1/2}\right| = t < 1$ so we get that $x \to H(x)e^{-sx} \in S_k$ for every $k \in R$.

3° Let $a \geq 0$. Since $\langle \delta(x-a), \varphi(x) \rangle = \varphi(a)$, we get at once

$$\delta(x-a) = \sum_{n=0}^{\infty} l_n(a) l_n(x) \,. \tag{8}$$

Because $l_n(0) = 1,\ n \in \mathbf{N}_0$, we have

$$\delta(x) = \sum_{n=0}^{\infty} l_n(x) \,. \tag{9}$$

Since for $n \in \mathbf{N}_0,\ |l_n(x)| \leq 1,\ x \geq 0$, ([2, p. 205, (3)]), we have $\delta(x-a) \in S'_r,\ r > 1/2$.

Note that (9) can be derived from (6) because $H'(x) = \delta(x)$.

From (9) and (5) we have

$$\delta'(x) = \sum_{n=0}^{\infty} \Big(n + \frac{1}{2} \Big) l_n(x) \,,$$
$$\delta''(x) = \sum_{n=0}^{\infty} \left(\frac{(n-1)n}{2} + n + \frac{1}{4} \right) l_n(x) \,.$$

4° Let $a > 0$. The mapping from $S(\overline{\mathbf{R}}_+)$ to $S(\overline{\mathbf{R}}_+)$ defined by

$$\varphi(t) \to \psi(t) = \varphi(a+t), \quad t \geq 0 \,,$$

is continuous. So for given $f(t) \in S'_+$ the distribution $f(t-a) \in S'_+$ is defined by

$$\langle f(t-a), \varphi(t) \rangle = \langle f(t), \varphi(t+a) \rangle \,.$$

Clearly, $\operatorname{supp} f(t-a) \subset [a, \infty)$ and

$$(f(t) * \delta(t-a))(x) = f(x-a) \,. \tag{10}$$

From (6) and (10) we have

$$H(x-a) = \begin{cases} 1, & x \geq a \\ 0, & x < a \end{cases} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n-1} 4(-1)^{n-j} l_j(a) + 2 l_n(a) \right) l_n(x).$$

5° Since for $s \in \mathbf{C}$, $\operatorname{Re} s > 0$,

$$\int_0^{\infty} e^{-st} t^m L_n(t) dt = (-1)^m \frac{d^m}{ds^m} \Big[\Big(1 - \frac{1}{s} \Big)^n \frac{1}{s} \Big]$$

(see [2, p. 191 (32) or 1.p.9]), we have

$$\int_0^\infty e^{-st} t^m L_n(t)dt = \frac{m!}{s^{m+1}} \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m+k}{m} \frac{1}{s^k}, \quad m, n \in \mathbf{N}_0 \,,$$

and by taking $s = 1/2$ we have for $m \in \mathbf{N}$,

(11)

$$x_+^m = \begin{cases} x^m, & x \geq 0 \\ 0, & x < 0 \end{cases} = 2^{m+1} m! \sum_{n=0}^\infty \left(\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m+k}{m} 2^k \right) l_n(x) \,.$$

3. The convolution equations in S'_+. The numerical approach

The problem which we investigate is the following:

$$f * g = h \,, \tag{12}$$

where f and h are given elements from S'_+ and g is unknown. If g exists, then by using the generalized Laplace transformation we get

$$g = \mathcal{L}^{-1}(\mathcal{L}h/\mathcal{L}f)$$

where $\mathcal{L}^{-1}$ is the inverse mapping for $\mathcal{L}$ from $H(\mathbf{R}_+)$ into S'_+ (see §2.).

This method of finding the solution is not of practical use from the numerical point of view, but from the theoretical one the use of the Laplace transformation gives the best known results for the existence of the solution ([6]). Let us recall from ([6] Ch2. §13) two conditions on f which imply the solvability of (12) for any given $h \in S'_+$.

(A) If $f = P(\delta) = \sum\limits_{k=0}^n a_k \delta^{(k)}$ and if $P(-iz) \neq 0$ in $\mathbf{R} + i\mathbf{R}_+$, then for any $h \in S'_+$ there exists $g \in S'_+$ such that

$$P(\delta) * g = h \,.$$

(B) If $F(z) = (\mathcal{L}f)(z)$, and $z \in \mathbf{R} + i\mathbf{R}_+$ has non-negative imaginary part, then for any $h \in S'_+$ (12) is solvable in S'_+ .

Clearly, equation (12) is solvable for any h iff there exists $G \in S'_+$ such that

$$f * G = \delta \quad (G \text{ is the fundamental solution}). \tag{13}$$

If G exists, then the solution of (12) is

$$g = G * h.$$

Let $f = \sum\limits_{n=0}^{\infty} a_n l_n$, $G = \sum\limits_{n=0}^{\infty} x_n l_n$, then from (9) and (2) we get that (13) is equivalent to the following system of equations:

$$
\begin{aligned}
&a_0x_0 = 1\\
&a_1x_0 + a_0x_1 - a_0x_0 = 1\\
&a_2x_0 + a_1x_1 + a_0x_2 - a_1x_0 - a_0x_1 = 1\\
&\vdots
\end{aligned}
\tag{14}
$$

or

$$
\begin{aligned}
&a_0x_0 = 1\\
&a_1x_0 + a_0x_1 = 2\\
&a_2x_0 + a_1x_1 + a_0x_2 = 3\\
&\vdots
\end{aligned}
\tag{15}
$$

If $a_0 \neq 0$ this system is solvable and it gives an explicit method of finding G (and thus of g) if we know that G exists in S'_+; for example in cases (A) or (B).

We shall present this method on a Volterra type equation. Denote by $W(\mathbf{R}_+)$ the space of all holomorphic functions of the form

$$f(z) = \lambda + \int\limits_{\mathbf{R}_+} \varphi(t)e^{izt}dt, \quad z \in \mathbf{R} + i\mathbf{R}_+,\ \lambda \in \mathbf{C},\ \varphi(t) \in L^1(\overline{\mathbf{R}}_+)\,.$$

$W(\mathbf{R}_+)$ is Wiener's algebra, a subalgebra of the algebra of holomorphic functions $H(\mathbf{R}_+)$ ([6, Ch.II, §13, Ch.I. §4]).

Its elements are the Laplace transforms of distributions of the form $\lambda\delta + \varphi(t)$, $\varphi \in L^1(\overline{\mathbf{R}}_+)$.

Denote by V_+ the space of these distributions.

If $f(z) \neq 0$ in $(\mathbf{R} + i\mathbf{R}_+) \cup \dot{\mathbf{R}}$ (where $\dot{\mathbf{R}}$ is the completion of the real line), then there exists $G \in V_+$ so that

$$(\lambda\delta + \varphi) * G = \delta\,.$$

In other words if $\int\limits_{\mathbf{R}_+} \varphi(t)e^{izt} \neq -\lambda$, $z \in (\mathbf{R} + i\mathbf{R}_+) \cup \dot{\mathbf{R}}$, then there exists a solution $g \in V_+$ of the integral equation

$$\lambda g(t) + \int\limits_0^{\infty} \varphi(t)g(x-t)dt = h(t), \quad t \geq 0\,,$$

for all $h \in V_+$.

We can solve numerically this equation by using the following algorithm.

Let $f = \lambda\delta + f_1,\ f_1 \in L^1(\overline{\mathbf{R}}_+)$, then $f * G = \delta$, i.e. $\lambda\delta * G + G * f_1 = \delta$, $\lambda G + G * f_1 = \delta$. From (2) we obtain

$$\sum_{n=0}^{\infty} \lambda x_n l_n + \sum_{n=0}^{\infty} \sum_{p+q=n} x_p a_q - \sum_{p+q=n-1} x_p a_q = \sum_{n=0}^{\infty} l_n \,.$$

This is equivalent to the system of equations:

$$\begin{aligned}
&\lambda x_0 + x_0 a_0 = 1\\
&\lambda x_1 + x_0 a_1 + x_1 a_0 - x_0 a_0 = 2\\
&\lambda x_2 + x_0 a_2 + x_1 a_1 + x_2 a_0 - x_0 a_1 - x_1 a_0 = 3\\
&\vdots
\end{aligned}$$

or in a shortened notation

$$\begin{aligned}
&x_0(a_0 + \lambda) = 1\\
&x_0(a_1 + \lambda) + x_1(a_0 + \lambda) = 2\\
&x_0(a_2 + \lambda) + x_1(a_1 + \lambda) + x_2(a_0 + \lambda) = 3\\
&\vdots
\end{aligned}$$

The coefficients of G are

$$x_n = \frac{1}{(a_0 + \lambda)} \left[(n+1) - \sum_{i=0}^{n-1} x_i (a_{n-i} + \lambda) \right], \quad n \in \mathbf{N}_0, \ \left(\sum_{0}^{-1} = 0 \right) \,.$$

The solution of Volterra's equation for any h from V_+ is

$$g = G * h \,.$$

From (2) we get

$$g = \sum_{n=0}^{\infty} x_n l_n * \sum_{n=0}^{\infty} b_n l_n = \sum_{n=0}^{\infty} \left(\sum_{p+q=n} x_p b_q - \sum_{p+q=n-1} x_p b_q \right) l_n \,.$$

Denote the coefficients of the last series by c_n. Then the coefficients of the solution g are

$$c_n = \frac{1}{(a_0 + \lambda)} \left(\sum_{p+q=n} - \sum_{p+q=n-1} \right) (n+1) b_q +$$

$$+\frac{1}{(a_0+\lambda)}\sum_{p+q=n}\sum_{i=0}^{n-1} x_i b_q (a_{n-i}+\lambda) -$$
$$-\sum_{p+q=n-1}\sum_{i=0}^{n-2} x_i b_q (a_{n-i}+\lambda), \quad n \in \mathbf{N}_0\,.$$

4. Properties of the solution

Concerning the convolution equation (12) the question is: which conditions on $f \in S'_+$ imply the existence of G in S'_+?

From (15) we get at once that the necessary condition is $a_0 \neq 0$.

The problem of finding general conditions on f is not simple. This will be shown by the following

Examples.

6°. Let $f(x) = \frac{1}{1-q}\exp((q+1)/(2(q-1))x), \quad x \geq 0$, where $|q| < 1$. From [2] we have

$$f(x) = \sum_{n=0}^{\infty} q^n l_n(x), \quad x \geq 0,$$

where the series converges uniformly to f on $\overline{\mathbf{R}}_+$ as well as in $L^p(\mathbf{R}_+)$ for any $p \geq 1$. For G we have

$$G = \sum_{n=0}^{\infty} [(n+1) - nq] l_n$$

in the sense of convergence in S'. More precisely $G \in S_k,\ k < -2$.

Moreover, if $h = \sum\limits_{n=0}^{\infty} b_n l_n$ then the solution of (15) is

$$g = \sum_{n=0}^{\infty}\left((1-q)\sum_{p=0}^{n} b_p + b_n\right) l_n\,.$$

So if h has “nice classical” properties this does not hold for the solution.

7°. We shall show in this example that if f has very fast coefficients the solution can be quite simple and it belongs to the same space as h.

From (15) we get

$$\begin{aligned}
&a_0x_0 = 1\\
&(a_1 - a_0)x_0 + a_0x_1 = 1\\
&(a_2 - a_1)x_0 + (a_1 - a_0)x_1 + a_0x_2 = 1\\
&(a_3 - a_2)x_0 + (a_2 - a_1)x_1 + (a_1 - a_0)x_2 + a_0x_3 = 1\\
&\vdots
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&a_0x_0 = 1\\
&(a_1 - 2a_0)x_0 + a_0x_1 = 0\\
&(a_2 - 2a_1 + a_0)x_0 + (a_1 - 2a_0)x_1 + a_0x_2 = 0\\
&(a_3 - 2a_2 + a_1)x_0 + (a_2 - 2a_1 + a_0)x_1 + (a_1 - 2a_0)x_2 + a_0x_3 = 0\\
&\vdots
\end{aligned}\tag{16}$$

Denote the coefficients of (16) by

$$\alpha_0 = a_0,\ \alpha_1 = a_1 - 2a_0,\ \alpha_2 = a_2 - 2a_1 + a_0,\ \alpha_3 = a_3 - 2a_2 + a_1, \dots$$

Assuming

$$\alpha_0 = a_0,\ \ \alpha_1 = a_0q,\ \ \alpha_2 = a_0q^2,\ \ \alpha_3 = a_0q^3 \dots ,$$

the fundamental solution of (16) is

$$G = (1/a_0)e^{-t/2}(1 - q(1 - t))\,.$$

If $h = \sum\limits_{n=0}^{\infty} b_nl_n$ then the solution of (16) is

$$g = 1/a_0 \sum_{m=0}^{\infty} [(b_m - b_{m-1}) - q(b_{m-1} - b_{m-2})]l_m, \quad b_{-1}, b_{-2} = 0\,.$$

If $h \in L_k$ it follows that $g \in L_k,\ k \in \mathbf{R}$; moreover if $h \in Le_k$ then $g \in Le_k,\ k > 0$.

Observe the convolution equation (12) in LG'_e when $f = \sum\limits_{n=0}^{\infty} a_nl_n \in LG'_e$ is fixed.

Proposition 4. *The convolution equation* (12) *is solvable in* LG'_e *for any* $h \in LG'_e$ *iff* $a_0 \neq 0$.

Proof. Clearly, a_0 must be different from zero. Let $h = \sum_{n=0}^{\infty} c_n l_n \in LG'_e$. If $g = \sum_{n=0}^{\infty} b_n l_n$, then the coefficients b_n must satisfy the system

$$\sum_{p+q=n} a_p b_q - \sum_{p+q=n-1} a_p b_q = c_n, \quad n \in \mathbf{N}_0,$$

i.e. $\sum_{p+q=n} a_p b_q = \tilde{c}_n$, where $\tilde{c}_n = \sum_{i=0}^{n} c_i,\ n \in \mathbf{N}_0$. This system is solvable since $a_0 \neq 0$.

Note that $\sum_{n=0}^{\infty} \tilde{c}_n l_n$ also belongs to LG'_e. We have to prove that $a_0 \neq 0$ implies that for some $k > 0$ and $C > 0$

$$|b_n| < Ck^n, \quad n \in \mathbf{N}_0\,. \tag{17}$$

Observe the functions $a(t) = \sum_{n=0}^{\infty} a_n t^n,\ c(t) = \sum_{n=0}^{\infty} \tilde{c}_n t^n$ which are analytic in the interval $\left(-\frac{1}{r}, \frac{1}{r}\right)$, where we choose $r > 0$ such that for some $C > 0$

$$|a_n|, |\tilde{c}_n| \leq Cr^n, \quad n \in \mathbf{N}_0\,.$$

Put $b(t) = \sum_{n=0}^{\infty} b_n t^n$. We have formally $a(t) \cdot b(t) = c(t)$.

Since $a_0 \neq 0$, we get that $1/a(t)$ is an analytic function in some neighbourhood of zero and so $(1/a(t))c(t) = b(t)$ is analytic in some neighbourhood of zero. This implies that for some $k > 0$ and $C > 0$ (17) holds. $\square$

This proposition implies that the natural frame for convulution equations of elements supported by $[0, \infty)$ is LG'_e.

The preceding proof also suggests a method of finding the fundamental solution for the convolution equation. Namely, for given $f \in S'_+$ we have to solve the equation $a(t)x(t) = d(t)$, in some interval $(-\varepsilon, \varepsilon)$, where $a(t) = \sum_{n=0}^{\infty} a_n t^n,\ d(t) = \sum_{n=0}^{\infty} (n+1)t^n$ and $x(t) = \sum_{n=0}^{\infty} x_n t^n$.

According to (15) the coefficients of $x(t)$ are the coefficients of the fundamental solution.

Proposition 5. *Let $f \in S'_+$ be as above and $a_0 \neq 0$. The convolution equation* (12) *is solvable in S'_+ iff the analytic function $1/a(t),\ t \in (-\varepsilon, \varepsilon)$, has the coefficients $y_n,\ n \in \mathbf{N}_0$, such that $|y_n| < Cn^k,\ n \in \mathbf{N}_0$, for some $C > 0$ and $k > 0$.*

5. Error estimate

At the end we shall give some remarks concerning the error estimate for the approximate solution of (12), $g_N = \sum\limits_{n=0}^{N} c_n l_n$, where $g = \sum\limits_{n=0}^{\infty} c_n l_n$ is the exact solution. Let $G_N = \sum\limits_{n=0}^{N} x_n l_n$ and $h_N = \sum\limits_{n=0}^{N} b_n l_n$. We have

$$g_N = G_N * h = G * h_N = G_N * h_N.$$

This implies that for finding the approximate solution g_N we need the approximations for h and G. Also, for $f \in LG'_e$ and $a_0 \neq 0$ we have $G_N \to G$ in $LG'_e,\ N \to \infty$, and so, $g_N \to g$ in $LG'_e,\ N \to \infty$.

If we have more informations on G and h then we can give the estimations for $g \to g_N$. For example, let $h \in L^p(\mathbf{R}_+),\ h_N \to h$ in $L^p,\ G \in L^q(\mathbf{R}_+),\ G_N \to G$ in L^q, where p and q are real numbers such that

$$p \geq 1,\ q \geq 1,\ \frac{1}{p} + \frac{1}{q} \geq 1\,.$$

Let $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Then $g \in L^r(\mathbf{R}_+)$ and

$$\left(\int_0^\infty |g_N(t) - g(t)|^r dt \right)^{1/r} \leq$$

$$\leq \left(\int_0^\infty |G(t)|^q dt \right)^{1/q} \cdot \left(\int |h_N(t) - h(t)|^p dt \right)^{1/p},$$

and

$$\left(\int_0^\infty |g_N(t) - g(t)|^r dt \right)^{1/r} \leq$$

$$\leq \left(\int_0^\infty |h(t)|^p dt \right)^{1/p} \cdot \left(\int_0^\infty |G(t) - G_N(t)|^q dt \right)^{1/q} .$$

Note, if $\frac{1}{p} + \frac{1}{q} = 1$, then $r = \infty$ and the left hand side of these inequalities becomes $\sup\left\{ |g_N(t) - g(t)|,\ t \in (0, \infty) \right\}$.

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