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Arithmetical expressions and asymptotic formulae for generalized totient functions

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1. Introduction

For a positive integer m let μ_m denote the multiplicative function such that for each prime power p^a $(\neq 1)$, $\mu_m(p^a) = -1$ if $a = m$, and $= 0$ otherwise. The function μ_m is the generalized Möbius function due to KLEE [7]. Clearly $\mu_1 = \mu$, the classical Möbius function. We can also write $\mu_m = \Omega_m(\mu)$, where $\Omega_m(\mu)$ is the mth convolute of μ. The mth convolute $\Omega_m(f)$ of an arithmetical function f is defined by

$$\Omega_m(f)(n) = \begin{cases} f(n^{1/m}) & \text{if } n \text{ is an } m\text{th power}, \\ 0 & \text{otherwise} \end{cases}$$

(see [8], p. 53). For an arithmetical function g and a positive integer s let $\mu^g_{m,(s)}$ denote the arithmetical function defined inductively by $\mu^g_{m,(1)} = \mu_m g$, $\mu^g_{m,(s)} = \mu^g_{m,(1)} \star \mu^g_{m,(s-1)}$ $(s \geq 2)$, where $\star$ denotes the Dirichlet convolution. Similarly, let $\rho^g_{m,(s)}$ denote the arithmetical function defined inductively by $\rho^g_{m,(1)} = \mu^2_m g$, $\rho^g_{m,(s)} = \rho^g_{m,(1)} \star \rho^g_{m,(s-1)}$ $(s \geq 2)$.

Let f and g be arithmetical functions. Then we define generalized Euler and Dedekind totient functions by

$$\Phi^{f,g}_{m,s} = f \star \mu^g_{m,(s)},$$

$$\Psi^{f,g}_{m,s} = f \star \rho^g_{m,(s)}.$$

For a real number a let ζ_a denote the arithmetical function given by $\zeta_a(n) = n^a$ for all n. Then we use the brief notations $\Phi^{f,g}_{m,s} = \Phi^{a,g}_{m,s}$, $\Psi^{f,g}_{m,s} = \Psi^{a,g}_{m,s}$ when $f = \zeta_a$.

These functions include as special cases a large number of earlier generalizations of Euler and Dedekind totient functions. Namely, if $F = F(x)$ is a nonconstant polynomial with integer coefficients, $N_F(n)$ the number of solutions (mod n) of $F(x) \equiv 0 \pmod{n}$, t a positive integer and $N_F^t(n) = N_F(n)^t$, then the function $\Phi_{m,s}^{a,g}$ with $a = t$, $g = N_F^t$, $s = 1$ reduces to the generalized Euler totient function $\varphi_{F,t}^{(m)}$ by CHIDAMBARASWAMY [1]. We recall that $\varphi_{F,t}^{(m)}(n)$ is defined to be the number of ordered t-tuples of integers $a_1, \ldots, a_t$ (mod n) such that $((F(a_1), \ldots, F(a_t)), n)_m = 1$, where $(F(a_1), \ldots, F(a_t))$ is the g.c.d. of $F(a_1), \ldots, F(a_t)$ and $((F(a_1), \ldots, F(a_t)), n)_m$ stands for the largest mth power common divisor of $(F(a_1), \ldots, F(a_t))$ and n. With $a = m = s = 1$ and $g \equiv 1$ the function $\Phi_{m,s}^{a,g}$ reduces to the classical Euler totient function. Further special cases of $\Phi_{m,s}^{f,g}$ can be found from the references of [1] and [6].

With $a = t$, $g = N_F^t$, $s = 1$, with $a = kt$, $g = N_F^t\left((\cdot)^k\right)$, $m = s = 1$ and with $a = vt$, $g = N_F^t\left((\cdot)^v\right)$, $m = 1$ the function $\Psi_{m,s}^{a,g}$ reduces respectively to the generalized Dedekind totient functions $\Psi_{F,t}^{(m)}$, $\psi_{F,t}^{(k)}$ and $\psi_{F,t}^{s,v}$ by CHIDAMBARASWAMY ([2], [2] and [3]). With $a = k$, $g \equiv 1$, $m = s = 1$, with $a = 1$, $g \equiv 1$, $s = 1$ and with $a = 1$, $g \equiv 1$, $m = 1$ the function $\Psi_{m,s}^{a,g}$ reduces respectively to the functions ψ_k, Ψ_m and $\psi_{(s)}$ by SURYANARAYANA ([9], [10] and [10]). If $a = m = s = 1$ and $g \equiv 1$, the function $\Psi_{m,s}^{a,g}$ is the classical Dedekind totient function.

The purpose of this paper is to give arithmetical expressions for the functions $\Phi_{m,s}^{f,g}$ and $\Psi_{m,s}^{f,g}$ and asymptotic formulae for the summatory functions of the functions $\Phi_{m,s}^{a,g}$ and $\Psi_{m,s}^{a,g}$. The section of arithmetical expressions is motivated by papers of CHIDAMBARASWAMY ([2],[3]), HANUMANTHACHARI [5] and SURYANARAYANA [10], who gave arithmetical expressions for the functions $\Psi_{F,t}^{(m)}$, $\psi_{F,t}^{(k)}$, $\psi_{F,t}^{u,v}$, Ψ_m, ψ_k and $\psi_{(s)}$, and the section of asymptotic formulae is motivated by a paper of CHIDAMBARASWAMY and SITARAMACHANDRARAO [4], who gave asymptotic formulae for the summatory functions of the functions $\varphi_{F,t}^{(m)}$, $\Psi_{F,t}^{(m)}$, $\psi_{F,t}^{(k)}$ and $\psi_{F,t}^{u,v}$.

2. Arithmetical expressions

In this section we shall give arithmetical expressions for the functions $\Phi_{m,s}^{f,g}$ and $\Psi_{m,s}^{f,g}$. These expressions contain as special cases the arithmetical expressions given for $\Psi_{F,t}^{(m)}$ and $\psi_{F,t}^{(k)}$ in [2], for $\psi_{F,t}^{s,v}$ in [3], for Ψ_m in [5] and [10], for ψ_k in [10] and for $\psi_{(s)}$ in [10]. We wish to point out that several authors consider expressions of the type of Theorem 7 as the definition of generalized Dedekind totient functions.

The following notations will be needed here. Let $\omega(n)$ denote the number of distinct prime factors of n. Then the arithmetical functions θ and $\theta_{(s)}$ are defined by $\theta(n) = 2^{\omega(n)}$ for all n and $\theta_{(s)} = \theta \star \cdots \star \theta$ (s factors). Further, the arithmetical functions θ_m and $\theta_{m,(s)}$ are defined by $\theta_m(n) = \theta(n_m)$, where $n_m = \Pi p^a$, the product being over $p^a \parallel n$ (i.e. $p^a \mid n$, $(p^a, n/p^a) = 1$) for which $a \geq m$, and $\theta_{m,(s)} = \theta_m \star \cdots \star \theta_m$ (s factors). The arithmetical function $\rho_{(s)}$ is defined to be the multiplicative function given by $\rho_{(s)}(p^a) = \binom{s}{a}$ for all prime powers p^a ([10], p. 109).

Theorem 1.

$$\Phi_{m,s}^{f,\mu_m g} = \Psi_{m,s}^{f,g},$$

$$\Psi_{m,s}^{f,\mu_m g} = \Phi_{m,s}^{f,g}.$$

Theorem 2. *If* $1 \leq i \leq s-1$, *then*

$$\Phi_{m,s}^{f,g} = \Phi_{m,s-i}^{f,g} \star \mu_{m,(i)}^{g},$$

$$\Psi_{m,s}^{f,g} = \Psi_{m,s-i}^{f,g} \star \rho_{m,(i)}^{g}.$$

Theorems 1 and 2 are direct consequences of the definitions.

Theorem 3. *If g is completely multiplicative, then*

$$\Psi_{m,s}^{f,g} = \Phi_{m,s}^{f,g} \star [\Omega_m(\theta_{(s)})]g.$$

PROOF. By the properties of the mth convolute (see [8], p. 53) and the formula $\mu^2 = \mu \star \theta$, we have

$$\begin{aligned}\Psi_{m,1}^{f,g} &= f \star \mu_m^2 g = f \star [\Omega_m(\mu)^2]g = f \star [\Omega_m(\mu^2)]g = f \star [\Omega_m(\mu \star \theta)]g \\ &= f \star [\Omega_m(\mu)]g \star [\Omega_m(\theta)]g = \Phi_{m,1}^{f,g} \star [\Omega_m(\theta)]g.\end{aligned}$$

Now, applying induction on s we obtain the theorem. We omit the details.

Theorem 4. *If g is completely multiplicative, then*

$$\Psi_{m,s}^{f,g} = \Phi_{1,s}^{f,g} \star \theta_{m,(s)} g.$$

PROOF. Considering prime powers it can be verified that $\mu_m^2 = \mu \star \theta_m$. Thus

$$\Psi_{m,1}^{f,g} = f \star \mu_m^2 g = f \star \mu g \star \theta_m g = \Phi_{1,1}^{f,g} \star \theta_{m,(1)} g.$$

Further, applying induction on s gives the theorem. We omit the details.

Theorem 5. *If g is completely multiplicative, then*

$$\Psi^{f,g}_{m,s} = f \star [\Omega_m(\rho_{(s)})]g.$$

PROOF. We have

$$\Psi^{f,g}_{m,s} = f \star [\Omega_m(\mu^2 \star \cdots \star \mu^2)]g.$$

By multiplicativity and considering prime powers it can be verified that

$$\mu^2 \star \cdots \star \mu^2 = \rho_{(s)}.$$

We thus arrive at our result.

Theorem 6. *If f and g are completely multiplicative, then*

$$\Psi^{f,g}_{m,s}(n^m) = \Psi^{f^m,g^m}_{1,s}(n),$$

$$\Phi^{f,g}_{m,s}(n^m) = \Phi^{f^m,g^m}_{1,s}(n).$$

PROOF. By Theorem 5

$$\begin{aligned}\Psi^{f,g}_{m,s}(p^{am}) &= \sum_{i=0}^{a} f\left(p^{(a-i)m}\right)\rho_{(s)}(p^i)g(p^{im}) \\ &= \sum_{i=0}^{a} f^m(p^{a-i})\rho_{(s)}(p^i)g^m(p^i),\end{aligned}$$

which proves the first equation. The second equation follows in a similar way applying the formula

$$\Phi^{f,g}_{m,s} = f \star [\Omega_m(\mu \star \cdots \star \mu)]g.$$

Theorem 7. *If f and g are completely multiplicative functions and $f(n) \neq 0$ for all n, then*

$$\Psi^{f,g}_{m,1}(n) = \sum_{d\delta=n} \frac{f(d)}{f((d,\delta))}\chi_m(\delta)g(\delta)\Phi^{f,g}_{m,1}((d,\delta)),$$

where $\chi_m(\delta) = 1$ if δ is an mth power, and $= 0$ otherwise.

Theorem 8. *If f is multiplicative and g is completely multiplicative, then*

$$\Psi^{f,g}_{m,1} = (\theta\chi_m g) \star \Phi^{f,g}_{m,1}.$$

Theorems 7 and 8 can be proved by considering the function values on prime powers. We omit the details.

Theorem 9. *If f is completely multiplicative, $f(n) \neq 0$ for all n and g is multiplicative, then*

$$\Phi^{f,g}_{m,1}(n) = f(n) \prod_{p^m | n} [1 - g(p^m)/f(p^m)],$$

$$\Psi^{f,g}_{m,1}(n) = f(n) \prod_{p^m | n} [1 + g(p^m)/f(p^m)].$$

PROOF. We have

$$(f \star \mu_m g)(p^a) = \begin{cases} f(p^a) & \text{if } p^m \nmid p^a, \\ f(p^a) - f(p^{a-m})g(p^m) & \text{if } p^m \mid p^a. \end{cases}$$

We thus obtain the first equation. The second equation can be verified by a similar argument.

3. Asymptotic formulae

In this section we shall give asymptotic formulae for the summatory functions of the functions $\Phi^{a,g}_{m,s}$ and $\Psi^{a,g}_{m,s}$. We shall combine and generalize the formulae given by CHIDAMBARASWAMY and SITARAMACHANDRARAO [4], who studied the functions $\varphi^{(m)}_{f,t}$, $\Psi^{(m)}_{f,t}$, $\psi^{(k)}_{f,t}$ and $\psi^{u,v}_{f,t}$.

Throughout this section let g be a given arithmetical function such that there are positive integers b and C such that

$$|g(n^m)| \leq C^{\omega(n)} n^b \text{ for all } n \text{ with } \gamma(n) = n, \tag{1}$$

where $\gamma(n)$ is the product of distinct prime factors of n. This condition is satisfied for $g = N^t_F((\cdot)^e)$, where e is any positive integer. For that function $b = t(me - 1)$ and $C = (\max\{h, u\})^t$, where h is the degree of F and u is the largest prime divisor common to the coefficients of F. With these types of g's we obtain the special cases given in [4].

Lemma 1 ([4], Lemma 2.2). *For every positive integer k,*

$$\sum_{n \leq x} k^{\omega(n)} = O(x(\log x)^{k-1}).$$

Lemma 2.

$$\sum_{n \leq x} |\rho^g_{m,(s)}(n)| = O(x^{\frac{b+1}{m}} (\log x)^{Cs-1}).$$

PROOF. If $s = 1$, we have

$$\sum_{n \le x} |\rho^g_{m,(s)}(n)| = \sum_{n \le x} |\mu^2_m(n) g(n)| = \sum_{n \le x^{1/m}} |\mu^2(n) g(n^m)|$$

$$\le \sum_{n \le x^{1/m}} \mu^2(n) C^{\omega(n)} n^b \le x^{b/m} \sum_{n \le x^{1/m}} C^{\omega(n)} = O(x^{\frac{b+1}{m}} (\log x)^{C-1}).$$

Assume Lemma 2 holds for $s-1$ ($s \ge 2$). Then

$$\begin{aligned} \sum_{n \le x} |\rho^g_{m,(s)}(n)| &\le \sum_{d\delta \le x} |\rho^g_{m,(1)}(d) \rho^g_{m,(s-1)}(\delta)| \\ &= \sum_{d \le x} |\rho^g_{m,(1)}(d)| \sum_{\delta \le x/d} |\rho^g_{m,(s-1)}(\delta)| \\ &= O(\sum_{d \le x} |\rho^g_{m,(1)}(d)| (x/d)^{\frac{b+1}{m}} (\log(x/d))^{C(s-1)-1}) \\ &= O(x^{\frac{b+1}{m}} (\log x)^{C(s-1)-1} \sum_{d \le x} |\rho^g_{m,(1)}(d)| d^{-\frac{b+1}{m}}). \end{aligned}$$

By partial summation and this lemma with $s = 1$,

$$\sum_{d \le x} |\rho^g_{m,(1)}(d)| d^{-\frac{b+1}{m}} = O((\log x)^C).$$

We thus arrive at our result.

Lemma 3. *For $a \ne 0$,*

$$\sum_{n \le x} |\rho^g_{m,(s)}(n)| n^{-a} = \begin{cases} O(1) & \text{if } b - ma < -1, \\ O((\log x)^{Cs}) & \text{if } b - ma = -1, \\ O(x^{\frac{b+1}{m} - a} (\log x)^{Cs-1}) & \text{if } b - ma > -1. \end{cases}$$

Lemma 4. *If $b - ma < m - 1$, then*

$$\sum_{n > x} |\rho^g_{m,(s)}(n)| n^{-a-1} = O(x^{\frac{b+1}{m} - a - 1} (\log x)^{Cs-1}).$$

Lemma 5.

$$\sum_{n \le x} |\rho^g_{m,(s)}(n)| \log n = O(x^{\frac{b+1}{m}} (\log x)^{Cs}).$$

Lemmata 3, 4 and 5 follow by partial summation and Lemma 2.

Remark. It is easily seen that $\rho^{\mu_m g}_{m,(s)} = \mu^g_{m,(s)}$, $\mu^{\mu_m g}_{m,(s)} = \rho^g_{m,(s)}$. It is also easily seen that the function g satisfies (1) if, and only if, the function $\mu_m g$ satisfies (1). Replacing the function g by the function $\mu_m g$ shows that Lemmata 2–5 hold for the function $\mu^g_{m,(s)}$ as well.

Theorem 10. *Let $b - ma < m - 1$.*
a) *If $a \geq 0$, then*

$$\sum_{n\leq x} \Psi^{a,g}_{m,s}(n) = \alpha x^{a+1} + \begin{cases} O(x^a) & \text{if } b - ma < -1, \\ O(x^a(\log x)^{C_s}) & \text{if } b - ma = -1, \\ O(x^{\frac{b+1}{m}}(\log x)^{C_s - 1}) & \text{if } b - ma > -1, \end{cases}$$

where

$$\alpha = \frac{1}{a+1}\sum_{n=1}^{\infty} \rho^g_{m,(s)}(n) n^{-a-1}.$$

b) *If $a < 0$, $a \neq -1$, then*

$$\sum_{n\leq x} \Psi^{a,g}_{m,s}(n) = \alpha x^{a+1} + O(x^{\frac{b+1}{m}}(\log x)^{C_s - 1}).$$

c)

$$\sum_{n\leq x} \Psi^{-1,g}_{m,s}(n) = O(x^{\frac{b+1}{m}}(\log x)^{C_s}).$$

Remark. Theorem 10 holds if the function $\Psi^{a,g}_{m,s}$ is replaced by the function $\Phi^{a,g}_{m,s}$. Then the function $\rho^g_{m,(s)}$ should be replaced by the function $\mu^g_{m,(s)}$. This can be verified replacing the function g by the function $\mu_m g$.

Proof. If $a \geq 0$, then

$$\begin{aligned} \sum_{n\leq x} \Psi^{a,g}_{m,s}(n) &= \sum_{d\delta\leq x} \delta^a \rho^g_{m,(s)}(d) = \sum_{d\leq x} \rho^g_{m,(s)}(d) \sum_{\delta\leq x/d} \delta^a \\ &= \sum_{d\leq x} \rho^g_{m,(s)}(d) \left\{ \frac{1}{a+1}(x/d)^{a+1} + O((x/d)^a) \right\} \\ &= \frac{1}{a+1} x^{a+1} \sum_{d=1}^{\infty} \rho^g_{m,(s)}(d) d^{-a-1} \\ &\quad + O(x^{a+1} \sum_{d>x} |\rho^g_{m,(s)}(d)| d^{-a-1}) \\ &\quad + O(x^a \sum_{d\leq x} |\rho^g_{m,(s)}(d)| d^{-a}). \end{aligned}$$

Applying Lemmata 3 and 4 we arrive at our result. The series in the definition of α converges absolutely by Lemma 4.

In the cases $a < 0$ $(a \neq -1)$ and $a = -1$ the proof goes through in a similar way to that in the case $a \geq 0$ applying the formulae

$$\sum_{n \leq x} n^a = \frac{1}{a+1} x^{a+1} + \zeta(-a) + O(x^a), \quad a < 0 \ (a \neq -1),$$

$$\sum_{n \leq x} n^{-1} = \log x + \gamma + O(x^{-1}),$$

where γ is the Euler constant, and Lemmata 2–5.

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