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On an application of the Zincenko method to the approximation of implicit functions

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Abstract. Abstract We use the Zincenko iteration to approximate implicit functions in Banach spaces. The nonlinear equations involved contain a nondifferentiable term. Our hypotheses are more general than ZABREJKO-NGUEN's [10], in this case.

I. Introduction

Let $E, \wedge$ be Banach spaces and denote by $U(x_0, R)$ the closed ball with center $x_0 \in E$ and of radius R in E. We will use the same symbol for the norm $\| \ \|$ in both spaces. Suppose that the nonlinear operators $F(x,\lambda)$ and $G(x,\lambda)$ with values in E defined for $x \in U(x_0,R)$ and $\lambda \in U(\lambda_0,S)$ are such that F is Frechet differentiable there, $F'(x_0,\lambda_0)^{-1}$ exists and

(1) $$\| F'(x_0,\lambda_0)^{-1}(F'(x,\lambda) - F'(y,\lambda)) \| \leq K_1(r,s) \| x-y \| ,$$

(2) $$\| F'(x_0,\lambda_0)^{-1}(F'(x_0,\lambda) - F'(x_0,\lambda_0)) \| \leq K_2(s) \| \lambda - \lambda_0 \| ,$$

(3) $$\| F'(x_0,\lambda_0)^{-1}(G(x,\lambda) - G(y,\lambda)) \| \leq K_3(r,s) \| x-y \| ,$$

for all $x,y \in U(x_0,r) \subset U(x_0,R)$ and $\lambda \in U(\lambda_0,s) \subset U(\lambda_0,s)$. Here K_1, K_2, and K_3 denote non–decreasing functions on the intervals $[0,R]\times[0,S], [0,R]$ and $[0,R]\times[0,S]$ respectively.

We use the Zincenko interation [11]

(4) $$x_{n+1}(\lambda) = x_n(\lambda) - F'(x_n(\lambda),\lambda)^{-1}(F(x_n(\lambda),\lambda) + G(x_n(\lambda),\lambda)), n \geq 0$$

to approximate a solution $x^\star(\lambda)$ of the equation

(5) $$F(x,\lambda) + G(x,\lambda) = 0.$$

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By x_0 we mean $x_0(\lambda)$. That is x_0 depends on the λ used in (4).

Our assumptions (1)–(3) generalize the ones made by Zabrejko–Nguen [10], Yamamoto [9] and Potra–Ptăk (in [6] (for $G = 0$)). Moreover, several authors have treated the case when $G = 0$ provided that K_1 and K_2 are constants (or not)[1], [2], [4], [5], [6].

We provide sufficient conditions for the convergence of iteration (4) to a locally unique solution $x^\star(\lambda)$ of equation (5) as well as several error bounds on the distances $\| x_{n+1}(\lambda) - x_n(\lambda) \|$ and $\| x_n(\lambda) - x^\star(\lambda) \|$.

We need to define the functions

$$a_s = K(s) \| F'(x_0, \lambda_0)^{-1}(F(x_0, \lambda) + G(x_0, \lambda)) \|, \quad (s = 0 \text{ if } \lambda = \lambda_0),$$

$$w_s(r) = \int_0^r K_1(t, s)dt, \quad K_4(s) = \int_0^s K_2(t)dt, \quad k(s) = (1 - K_4(s))^{-1}$$

provided that

$$K_4(S) < 1, \quad \varphi_s(r) = a_s + K(s) \int_0^r w_s(t)dt - r,$$

$$\psi_s(r) = K(s) \int_0^r K_3(t, s)dt, \quad \chi_s(r) = \varphi_s(r) + \varphi_s(r),$$

and the iteration

$$\begin{aligned} y_{n+1}(\lambda) &= y_n(\lambda) - F'(x_0, \lambda_0)^{-1}(F(y_n(\lambda), \lambda) + G(y_n(\lambda), \lambda)), \\ y_0 &= x_0, \quad n \geq 0. \end{aligned} \tag{6}$$

II. Convergence results

We can now formulate the following result:

Theorem 1. *Suppose that the function $\chi_s(r)$ has a unique zero $\rho^\star = \rho_s^\star$ in $[0, R]$, and $\chi_s(R) \leq 0$. Then*

(a) *equation (5) has a unique solution $x^\star(\lambda) \in U(x_0, R)$ with $x^\star(\lambda) \in U(x_0, \rho^\star)$;*

(b) *the following estimates are true*

$$\| y_{n+1}(\lambda) - y_n(\lambda) \| \leq v_{n+1} - v_n \tag{7}$$

and

$$\| y_n(\lambda) - x^\star(\lambda) \| \leq \rho^\star - v_n \tag{8}$$

where the scalar sequence $\{v_n\}, n \geq 0$ *is monotonically increasing and convergent to* $\rho^\star$ *with*

$$v_{n+1} = d_s(v_n),\ n \geq 0,\ v_0 = 0 \qquad (9)$$
$$d_s(r) = r + \chi_s(r).$$

Proof. It is simple calculus to show that the sequence $\{v_n\},\ n \geq 0$ is monotonically increasing and convergent to $\rho^\star$ (see also, [10, v. 675]). We will show using induction on n that the estimate (7) is true, from which (8) will follow immediately.

From (6) for $n = 0$ we get

$$\| y_1(\lambda) - y_0 \| = \| F'(x_0, \lambda_0)^{-1}(F(x_0, \lambda) + G(x_0, \lambda)) \| \leq a_s = d_s(0) = v_1 - v_0.$$

That is, the estimate (7) is true for $n = 0$. Let us assume that (7) is true for $n < k$. Then by (6), (1), (3), [10, p. 674] and the induction hypothesis we get

$$\begin{aligned}
&\| y_{k+1}(\lambda) - y_k(\lambda) \| \leq \| y_k(\lambda) - y_{k-1}(\lambda) - F'(x_0, \lambda_0)^{-1}(F(y_k(\lambda), \lambda) - \\
&- F(y_{k-1}(\lambda), \lambda)) \| + \| F'(x_0, \lambda_0)^{-1}(G(y_k(\lambda), \lambda) - G(y_{k-1}(\lambda), \lambda)) \| \leq \\
&\leq \int_0^1 \| F'(x_0, \lambda_0)^{-1}(F'((1-t)y_{k-1}(\lambda) + ty_k(\lambda)) - F'(x_0, \lambda_0)) \| \cdot \\
&\cdot \|y_k(\lambda) - y_{k-1}(\lambda)\| dt + \|F'(x_0, \lambda_0)^{-1}(G(y_k(\lambda), \lambda) - G(y_{k-1}(\lambda), \lambda))\| \leq \\
&\leq \int_0^1 w_s((1-t)v_{k-1} + tv_k)(v_k - v_{k-1})dt + \int_{v_{k-1}}^{v_k} K_3(t, s)dt \leq \\
&\leq K(s)\left[\int_{v_{k-1}}^{v_k} w_s(t)dt + \int_{v_{k-1}}^{v_k} K_3(t, s)dt\right] = \\
&= d_s(v_k) - d_s(v_{k-1}) = v_{k+1} - v_k.
\end{aligned}$$

That is, the estimate (7) is true for $n = k$. Hence the sequence $\{y_n(\lambda)\}$ is a Cauchy sequence in a Banach spacee and as such it converges to some $x^\star(\lambda) \in U(x_0, \rho^\star) \subset U(x_0, R)$. By letting $n \to \infty$ in (6) we deduce that $x^\star(\lambda)$ is a solution of equation (5). We will show that $x^\star(\lambda)$ is the unique solution of equation (5) in $U(x_0, R)$, by considering the sequences

$$z_{n+1}(\lambda) = z_n(\lambda) - F'(x_0, \lambda_0)^{-1}(F(z_n(\lambda), \lambda) + G(z_n(\lambda), \lambda)), \quad z_0 \in U(x_0, R), n \geq 0 \qquad (10)$$

and

$$w_{n+1} = d_s(w_n), \quad n \geq 0, \quad w_0 = R. \qquad (11)$$

It is enough to show

$$\| y_n(\lambda) - z_n(\lambda) \| \leq w_n - v_n, \quad n \geq 0. \tag{12}$$

It is simple calculus to show that the scalar sequence given by (11) is monotonically convergent to $\rho^\star$. Hence, if for z_0 we choose the second solution $y^\star(\lambda) \in U(x_0, r)$ of equation (5) then by (12)

$$\| x^\star(\lambda) - y^\star(\lambda) \| \leq w_n - v_n.$$

That is, $x^\star(\lambda) = y^\star(\lambda)$.

For $n = 0$, (12) becomes $\| y - z_0 \| \leq R - 0 = R$. Hence, (12) is true for $n = 0$. Let us assume that (12) holds for $n \leq k$ then by (6), (10) as before we get

$$\begin{aligned}
&\| y_{k+1}(\lambda) - z_{k+1}(\lambda) \| \leq \| z_k(\lambda) - y_k(\lambda) - F'(x_0, \lambda_0)^{-1}(F(z_k(\lambda), \lambda) - \\
&- F(y_k(\lambda), \lambda)) \| + \| F'(x_0, \lambda_0)^{-1}(G(z_k(\lambda), \lambda) - G(y_k(\lambda), \lambda)) \| \leq \\
&\leq \int_0^1 \| F'(x_0, \lambda_0)^{-1}(F'((1-t)y_k(\lambda) + tz_k(\lambda)) - F'(x_0, \lambda_0)) \| \cdot \\
&\cdot \| z_k(\lambda) - y_k(\lambda) \| dt + \int_{v_k}^{w_k} K_3(t, s)dt \leq \int_0^1 w_s((1-t)v_k + tw_k) \\
&(w_k - v_k)dt + \int_{v_k}^{w_k} K_3(t, s)dt \leq K(s) \left[\int_{v_k}^{w_k} w_s(t)dt + \right. \\
&\left. + \int_{v_k}^{w_k} K_3(t, s)dt \right] = d_s(w_k) - d_s(v_k) = w_{k+1} - v_{k+1}.
\end{aligned}$$

That completes the proof of the theorem.

We can now formulate the main result:

Theorem 2. *Suppose that the hypotheses of Theorem 1 are true. Then*

(a) *the sequence $\{\rho_n\}, n \geq 0$ given by*

$$\rho_{n+1} = \rho_n + u_s(\rho_n), \quad n \geq 0, \quad \rho_0 = 0 \text{ with } u_s(r) = -\frac{\chi_s(r)}{\varphi_s'(r)}$$

is monotonically increasing and converges to $\rho^\star$.

(b) *The iterates generated by (4) are well defined for all $n \geq 0$ and remain in $U(x_0, \rho^\star)$.*

(c) *Moreover the following estimates are true*

$$\| x_{n+1}(\lambda) - x_n(\lambda) \| \leq \rho_{n+1} - \rho_n, \quad n \geq 0 \tag{13}$$

and

$$(14) \qquad \| x_n(\lambda) - x^\star(\lambda) \| \le \rho^\star - \rho_n , \quad n \ge 0.$$

PROOF. Part (a) can be shown exactly as in Proposition 3 in [10, p. 677]. We will only show (13), since (14) will follow then from it immediately. For $n = 0$ we get $\| x_1(\lambda) - x_0 \| \le a_s = \rho_1 - \rho_0$. That is, (13) is true for $n = 0$. Let us assume that (13) is true for $n < k$. By the induction hypothesis

$$\| x_k(\lambda) - x_0 \| \le \sum_{j=1}^{k} \| x_j(\lambda) - x_j(\lambda) \| \le \sum_{j=1}^{k} (\rho_j - \rho_{j-1}) = \rho_k,$$

the Banach lemma on invertible operators, (2) and the estimate

$$\begin{aligned} &\| F'(x_0,\lambda_0)^{-1}(F'(x_k(\lambda),\lambda) - F'(x_0,\lambda_0)) \| \le \\ &\qquad \le K(s) w_s(\rho_k) < K(s) w_s(\rho^\star) = \rho_s'(\rho^\star) + 1 \le 1, \end{aligned}$$

it follows that $F'(x,\lambda)$ is invertible for all $\lambda \in U(\lambda_0, S)$, $\; x \in U(x_0, R)$ and

$$\begin{aligned} &\| F'(x_k(\lambda),\lambda)^{-1} F'(x_0,\lambda_0) \| \le \\ &\le \| [I + F'(x_0,\lambda)^{-1}(F'(x,\lambda) - F'(x_0,\lambda_0)))]^{-1} \| \cdot \\ &\cdot \| F'(x_0,\lambda)^{-1} F'(x_0,\lambda_0) \| \le -\frac{K(s)}{\varphi_s'(\rho_k)}. \end{aligned}$$

Then by (4), (1)–(3), (15) and the induction hypothesis we get

$$\begin{aligned} &\| x_{k+1}(\lambda) - x_k(\lambda) \| = \| F'(x_k(\lambda),\lambda)^{-1}(F(x_k(\lambda),\lambda) + G(x_k(\lambda_),\lambda)) \| = \\ &= \| F'(x_k(\lambda),\lambda)^{-1}(F(x_k(\lambda),\lambda) - F(x_{k-1}(\lambda),\lambda) - F'(x_{k-1}(\lambda),\lambda)(x_k(\lambda) - \\ &\quad - x_{k-1}(\lambda) + G(x_k(\lambda),\lambda) - G(x_{k-1}(\lambda),\lambda))) \| \le \\ &\le F'(x_k(\lambda),\lambda)^{-1} F'(x_0,\lambda_0) \| \Big[\int_0^1 \| F'(x_0,\lambda_0)^{-1}(F'((1-t)x_{k-1}(\lambda) + \\ &\quad + \; t x_k(\lambda)) - F'(x_{k-1}(\lambda)) \| \cdot \| x_k(\lambda) - x_{k-1}(\lambda) \| \, dt + \\ &\quad + \| F'(x_0,\lambda_0)^{-1}(G(x_k(\lambda),\lambda)) - G(x_{k-1}(\lambda)\lambda)) \| \Big] \le \\ &\le -\frac{K(s)}{\varphi_s'(\rho_k)} \int_0^1 (w_s((1-t)\rho_{k-1} + t\rho_k) - w_s(\rho_{k-1}))(\rho_k - \rho_{k-1}) dt - \\ &\quad - \frac{1}{\varphi_s'(\rho_k)} (\psi_s(\rho_k) - \psi_s(\rho_{k-1})) \le \end{aligned}$$

$$\leq -\frac{\varphi_s(\rho_k) - \varphi_s(\rho_{k-1}) - \varphi_s'(\rho_{k-1})(\rho_k - \rho_{k-1}) + \psi_s(\rho_{k-1}) - \psi_s(\rho_{k-1})}{\varphi_s'(\rho_k)} =$$

$$= \rho_{k+1} - \rho_k .$$

Hence (13) is true for $n = k$. That completes the proof of the theorem.

We will now derive some a posteriori error bounds for iteration (4). Let $r_{n,s} = r_n = \| x_n(\lambda) - x_0 \|$,

$$q_{n,s}(r) = q_n(r) = K_1(r_n + r, s), \quad f_{n,s}(r) = f_n(r) = K_3(r_n + r, s)$$

for $r \in [0, R - r_n]$ and set

$$a_{n,s} = a_n = \| x_{n+1}(\lambda) - x_n(\lambda) \|, b_{n,s} = b_n = K(s)(1 - K(s)w_s(r_n))^{-1}.$$

Without loss of generality we assume than $a_n > 0$.

Then exactly as in Theorem 2 in [9, p. 989] we can show

Theorem 3. *Suppose that the hypotheses of Theorem 1 are satisfied. Then*

(a) *the equation*

$$r = a_n + b_n \int_0^r (r - t)q_n(t) + f_n(t)]dt$$

has a unique positive zero $\rho^\star_{n,s} = \rho^\star_n$ *in the interval* $[0, R - r_n]$, $n \geq 0$ *and* $\rho^\star_0 = \rho^\star$.

(b) *The following estimates are true:*

$$\begin{aligned} \| x_n(\lambda) - x^\star(\lambda) \| &\leq \rho^\star_n \\ &\leq (\rho^\star - \rho_n)a_n/\Delta\rho_n, \quad n \geq 0, \\ &\leq (\rho^\star - \rho_n)a_{n-1}/\Delta\rho_{n-1}, \quad n \geq 1, \\ &\leq \rho^\star - \rho_n, \quad n \geq 0, \end{aligned} \tag{16}$$

where $\Delta\rho_n = \rho_{n+1} - \rho_n$.

That is, our bound (16) is sharper then Miel–type bounds [3], [7] and more general than the corresponding one in [9, p. 989].

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