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On projective and injective objects in the category of abstract objects

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1. Introduction

In this paper we consider projective and injective objects in the category of abstract objects. We state that an object (X, G, F) is a projective object in the category OA iff G is a free group acting freely on the set X (Theorem 1) and we determine the form of projective objects in the subcategories of OA (Corollaries 1, 2, 3). We also determine the form of injective objects in OA and subcategories of OA (Theorems 2, 3, Corollary 4).

The present results constitute a part of the author's research on categorial properties in the theory of abstract objects (cf. [5], [6], [7]). The categorial aspects of this theory were explored by M. KUCHARZEWSKI in the paper [2].

2. Preliminaries

2.1. An abstract object (object) is a triple (X, G, F) where X is a nonempty set, G is a group and $F : X \times G \to X$ is a mapping satisfying the translation and identity conditions (cf. [3], p.12).

An equivariant mapping from the object (X, G, F) into the object (Y, H, f) is a pair (α, φ), where $\alpha : X \to Y$ is a mapping and $\varphi : G \to H$ is a homomorphism which satisfies the condition:

$$\alpha(F(x, g)) = f(\alpha(x), \varphi(g)), \quad x \in X, \quad g \in G \tag{1}$$

Composition of equivariant mappings is also defined in a well-known manner.

The category with abstract objects as "objects" and with equivariant mappings as "morphisms" is called the category OA of abstract objects (see [3], p.18–19). We also need the category PK, OAT, PJ, OS of Klein spaces, transitive abstract objects, homogeneous spaces and scalars, respectively (cf. [3], p.19; [5], p.12).

The category whose objects are the geometric objects of a fixed Klein space and whose morphisms are the equivariant mappings of the form (α, id_G) is called the Klein geometry of the group G or briefly G-geometry and is denoted by OG (cf. [3], p.10).

2.2. Let $\mathcal{B}$ be a subcategory of a category $\mathcal{A}$ such that every morphism $f \in \mathcal{B}$ is an epimorphism (monomorphism). An object $P \in \mathcal{A}^\circ$ ($I \in \mathcal{A}^\circ$) will be called a $\mathcal{B}$-projective ($\mathcal{B}$-injective) object if for each morphism $f : A \to B$, $f \in \mathcal{B}$, and each morphism $g : P \to B$ ($g : A \to I$), $g \in \mathcal{A}$, there exists a morphism $h : P \to A$ ($h : B \to I$), $h \in \mathcal{A}$, such that $fh = g$ ($hf = g$) (cf. [8], p.195–196).

In our consideration $\mathcal{B}$ will be the category of all epimorphisms (monomorphisms) of the category OA (PK, OAT, PJ, OS).

3. Projective objects in the category OA and its subcategories

Lemma ([7]). *Let (X, G, F) be an object of the category OA, where G acts freely on X. Let $\{W_s\}$, $s \in S$ (where S is a set of indices) be a family of transitive fibres of X such that $\bigcup_{s \in S} W_s = X$ and $W_s \cap W_{s'} = \emptyset$ for $s \neq s'$, $s, s' \in S$. For every fibre W_s we fix exactly one point denoted by x_s.*

For any object (Y, H, f) of OA, any homomorphism $\varphi : G \to H$ and any mapping $\alpha : X \to Y$, the pair (α, φ) is a morphism of OA iff there exists $y_s \in Y$, $s \in S$, such that

$$\alpha(x) = f(y_s, \varphi(g_x)), \quad s \in S, \tag{2}$$

where $x \in W_s$, $g_x \in G$ and $F(x_s, g_x) = x$.

Theorem 1. *An object (X, G, f) is a projective object in the category OA iff G is a free group and acts freely on X.*

Proof. Let (X, G, F) be a projective object in the category OA and $\varphi : H \to K$ be an arbitrary epimorphism of groups. Let us consider an epimorphism $(\alpha, \varphi) : (\{x\}, H, F_1) \to (\{x\}, K, F_2)$ and an arbitrary morphism $(\beta, \psi) : (X, G, F) \to (\{x\}, K, F_2)$.

The object $(\{x\}, K, F_2)$ is a scalar and therefore a morphism $(\gamma, \sigma) : (X, G, F) \to (\{x\}, H, F_1)$ such that $(\alpha, \varphi) \circ (\gamma, \sigma) = (\beta, \psi)$ exists iff there exists a homomorphism $\sigma : G \to H$ such that $\varphi\sigma = \psi$. Hence G is a free group as a projective object in the category of groups Gr.

Now let us consider an epimorphism $(\alpha, \mathrm{id}_G) : (G, G, L_G) \to (\{x\}, G, F_1)$, where $(\{x\}, G, F_1)$ is a scalar and L_G is the left translation on the group G and the morphism $(\beta, \mathrm{id}_G) : (X, G, F) \to (\{x\}, G, F_1)$.

A morphism $(\gamma, \varphi) : (X, G, F) \to (\{x\}, G, F_1)$ such that $(\alpha, \mathrm{id}_G) \circ (\gamma, \varphi) = (\beta, \mathrm{id}_G)$ is of the form (γ, id_G). So the stability group G_x of x is a subgroup of the stability group $G_{\gamma(x)}$ of $\gamma(x)$ for every $x \in X$ (cf. [9]). Because $G_{\gamma(x)} = \{e\}$, so G acts freely on X.

Now assume that in the object (X, G, F) G is a free group acting freely on X.

Let $(\alpha, \varphi) : (Y, H, F_1) \to (Z, K, F_2)$ be an arbitrary epimorphism and $(\beta, \psi) : (X, G, F) \to (Z, K, F_2)$ an arbitrary morphism in the category OA. G is a free group, so there exists a homomorphism $\sigma : G \to H$, such that

$$\varphi\sigma = \psi. \tag{3}$$

There exists also (lemma – (2)) $z_s \in Z$, $s \in S$, such that $\beta(x) = F_2 \times (z_s, \varphi(g_x))$, for $x \in X$. Because α is a surjection, there exists $y_s \in Y$, $s \in S$, such that $\alpha(y_s) = z_s$, $s \in S$. We define $\gamma : X \to Y$ by $\gamma(x) := F_1(y_s, \sigma(g_x))$, $x \in X$.

The pair $(\gamma, \sigma) : (X, G, F) \to (Y, H, F_1)$ is a morphism in OA (lemma). For every $x \in X$ we have ((1), (3)) $\alpha(\gamma(x)) = \alpha(F_1(y_s, \sigma(g_x))) = F_2(\alpha(y_s),\ \varphi(\sigma(g_x))) = F_2(z_s, \psi(g_x)) = \beta(x)$. □

Analogically we can prove

Corollary 1. *An object (X, G, F) is a projective object in the category OG iff the group G acts freely on X. An object (X, G, F) is a projective object in the category of scalars OS iff G is a free group.*

Corollary 2. *Every object (X, G, F) of the category PK (PJ) where G is a free group and acts freely on X is a projective object in the category PK (PJ).*

Because an object is 1-transitive if and only if it is transitive and the group G acts freely on X, we obtain

Corollary 3. *An object (X, G, F) is a projective object in the category OAT iff it is a 1-transitive object and G is a free group.*

4. Injective objects in the category OA and its subcategories

Theorem 2. *An object (X, G, F) is an injective object in the category OA $(PK,\ OAT,\ PJ)$ iff X is a singleton set and $G = \{e\}$.*

PROOF. Let (X, G, F) be an injective object in the category OA $(PK,\ OAT,\ PJ)$, and $\varphi : H \to K$ be an arbitrary group-monomorphism.

Let us consider the monomorphism $(\varphi,\varphi):(H,H,L_H)\to(K,K,L_K)$ (L_H and L_K are the left tarnslations on the groups H and K).
If $(\beta,\psi):(H,H,L_H)\to(X,G,F)$ is an arbitrary morphism of OA (PK, OAT,PJ) then a morphism $(\gamma,\sigma):(K,K,L_K)\to(X,G,F)$ satisfying the condition $(\gamma,\sigma)\circ(\varphi,\varphi)=(\beta,\psi)$ exists iff there exists a homomorphism $\sigma:K\to G$ such that $\sigma\varphi=\psi$. Hence $G=\{e\}$ is an injective object in the category of groups (cf. [1]).

In the case of the categories OAT and PJ we obtain that X is a singleton.

Let us consider the objects (R_*,R_+,F_1) and (R_*,R_*,F_2) where R_* and R_+ denote the multiplicative group of the real and of the positive real numbers respectively, and $F_1(x,g):=g\cdot x$ for $x\in R_*$, $g\in R_+$, $F_2(x,g):=g\cdot x$ for $x\in R_*$ and $g\in R_*$.

The morphism $(\mathrm{id}_{R_*},\varphi):(R_*,R_+,F_1)\to(R_*,R_*,F_2)$ where $\varphi(g):=g$ for $g\in R_+$ is a monomorphism in the category OA (PK). Assume that $x_1,x_2\in X$ and $x_1\neq x_2$, and consider the morphism $(\beta,\psi):(R_*,R_+,F_1)$ $\to(X,\{e\},F)$, where $\psi(g):=e$, for $g\in R_+$ and

$$\beta(x):=\begin{cases} x_1, & \text{for } x\in(-\infty,0)\\ x_2, & \text{for } x\in(0,+\infty).\end{cases}$$

There exists no morphism $(\gamma,\sigma):(R_*,R_*,F_2)\to(X,\{e\},F)$ satisfying the condition of the definition of an injective object because γ is constant on R_* (the object (R_*,R_*,F_2) is transitive).

It is easy to verify that the object $(\{x\},\{e\},F)$ is an injective object in the category OA (PK,OAT,PJ). □

Analogically we can prove

Corollary 4. *An object (X,G,F) is an injective object in the category OS if and only if $G=\{e\}$.*

In the case of the category OG we have

Theorem 3. *An object (X,G,F) is an injective object in the category OG iff X contains at least one one-element transitive fibre.*

For a proof, see [5].

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