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Simulation and representation by ν_i^*-products of automata

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1. Introduction

It is proved in [2] that the generalized product is a proper generalization of the generalized ν_1-product from the point of view of homomorphic simulation. On the basis of this result it can be seen easily that similar statement holds for the homomorphic representation. Using results in [1] and [3], this paper shows that the generalized product is equivalent to the generalized ν_i-product from the point of view of isomorphic and homomorphic simulation if and only if $i > 1$. Moreover, we prove that in the class of monotone automata the generalized product is equivalent to the generalized ν_i-product with $i > 1$ from the point of view of homomorphic representation. It remains an open problem whether or not this result can be extended to the class of all automata.

2. Basic notions

For any finite nonempty set X let X^* denote the *free monoid* of all *words* over X (including the *empty word* λ). Moreover, denote by X^+ $(= X^* - \{\lambda\})$ the *free semigroup* of all nonempty words over X. The *length* of a word $p = x_1 \dots x_n \in X^+$ is denoted by $|p|$ $(= n)$. The length of the empty word λ is zero per definitionem. Finally, we put $p^0 = \lambda,\ p^n = p^{n-1}p\ \ (p \in X^*,\ n > 0)$.

By an *automaton* we mean a system $\mathfrak{A} = (A, X, \delta)$ where A is the (nonempty finite) *set of states*, X is the (nonempty finite) *set of inputs,* and $\delta : A \times X \to A$ is the *transition function*. We extend δ to a function $A \times X^* \to A$ in the usual way, i.e.

$$\delta(a, \lambda) = a, \quad \delta(a, px) = \delta(\delta(a, p), x) \quad (a \in A,\ p \in X^*,\ x \in X).$$

We can consider an automaton a special algebraic structure. In this sense we speak about *subautomata, homomorphism,* and *isomorphism* of automata. We say that an automaton $\mathfrak{A}$ *homomorphically (isomorphically) represents* an automaton $\mathfrak{B}$ iff $\mathfrak{A}$ has a subautomaton which can be mapped homomorphically (isomorphically) onto $\mathfrak{B}$. Let $\mathfrak{A} = (A, X, \delta)$ and $\mathfrak{B} = (B, Y, \delta')$ be automata. We say that $\mathfrak{A}$ *homomorphically simulates* $\mathfrak{B}$ if there are a subset $A' \subseteq A$, a surjective mapping $h_1 : A' \to B$ and a (not necessarilly surjective) mapping $h_2 : Y \to X^*$ with

$$h_1(\delta(a, h_2(y))) = \delta'(h_1(a), y) \quad (a \in A',\ y \in Y).$$

(It is understood that $\delta(a, h_2(y)) \in A'$ holds for every pair $a \in A',\ y \in Y$.) If h_1 is bijective then $\mathfrak{A}$ *isomorphically simulates* $\mathfrak{B}$. It can be seen easily that the concept of homomorphic (isomorphic) simulation is a natural extension of that of homomorphic (isomorphic) representation.

Let $\mathfrak{A} = (A, X, \delta)$ be an automaton. We say that $\mathfrak{A}$ is *discrete* if $\delta(a, x) = a$ for all $a \in A$ and $x \in X$. $\mathfrak{A}$ is *monotone* if there is a partial ordering $\leq$ on its state set A such that $a \leq \delta(a, x)$ for all $a \in A$ and $x \in X$. Finally, we refer to the automaton

$$\mathfrak{E} = (\{0, 1\}, \{x_1, x_2\}, \delta_{\mathfrak{E}}),$$

$$\delta_{\mathfrak{E}}(0, x_1) = 0, \quad \delta_{\mathfrak{E}}(0, x_2) = \delta_{\mathfrak{E}}(1, x_1) = \delta_{\mathfrak{E}}(1, x_2) = 1$$

as the (two state) *elevator*. Obviously, the elevator is a monotone automaton.

Let $\mathfrak{A}_t = (A_t, X_t, \delta_t)$ $(t = 1, \ldots, k,\ k \geq 1)$ be automata. Take a finite nonempty set X and a system of *feedback functions*

$$\varphi_t : A_1 \times \cdots \times A_k \times X \to X_t^* \qquad (t = 1, \ldots, k).$$

We let $\mathfrak{A} = (A, X, \delta) = \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_k(X, \varphi)$ be the automaton with $A = A_1 \times \cdots \times A_k$,

$$\delta((a_1, \ldots, a_k), x) = (\delta_1(a_1, \varphi_1(a_1, \ldots, a_k, x)), \ldots, \delta_k(a_k, \varphi_k(a_1, \ldots, a_k, x)))$$

$((a_1, \ldots, a_k) \in A,\ x \in X)$. The automaton $\mathfrak{A}$ is called the *generalized product* or *g^*-product* of $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ (with respect to X and φ).

Especially, if φ_t has the form $\varphi_t : A_1 \times \cdots \times A_k \to X_t$ $(t = 1, \ldots, k)$ then we speak about a *general product* or *g-product*.

We also use the feedback functions in the following extended sense: For arbitrary $(a_1, \ldots, a_k) \in A,\ p \in X^*,\ x \in X,\ t\ (= 1, \ldots, k)$ let

$$\varphi_t(a_1, \ldots, a_k, \lambda) = \lambda,$$

$$\varphi_t(a_1, \dots, a_k, px) = \varphi_t(a_1, \dots, a_k, p)\varphi_t(b_1, \dots, b_k, x)$$

where

$$b_s = \delta_s(a_s, \varphi_s(a_1, \dots, a_k, p)) \qquad (1 \le s \le k).$$

Let i be an arbitrary natural number. Moreover, let us given a g^*-product $\mathfrak{A} = \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_k(X, \varphi)$ such that for each t $(= 1, \dots, k)$ a set $\gamma(t) \subseteq \{1, \dots, k\}$ with $|\gamma(t)| \le i$ is specified, so that φ_t does not depend on the state variables a_s with $s \notin \gamma(t)$ $(1 \le s \le k)$. Then we write $\mathfrak{A} = \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_k(X, \varphi, \gamma)$ and call $\mathfrak{A}$ a *generalized ν_i-product* or *ν_i^*-product*. Especially, if we have the form $\varphi_t : A_1 \times \cdots \times A_k \times X \to X_t$ $(t = 1, \dots, k)$ then $\mathfrak{A}$ is a *ν_i-product*. In addition, if $X_1 = \cdots = X_k = X$ and $\varphi_t(a_1, \dots, a_k, x) = x$ $(t = 1, \dots, n,\ (a_1, \dots, a_k) \in A_1 \times \cdots \times A_k,\ x \in X)$ then we speak about the *direct product* $\mathfrak{A}_1 \times \cdots \times \mathfrak{A}_k$.

If every component of a product (generalized product) of automata is the same then it is a *power* (*generalized power*) of automata.

By a class $\mathcal{K}$ of automata we shall always mean a nonempty class. Let $\mathcal{K}$ be a class of automata. We say that $\mathcal{K}$ is *isomorphically (homomorphically) S-complete* with respect to the g^*-product (g-product, ν_i^*-product, ν_i-product) if every automaton can be simulated isomorphically (homomorphically) by a g^*-product (g-product, ν_i^*-product, ν_i-product) with components from $\mathcal{K}$. The following results hold.

Theorem 2.1 (Gécseg [4], [5]). *Let $\mathcal{K}$ be a class of automata. $\mathcal{K}$ is isomorphically (homomorphically) S-complete with respect to the g^*-product iff $\mathcal{K}$ contains a nonmonotone automaton.*

Theorem 2.2 (Dömösi–Ésik [1], Dömösi–Imreh [3]). *Let $\mathcal{K}$ be a class of automata. $\mathcal{K}$ is isomorphically (homomorphically) S-complete with respect to the ν_1^*-product iff $\mathcal{K}$ contains a nonmonotone automaton.*

Some more notation. Let $\mathcal{K}$ be a class of automata. We define the following classes.

$\mathbf{P}_g(\mathcal{K})$:= all g-products of automata from $\mathcal{K}$;

$\mathbf{P}_g^*(\mathcal{K})$:= all g^*-products of automata from $\mathcal{K}$;

$\mathbf{P}_{\nu_i}(\mathcal{K})$:= all ν_i-products of automata from $\mathcal{K}$;

$\mathbf{P}_{\nu_i}^*(\mathcal{K})$:= all ν_i^*-products of automata from $\mathcal{K}$;

$\mathbf{IS}(\mathcal{K})$:= all automata which can be represented isomorphically by automata from $\mathcal{K}$;

$\mathbf{HS}(\mathcal{K})$:= all automata which can be represented homomorphically by automata from $\mathcal{K}$;

$\mathbf{IS}^*(\mathcal{K})$:= all automata which can be simulated isomorphically by automata from $\mathcal{K}$;

$\mathbf{HS}^*(\mathcal{K})$:= all automata which can be simulated homomorphically by automata from $\mathcal{K}$.

Let $\mathbf{O}_1$ and $\mathbf{O}_2$ be one of the operators $\mathbf{IS}, \mathbf{HS}, \mathbf{IS}^*, \mathbf{HS}^*$ and $\mathbf{P}_g, \mathbf{P}_g^*$, $\mathbf{P}_{\nu_i}, \mathbf{P}_{\nu_i}^*$ $(i = 1, 2 \dots)$. For every class $\mathcal{K}$ of automata we define $\mathbf{O}_1\mathbf{O}_2(\mathcal{K})$ as the class $\mathbf{O}_1(\mathbf{O}_2(\mathcal{K}))$. We shall use the following consequence of results in [4] and [5].

Theorem 2.3 (Gécseg [4], [5]). $\mathbf{IS}^*\mathbf{P}_g^*(\{\mathfrak{E}\})$ *is the class of all monotone automata (where* $\mathfrak{E}$ *denotes the elevator).*

Theorem 2.4 (Dömösi–Gécseg [2]). $\mathbf{HS}^*\mathbf{P}_{\nu_1}^*(\{\mathfrak{E}\})$ *is not the class of all monotone automata. Therefore,* $\mathbf{HS}^*\mathbf{P}_{\nu_1}^*(\{\mathfrak{E}\})$ *is a proper subclass of* $\mathbf{HS}^*\mathbf{P}_g^*(\{\mathfrak{E}\})$.

3. ν_i^*-product and simulation

We start our investigations with

Theorem 3.1. *Every monotone automaton can be simulated isomorphically by a* ν_2*-power of the elevator.*

Proof. Let $\mathfrak{A} = (A, X, \delta)$ be a monotone automaton and denote by $\leq$ a partial ordering on A with $a \leq \delta(a, x)$ $(a \in A,\ x \in X)$. Take an arrangement $a_1, \dots, a_n$ of elements of A for which $a_i \neq a_j$ and $a_i \leq a_j$ imply $i < j$ $(1 \leq i, j \leq n)$. Then it is clear that $\delta(a_t, x) \notin \{a_1, \dots, a_{t-1}\}$ $(a_t \in A,\ x \in X)$. We construct an automaton $\mathfrak{B}$ which isomorphically simulates $\mathfrak{A}$, where $\mathfrak{B}$ is a subautomaton of a ν_2-power of $\mathfrak{E}$. If $n \leq 2$, then such a $\mathfrak{B}$ obviously exists. Thus we may suppose that $n > 2$. Let us use the short notation $d_1 \dots d_{2n}$ for $(d_1, \dots, d_{2n}) \in \{0,1\}^{2n}$ and let $B = \{1^t 0^{n-t} 1^t 0^{n-t} \mid t = 1, \dots, n\}$ $(\subseteq \{0,1\}^{2n})$. Moreover, let

$$B' = \{1^{s+t} 0^{n-s-t} 1^t 0^{n-t} \mid t = 1, \dots, n,\ s = 0, \dots, n-t\}\ (\supseteq B)$$

and for arbitrary $1^{s+t}0^{n-s-t}1^t0^{n-t} \in B'$ use the short notation $b_{s+t}b_t$. Construct the automaton $\mathfrak{B} = (B', X', \delta')$, where $X' = A \times X \cup A \cup \{*\}$ ($*$ is arbitrary with $* \notin A \cup A \times X$). Furthermore, for all $b_t b_t \in B$ $(\subseteq B')$, $b_{s+t}b_t \in B'$, $(a_r, x) \in A \times X$, $a_r \in A$,

$$\delta'(b_t b_t, (a_r, x)) = b_{t+1} b_t, \quad \text{if } r > t \text{ and } \delta(a_t, x) = a_r,$$

$$\delta'(b_{s+t} b_t, a_r) = b_{s+t+1} b_t, \quad \text{if } s > 0 \text{ and } r > s + t,$$

$$\delta'(b_{s+t} b_t, *) = b_{s+t} b_{s+t},$$

and in all other cases

$$\delta'(b', x') = b' \quad (b' \in B',\ x' \in X').$$

We show that for arbitrary $b_t b_t \in B\,(\subseteq B'),\ a_r \in A,\ x \in X$

(i) $\delta'(b_t b_t, (a_r, x)a_r^{n-2}*) = b_r b_r, \quad \text{if } \delta(a_t, x) = a_r,$

(ii) $\delta'(b_t b_t, (a_r, x)a_r^{n-2}*) = b_t b_t, \quad \text{if } \delta(a_t, x) \neq a_r.$

For this, observe that $\delta'(b_t b_t, (a_r, x)) = b_t b_t$ if $r \leq t$ or $\delta(a_t, x) \neq a_r$. Furthermore, $\delta'(b_t b_t, (a_r, x)) = b_{t+1} b_t$, if $r > t$ and $\delta(a_t, x) = a_r$. It is also clear that $\delta'(b_t b_t, a_r^{n-2}) = b_t b_t$. Moreover, if $r > t$, then $\delta'(b_{t+1} b_t, a_r^{n-2}) = b_r b_t$. Finally, $\delta'(b_t b_t, *) = b_t b_t$, and if $r \geq t$, then $\delta'(b_r b_t, *) = b_r b_r$. Taking into consideration

$$\delta(a_t, x) \notin \{a_1, \ldots, a_{t-1}\} \quad (a_t \in A, x \in X),$$

we obtain that our construction has properties (i) and (ii) above. This means that under

$$p_x = (a_n, x)a_n^{n-2} * (a_{n-1}, x)a_{n-1}^{n-2} * \ldots * (a_1, x)a_1^{n-2} * \ (\in (X')^+)$$

we have $\delta'(b_t b_t, p_x) = b_r b_r$ if and only if $\delta(a_t, x) = a_r$. Define $h_1 : B \to A$ $(B \subseteq B')$ and $h_2 : X \to (X')^*$ by

$$h_1(b_t b_t) = a_t, \quad h_2(x) = p_x \quad (b_t b_t \in B,\, x \in X).$$

Therefore, for arbitrary $b_t b_t \in B$, $x \in X$,

$$h_1(\delta'(b_t b_t, h_2(x))) = \delta(h_1(b_t b_t), x).$$

Thus, $\mathfrak{B}$ isomorphically simulates $\mathfrak{A}$ (with respect to h_1 and h_2).

Now we show that $\mathfrak{B}$ is a subautomaton of a ν_2-power $\mathfrak{E}^{2n}(X', \varphi, \gamma)$ of $\mathfrak{E}$. Let

$$\gamma(1) = \emptyset, \quad \gamma(t) = \{t-1, n+t-1\} \quad (t = 2, \ldots, n),$$
$$\gamma(n+s) = \{s\} \quad (s = 1, \ldots, n).$$

Moreover, for arbitrary $(l_1, \ldots, l_{2n}) \in \{0,1\}^{2n},\ a_r \in A,\ x \in X$, let

$\varphi_{t+1}(l_1, \ldots, l_{2n}, (a_r, x)) = x_2,$

if $l_t = l_{n+t} = 1,\ r \geq t+1$ and $\delta(a_t, x) = a_r \quad (t = 1, \ldots, n-1),$

$\varphi_{t+1}(l_1, \ldots, l_{2n}, a_r) = x_2,$

if $l_t = 1,\ l_{n+t} = 0$ and $r \geq t+1 \quad (t = 1, \ldots, n-1),$

$\varphi_{n+t}(l_1, \ldots, l_{2n}, *) = x_2, \quad \text{if } l_t = 1 \quad (t = 1, \ldots, n).$

In all other cases let

$$\varphi_s(l_1,\ldots,l_{2n},x') = x_1 \ ((l_1,\ldots,l_{2n}) \ \in \{0,1\}^{2n}, \ x' \in X', \ s = 1,\ldots,2n).$$

Denote by δ'' the transition function of the ν_2-power $\mathfrak{E}^{2n}(X',\varphi)$, and let $l_1 \ldots l_{2n} \in B'$, $x' \in X'$ be arbitrary. It is easy to show that $\delta''(l_1 \ldots l_{2n}, x') \neq l_1 \ldots l_{2n}$ holds in the following cases only.

(1) $l_1 \ldots l_{2n} = b_t b_t$ $(t \in \{1,\ldots,n\})$, $x' = (a_r, x) \in A \times X$, $t < r \leq n$ and $\delta(a_t, x) = a_r$.
Then $\delta''(l_1 \ldots l_{2n}, x') = b_{t+1} b_t$.

(2) $l_1 \ldots l_{2n} = b_{s+t+1} b_t$ $(t \in \{1,\ldots,n-1\}, \ s \in \{0,\ldots,n-t-1\})$, $x' = a_r \in A, \ s+t+1 < r$.
Then $\delta''(l_1 \ldots l_{2n}, x') = b_{s+t+2} b_t$.

(3) $l_1 \ldots l_{2n} = b_{s+t+1} b_t$ $(t \in \{1,\ldots,n-1\}, \ s \in \{0,\ldots,n-t-1\})$, $x' = *$.
Then $\delta''(l_1 \ldots l_{2n}, x') = b_{s+t+1} b_{s+t+1}$.

Thus we obtained that the transition function δ' of $\mathfrak{B}$ is the restriction of δ'' to $B' \times X'$. □

Now we are ready to prove

Theorem 3.2. *The generalized ν_2-product is equivalent to the generalized product from the point of view of homomorphic (isomorphic) simulation.*

PROOF. Let $\mathcal{K}$ be any class of automata. If $\mathcal{K}$ has a nonmonotone automaton then by Theorem 2.1 and Theorem 2.2 our statement holds. If $\mathcal{K}$ has only discrete automata then Theorem 3.2 is holding trivially. Otherwise $\mathcal{K}$ is a class of monotone automata in which there exists an $\mathfrak{A} = (A, X, \delta)$ with $a \neq \delta(a,x)$ and $\delta(a,xx) = \delta(a,x)$ for some $a \in A, \ x \in X$. Obviously, $\mathfrak{A}$ isomorphically simulates $\mathfrak{E}$ under the mappings

$$h_1 : \{a, \delta(a,x)\} \to \{0,1\} \quad \text{and} \quad h_2 : \{x_1, x_2\} \to X^*$$

given by

$$h_1(a) = 0, \quad h_1(\delta(a,x)) = 1, \quad h_2(x_1) = \lambda \quad \text{and} \quad h_2(x_2) = x.$$

From this it trivially follows that every ν_2-power of $\mathfrak{E}$ can be simulated isomorphically by a ν_2^*-power of $\mathfrak{A}$. Thus, using Theorem 3.1, we obtain that all monotone automata can be simulated isomorphically by a ν_2^*-power of $\mathfrak{A}$ which, by Theorem 2.3, completes the proof. □

Obviously,

$$\mathbf{HS}^*\mathbf{P}^*_{\nu_i}(\mathcal{K}) \subseteq \mathbf{HS}^*\mathbf{P}^*_{\nu_{i+1}}(\mathcal{K}) \subseteq \mathbf{HS}^*\mathbf{P}^*_{g}(\mathcal{K})$$

for every class $\mathcal{K}$ of automata. Therefore, by Theorem 2.4, we have

Corollary 3.3. *The generalized ν_i-product is equivalent to the generalized product from the point of view of homomorphic (isomorphic) simulation if and only if $i > 1$.*

4. ν_i^*-product and homomorphic representation

For a fixed X, let $\mathcal{L}_X$ be the class of all automata $\mathfrak{A}=(\{0,\dots,n\},X,\delta)$ $(n=1,2\dots)$, $\delta(0,x)=0$, $\delta(n,x)=n$ and

$$\delta(j,x)\in\begin{cases}\{j,j+1\}, & \text{if } 0<j<n-1,\\ \{0,n-1,n\}, & \text{if } j=n-1 \text{ and } n>1\end{cases}$$

for all $x\in X$. We have

Lemma 4.1. *Every automaton in $\mathcal{L}_X$ can be represented isomorphically by a ν_2-power of the elevator.*

Proof. Let $\mathfrak{A}=(\{0,\dots,n\},X,\delta)\in\mathcal{L}_X$. If $n=1$, then $\mathfrak{A}$ can be represented isomorphically by a quasi-direct power of the elevator with a single factor. Thus, we may suppose that $n>1$.

Consider the ν_2-power $\mathfrak{E}^{n+1}(X,\varphi,\gamma)$ of $\mathfrak{E}$ given in the following way: Let $\gamma(1)=\emptyset$, $\gamma(t)=\{t-1\}$ if $1<t\le n$, and $\gamma(n+1)=\{n-1,n\}$. Moreover, for arbitrary $(l_1,\dots,l_{n+1})\in\{0,1\}^{n+1}$, $x\in X$ and t $(=1,\dots,n+1)$,

$$\varphi_t(l_1,\dots,l_{n+1},x)=\begin{cases}x_2, & \text{if } 1<t<n,\ l_{t-1}=1 \text{ and } \delta(t-1,x)=t,\\ & \text{or}\\ & t=n,\ l_{n-1}=1 \text{ and } \delta(n-1,x)\in\{0,n\},\\ & \text{or}\\ & t=n+1,\ l_{n-1}=1,\ l_n=0 \text{ and } \delta(n-1,x)=\\ & =0,\\ x_1 & \text{otherwise.}\end{cases}$$

One can verify by a trivial computation that the mapping $h: A\to\{0,1\}^{n+1}$ given by

$$h(i)=\begin{cases}(1^i0^{n+1-i}), & \text{if } 1\le i\le n,\\ (1^{n+1}), & \text{if } i=0\end{cases}$$

is an isomorphism of $\mathfrak{A}$ into $\mathfrak{E}^{n+1}(X,\varphi,\gamma)$. $\square$

Lemma 4.2. *Every monotone automaton can be represented homomorphically by a direct product of automata from* $\mathcal{L}_X$.

Proof. Let $\mathcal{M}_{X,n}$ be the subset of all monotone automata with input alphabet X and at most n states. Moreover, $\mathcal{L}_{X,n}$ consists of all automata from $\mathcal{L}_X$ having at most $n+1$ states. We shall show that $\mathcal{M}_{X,n}$ is contained by the equational class generated by $\mathcal{L}_{X,n}$. Since $\mathcal{L}_{X,n}$ is a finite class of finite automata and every automaton in $\mathcal{M}_{X,n}$ is finite, this will imply that each automaton in $\mathcal{M}_{X,n}$ is a homomorphic image of a subautomaton of a direct product with finitely many factors from $\mathcal{L}_{X,n}$.

In order to prove the above claim it is enough to show that if an equation does not hold in an automaton from $\mathcal{M}_{X,n}$, then there is an automaton in $\mathcal{L}_{X,n}$ in which the given equation does not hold either.

Let $\mathfrak{A} = (A, X, \delta) \in \mathcal{M}_{X,n}$ be arbitrary, and denote by $\leq$ a partial ordering on A for which $a \leq \delta(a, x)$ $(a \in A,\ x \in X)$. Assume that an equation $zp = zq$ does not hold in $\mathfrak{A}$. Then there is an $a_1 \in A$ such that $\delta(a_1, p) \neq \delta(a_1, q)$. Let $\{a_1, \dots, a_k\}$ be the set of all states which can be given in the form $\delta(a_1, p')$, where p' is a prefix of p. The set $\{b_1(= a_1), b_2, \dots, b_l\}$ of states is given in a similar way for q. We may suppose that $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_l$. Let us distinguish the following cases.

(i) $k < l$ and $a_i = b_i$ $(i = l, \dots, k)$.

(ii) $l < k$ and $a_i = b_i$ $(i = 1, \dots, l)$.

(iii) None of (i) and (ii) holds.

In case (i) take the automaton $\mathfrak{B} = (B, X, \delta')$ with $B = \{0, 1, \dots, l\}$. Moreover, for all i $(1 \leq i < l)$ and $x \in X$, $\delta'(i, x) = i+1$ iff $\delta(b_i, x) = b_{i+1}$. In all other cases $\delta'(i, x) = i$. Then $\mathfrak{B} \in \mathcal{L}_{X,n}$ and $\delta'(1, p) = a_k \neq b_l = \delta'(1, q)$.

Case (ii) can be treated in a similar way.

Finally, it can be verified in a trivial manner that in case (iii) there is an $i < \min\{k, l\}$ such that $a_j = b_j$ $(j = 1, \dots, i)$ and the elements of $\{a_i, a_{i+1}, b_{i+1}\}$ are pairwise distinct.

Now let $\mathfrak{B} = (B, X, \delta')$ be the following automaton: $B = \{0, 1, \dots, i+1\}$, $\delta'(u, x) = u + 1$ iff $1 \leq u \leq i$ and $\delta(a_u, x) = a_{u+1}$, $\delta'(i, x) = 0$ iff $\delta(a_i, x) = b_{i+1}$. In all other cases $\delta'(i, x) = i$. Obviously, $\mathfrak{B} \in \mathcal{L}_{X,n}$ and $\delta'(1, p) = i + 1 \neq 0 = \delta'(1, q)$.

In all of the above three cases we found an automaton $\mathfrak{B} = (B, X, \delta')$ in $\mathcal{L}_{X,n}$ such that $\delta'(1, p) \neq \delta'(1, q)$. Therefore, the equation $zp = zq$ does not hold in $\mathcal{L}_{X,n}$.

Next take $\mathfrak{B} = (B, X, \delta')$ with $B = \{0, 1\}$, $\delta'(0, x) = 0$ and $\delta'(1, x) = 1$, where $x \in X$ is arbitrary. Then $\mathfrak{B} \in \mathcal{L}_{X,n}$. Moreover, for every $p \in X^*$, $\delta'(0, p) = 0 \neq 1 = \delta'(1, p)$. Therefore, none of the equations of the form $z_1 p = z_2 q$ $(p, q \in X^*)$ holds in $\mathcal{L}_{X,n}$. Since automata equations have only the forms $zp = zq$ and $z_1 p = z_2 q$ $(p, q \in X^*)$, this ends the proof of Lemma 4.2. $\square$

By Lemma 4.1. and 4.2 we obtain

Theorem 4.3. *Every monotone automaton can be represented homomorphically by a ν_2-power of the elevator.*

By an easy proof one can show the following consequence of this result.

Corollary 4.4. *If $\mathcal{K}$ is a class of monotone automata then*

$$\mathbf{HSP}^*_{\nu_2}(\mathcal{K}) = \mathbf{HSP}^*_g(\mathcal{K}).$$

It remains an open problem whether a similar statement holds for an arbitrary class of automata.

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