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Necessary and sufficient conditions for imbedding in the theory of best approximation by polynomials

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1. Introduction

The imbedding of Hölder's class of functions into another class of functions has been considered by P. L. ULJANOV [9], M. TIMAN [8], L. LEINDLER [7]. By virtue of Theorem 1 in [8] we can deduce a sufficient condition for the imbedding of that type in the case of a system satisfying a Nikolskii–type inequality. It remained open to discuss the necessity of the condition for a given special system. In the case of the trigonometric system, Timan [8] gave the solution of this problem by a very nice example. In [5], by another method, we considered a necessary condition in the case of the system of orthonormal polynomials with respect to the exponential weight. In the present paper we investigate this problem for the system of Jacobi polynomials.

2. Sufficient and necessary conditions for imbedding

Let $P_n(u,v,x)$ be the n–th orthonormal Jacobi polynomial with respect to the parameters $u,v>-1$. Then the system

$$\{J_n(u,v,\theta)\} := \left\{(1-\cos\theta)^{\frac{u+1/2}{2}}(1+\cos\theta)^{\frac{v+1/2}{2}}P_n(u,v,\cos\theta)\right\}$$

is orthonormal on $[0,\pi]$. Let $L^P[0,\pi]$ be the usual Banach space of measurable functions on $[0,\pi]$ with the norm $\|f\|_p=\left\{\int\limits_0^\pi |f(x)|^p dx\right\}^{1/p}$ $(1\le p<\infty)$. For a function $f\in L^p[0,\pi]$, let

$$E_n(f)_p = E_n(u,v,f)_p =$$

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$$= \inf_{(\lambda_k)} \|f - \sum_{k=0}^{n} \lambda_k \mathcal{J}_k(u,v)\|_p \qquad (n = 0, 1, \dots)$$

Let $\alpha = \{\alpha_n \downarrow 0\}$ be a decreasing sequence of nonnegative numbers tending to zero.
Define

$$E_n^{u,v}(\alpha, p) = \{f \in L^p[0,\pi] : E_n(u,v,f) \le c(f)\alpha_n\}$$

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where (and later) $c(f)$ $(c(x, \dots))$ denotes a constant depending only on its ent variables, furthermore c will denote an absolute constant (may be different Ni-kolskii–type inequality (see definition in [8]). Therefore we obtain immediately from Theorem 1 of [8] the following

Theorem 1. *Let* $1 \le p < q < \infty, \quad u, v > -1.$ *If*

$$\sum_{n=1}^{\infty} n^{q/p-2} \alpha_n^q < \infty, \tag{1}$$

then

$$E^{u,v}(\alpha, p) \subset L^q[0,\pi]. \tag{2}$$

For an investigation of the necessity of condition (1), we distinguish two cases : $q \le 2$ and $2 < q < \infty$.

Theorem 2. *Let* $1 \le p < q \le 2$ *and* $u, v \ge -1/2$. *Then* (1) *is necessary for imbedding* (2).

Proof. We apply the method used in [8]. Suppose that (1) does not hold. In [8] the author introduced the numbers:

$$a_n = \left\{ \sum_{\nu=n}^{\infty} \frac{(\nu + n - 1)(\alpha_\nu^p - \alpha_{\nu+1}^p)}{(\nu+1)} \right\}^{1/p} \qquad (n = 1, 2, \dots)$$

and proved that

$$\left\{ a_{n+1}^p (n+1)^{p-1} + \sum_{\nu=n+1}^{\infty} a_\nu^p \, \nu^{p-2} \right\}^{1/p} \le \alpha_n \qquad (n = 1, 2, \dots) \tag{3}$$

and

$$\sum_{n=1}^{\infty} a_n^q \, n^{q-2} = \infty. \tag{4}$$

Now, for $M > n+1 > 1$ we have by Abel's transform

$$\sum_{\nu=n+1}^{M} a_\nu \mathcal{J}_\nu(\theta)\mathcal{J}_\nu(0) =$$

(5)

$$= \sum_{\nu=n+1}^{M} \phi_\nu(\theta)(a_\nu - a_{\nu+1}) - \phi_{M-1}(\theta)a_{M-1} + \phi_M(\theta)a_M$$

where ϕ_ν is determined by the Christoffel–Darboux formula:

$$\phi_\nu(\theta) = \sum_{i=0}^{\nu} \mathcal{J}_i(\theta)\mathcal{J}_i(0) =$$

$$= \frac{\gamma_\nu}{\gamma_{\nu+1}} \;\frac{\mathcal{J}_\nu(0)\mathcal{J}_{\nu+1}(\theta) - \mathcal{J}_{\nu+1}(\theta)\mathcal{J}_\nu(0)}{\theta},$$

(see. e.g. [2], p.24). Since for $u, v \geq -1/2$, the system $\{\mathcal{J}_n(u,v)\}$ is bounded on $[0,\pi]$ (see [1], T.1.1.), and $|\gamma_\nu/\gamma_{\nu+1}| \leq c \ \ (\nu = 0,1,\dots)$ (see [2], p.49), from (5) we have

$$\left|\sum_{\nu=n+1}^{\infty} a_\nu \mathcal{J}_\nu(0)\mathcal{J}_\nu(\theta)\right| \leq c\, a_{n+1}/\theta \qquad (0 < \theta \leq \pi).$$

This means that the series

$$f(\theta) := \sum_{\nu=0}^{\infty} a_\nu \mathcal{J}_\nu(0)\mathcal{J}_\nu(\theta)$$

is convergent at every $0 < \theta \leq \pi$ and for its partial sum $s_n(\theta)$ the estimate

$$|f(\theta) - s_n(\theta)| \leq c\, a_{n+1}/\theta \qquad (0 < \theta \leq \pi, \ \ n = 1,2,\dots)$$

holds. By the last estimate, using the totally analogous method applied in [4], p.60 we get

$$E_n(u,v,f)_p \leq c \left\{ a_{n+1}^p (n+1)^{p-1} + \sum_{\nu=n+1}^{\infty} a_\nu^p\, \nu^{p-2} \right\}^{1/p}.$$

So, by (3)

$$E_n(u,v,f)_p \leq c\,\alpha_n \qquad (n + 0,1,\dots)$$

which means that $f \in E^{u,v}(\alpha, p)$.

On the other hand, since the system $\{\mathcal{J}_n(u,v)\}$ is bounded, we have by (4) and Paley's theorem (see. e.g. [10], p.120) that $f \notin L^q[0,\pi]$ and so the proof of T.1. is completed.

Remark 1. For $q > 2$ we can not apply the above method to the proof of the necessity of condition (1). However, by an additional condition on $\{\alpha_n\}$ we have

Theorem 3. *Let* $1 \leq p < q < \infty$; $u, v \geq -1/2$. *Let* $\alpha = \{\alpha_n \downarrow 0\}$ *be a sequence satisfying*

$$n\alpha_n \leq c\, m\, \alpha_m \quad \text{for} \quad 1 \leq n < m. \tag{6}$$

Then (1) is necessary for imbedding (2).

PROOF. Suppose that (1) does not hold, that is

$$\sum_{n=1}^{\infty} n^{q/p-2} \alpha_n^q = \infty. \tag{7}$$

By virtue of Lemma 5 in [7] (applied to the interval [1/4, 5/4] and the case $\phi_n \equiv 1$), there exists a function $f_0 \in L^p[1/4, 5/4]$ having the following properties

$$f_0(x) = 0, \qquad x \in [3/4, 5/4] \tag{8}$$

$$\int_{1/4}^{1/4+h} |f_0(x)|^p dx \leq c\, \alpha_{2^k}^p \qquad (0 < h < 2^{-(k+2)},\ k = 1, 2, \dots) \tag{9}$$

$$\omega(f_0, 2^{-k})_p \leq c\, \alpha_{2^k}, \qquad k = 1, 2, \dots \tag{10}$$

where

$$\omega(f_0, \delta)_p := \sup_{0<h\leq\delta} \left(\int_{1/4}^{5/4-h} |f(x+h) - f(x)|^p dx \right)^{1/p} \qquad (0 < \delta < 1),$$

$$f_0 \notin L^q[1/4, 5/4] \tag{11}$$

Now, we introduce the function

$$g_0(\theta) = \begin{cases} f_0(\theta)\rho_{u,v}^p(\theta), & \theta \in [1/4, 5/4] \\ 0, & \theta \in [0, \pi] \setminus [1/4, 5/4] \end{cases} \tag{12}$$

where

$$\rho_{u,v}(\theta) = (1 - \cos\theta)^{\frac{1+u}{2}} (1 + \cos\theta)^{\frac{1+v}{2}}$$

We compute the modulus defined in [6] for this function:

$$\Omega(g_0,\delta) = \sup_{0<h\le\delta}\left(\int_0^2 |g_0\rho_{u,v}^{-p}(x+h) - g_0\rho_{u,v}^{-p}(x)|^p \rho_{u,v}^p(x)dx\right)^{1/p} +$$

$$+\sup_{0<h\le\delta}\left(\int_{3/2}^{\pi} |g_0\rho_{u,v}^{-p}(x-h) - g_0\rho_{u,v}^{-p}(x)|^p \rho_{u,v}^p(x)dx\right)^{1/p} \qquad (0<\delta\le 1).$$

From (8), (9), (10) and (12) by simple computation we get

$$\Omega(g_0, 2^{-k}) \le c\,2^{-k} \qquad (k\ge 2).$$

From this, using Theorem 3 in [6] we obtain

$$E_{2^k}(u,v,g_0)_p \le c\,\Omega(g_0,2^{-k}) \le c\,\alpha_{2^k} \qquad (k\ge 2).$$

The last estimate implies together with (6) that

$$E_n(u,v,g_0)_p \le c\,\alpha_n \qquad (n=0,1,\dots),$$

and so $g_0 \in E^{u,v}(\alpha,p)$.

Finally, from (11), (12) and the fact that $\rho_{u,v}(\theta) \ge c(u,v)$ (>0) for $\theta\in[1/4,5/4]$, it follows that $g_0 \notin L^q[0,\pi]$. So, the necessity of condition (1) is proved.

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