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Sasakian space forms and geodesic spheres and tubes

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Abstract. We prove that a connected Sasakian manifold of dimension at least five is a Sasakian space form if and only if its small geodesic spheres (resp. geodesic tubes) satisfy certain specific conditions of Ricci-semi-symmetric or semi-parallel type.

1. Introduction

The idea to investigate Ricci-semi-symmetry and semi-parallelity conditions on geodesic spheres and tubes originated from [3]. In this article it is proved that a connected Riemannian manifold of dimension at least four is a real space form if and only if its small geodesic spheres or tubes are semi-parallel or semi-symmetric. In two later papers it is also shown that these conditions can be adapted to characterize complex space forms [11] and quaternionic space forms [12].

We recall that a hypersurface is *semi-parallel* if $\tilde{R}_{XY} \cdot \sigma = 0$, and that it is *Ricci-semi-symmetric* if $\tilde{R}_{XY} \cdot \tilde{\rho} = 0$ for all vectors X, Y tangent to the hypersurface. Here, $\tilde{R}$, $\tilde{\rho}$ and σ denote the Riemann curvature tensor, the corresponding Ricci tensor of type $(0,2)$ and the second fundamental form of the hypersurface concerned and $\tilde{R}_{XY}$ acts as a derivation.

In this paper the Sasakian case is dealt with. As in [11], [12] we have also here to include some natural restrictions both on the tangent vectors X, Y as on the points of the spheres or tubes where we consider these vectors.

First, the vectors X, Y not only have to be horizontal in the Sasakian sense, that is, orthogonal to the characteristic vector field ξ of the Sasakian manifold $(M^n, g, \varphi, \xi, \eta)$, but also orthogonal to $\varphi\gamma'$, where γ denotes the

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radial geodesic leading to the point at which X, Y are tangent to the sphere or tube.

Further, we only consider both conditions at special points of the hypersurfaces. For geodesic spheres, these are the intersection points with horizontal geodesic rays and are called *φ-geodesic points*. For geodesic tubes we distinguish two cases: if the center geodesic ϑ of the tube is tangent to ξ, we again look at φ-geodesic points; if ϑ is a horizontal geodesic, we restrict to these points who lie on the geodesic ray with initial velocity $\varphi\dot{\vartheta}$.

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2. Preliminaries

In this section we collect some basic material concerning Sasakian manifolds, spheres and tubes. Let (M, g) be an n-dimensional, connected, smooth Riemannian manifold. Denote by ∇ the Levi Civita connection and by R and ρ the corresponding Riemann curvature tensor and Ricci tensor, respectively. We use the sign convention

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for tangent vector fields X, Y on M.

In this paper we always consider *Sasakian manifolds*. They are characterized by the fact that they admit a unit Killing vector field ξ such that the Riemann curvature tensor satisfies the condition

$$R_{XY}\xi = \eta(X)Y - \eta(Y)X \tag{1}$$

for all tangent vectors X, Y on M and where η denotes the metric dual one-form of ξ, that is, $\eta(X) = g(X, \xi)$. The vector field ξ is called the *characteristic vector field* of the Sasakian manifold. Another important structure tensor on M is defined by

$$\varphi = -\nabla\xi. \tag{2}$$

This (1,1)-tensor has the properties

$$\varphi^2 = -I + \eta \otimes \xi, \tag{3}$$

$$\varphi\xi = 0 = \eta \circ \varphi, \tag{4}$$

$$(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{5}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{6}$$

From (5) it follows by direct computation that

$$R_{XY}\varphi Z = \varphi R_{XY}Z + g(\varphi Y, Z)X - g(\varphi X, Z)Y + g(Y, Z)\varphi X - g(X, Z)\varphi Y \tag{7}$$

and with respect to the Ricci tensor we have the following relations:

$$\rho(X, \xi) = 2\ell\, \eta(X), \tag{8}$$

$$\rho(\varphi X, \varphi Y) = \rho(X, Y) - 2\ell\, \eta(X)\eta(Y), \tag{9}$$

where $\dim M = 2\ell+1$. For more details about Sasakian geometry we refer to [1] and [20].

Since ξ is a unit vector Killing field, its integral curves are geodesics, called *ξ-geodesics*. Further, this implies that a geodesic which is orthogonal to ξ at one of its points, remains orthogonal to ξ at all of its points. These geodesics are called *φ-geodesics* or *horizontal* geodesics, since tangent vectors X on M are called *horizontal* if $\eta(X) = g(X, \xi) = 0$.

Further, we note that Sasakian manifolds can be fibred locally over Kählerian base spaces, the ξ-geodesics acting as fibers. (See [16] for more details.) For a sufficiently small neighbourhood $\mathcal{U}$ in M, we thus have a mapping $\pi : \mathcal{U} \to \mathcal{U}/\xi = \bar{\mathcal{U}}$ which induces a Kähler structure $(J, \bar{g})$ on $\bar{\mathcal{U}}$ by

$$(JX)^* = \varphi X^*, \tag{10}$$

$$\bar{g}(X, Y) = g(X^*, Y^*), \tag{11}$$

where X, Y are tangent vectors on $\bar{\mathcal{U}}$ and X^* denotes a horizontal lift of $X \in T_{\bar{q}}\bar{\mathcal{U}}$, that is, $X^* \in T_q\mathcal{U}$ for some $q \in \mathcal{U}$ with $\pi q = \bar{q}$ and $\pi_* X^* = X$, $\eta(X^*) = 0$. The construction of J and $\bar{g}$ is independent of the choice of the point q. We have the following relations:

$$(\bar{\nabla}_X Y)^* = \nabla_{X^*}Y^* - \eta(\nabla_{X^*}Y^*)\xi, \tag{12}$$

$$(\bar{R}_{XY}Z)^* = R_{X^*Y^*}Z^* + g(\varphi X^*, Z^*)\varphi Y^* - g(\varphi Y^*, Z^*)\varphi X^* + 2g(\varphi X^*, Y^*)\varphi Z^*, \tag{13}$$

$$\bar{\rho}(X, Y) \circ \pi = \rho(X^*, Y^*) + 2g(X^*, Y^*). \tag{14}$$

Deriving the last two expressions, we obtain

$$(\bar{\nabla}_X \bar{R})_{YZVW} \circ \pi = (\nabla_{X^*}R)_{Y^*Z^*V^*W^*}, \tag{15}$$

$$(\bar{\nabla}_X \bar{\rho})_{YZ} \circ \pi = (\nabla_{X^*}\rho)_{Y^*Z^*}. \tag{16}$$

Taking derivatives once again, gives

$$(17)\quad (\bar{\nabla}^2_{YX}\bar{R})_{UVWZ}\circ\pi = (\nabla^2_{Y^*X^*}R)_{U^*V^*W^*Z^*} + T(U^*,V^*,W^*,Z^*) - T(V^*,U^*,W^*,Z^*) + T(W^*,Z^*,U^*,V^*) - T(Z^*,W^*,U^*,V^*),$$

$$(18)\quad (\bar{\nabla}^2_{YX}\bar{\rho})_{ZW}\circ\pi = (\nabla^2_{Y^*X^*}\rho)_{Z^*W^*} + \Theta(Z^*,W^*) + \Theta(W^*,Z^*),$$

where

$$\begin{aligned} T(U^*,V^*,W^*,Z^*) &= \eta(\nabla_{Y^*}U^*)(\nabla_{X^*}R)_{\xi V^*W^*Z^*} \\ &= g(U^*,\varphi Y^*)\Big(R_{\varphi X^*V^*W^*Z^*} \\ &\quad - g(W^*,\varphi X^*)g(Z^*,V^*) - g(Z^*,\varphi X^*)g(W^*,V^*)\Big), \\ \Theta(Z^*,W^*) &= \eta(\nabla_{Y^*}Z^*)(\nabla_{X^*}\rho)_{\xi W^*} \\ &= g(Z^*,\varphi Y^*)\Big(\rho(\varphi X^*,W^*) - 2\ell g(\varphi X^*,W^*)\Big). \end{aligned}$$

A plane section T_mM is called a *φ-section* if it is spanned by a basis of the form $\{u,\varphi u\}$, where $u\in T_mM$ is a horizontal unit tangent vector. The sectional curvature $K(u,\varphi u)$ of a φ-section is called a *φ-sectional curvature*. For $\dim M\geq 5$ it is well-known that if $K(u,\varphi u)$ is pointwise constant, then it is globally constant on the manifold. In that case, (M,g,φ,ξ,η) is a space of constant φ-sectional curvature or a *Sasakian space form*. These spaces are characterized by their curvature tensor

$$(19)\quad \begin{aligned} R_{XY}Z = {} & \frac{c+3}{4}\Big\{g(X,Z)Y - g(Y,Z)X\Big\} \\ & + \frac{c-1}{4}\Big\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ & - g(Z,\varphi Y)\varphi X + g(Z,\varphi X)\varphi Y \\ & - 2g(X,\varphi Y)\varphi Z - g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi\Big\}, \end{aligned}$$

where c denotes the constant φ-sectional curvature.

We will frequently need the following characterization of Sasakian space forms [17].

Theorem 2.1. *A connected Sasakian manifold (M^n,g,φ,ξ,η) of dimension $n\geq 5$ is a Sasakian space form if and only if $R_{u\varphi u}u$ is proportional to φu for every horizontal tangent vector u on the manifold.*

This theorem is similar to the following one in Kähler geometry [17].

Theorem 2.2. *Let (M^n, g, J) be a connected Kähler manifold of dimension $n \geq 4$. Then M is a complex space form if and only if $R_{xJx}x$ is proportional to Jx for any vector x tangent to M.*

Another useful characterization, in terms of the base spaces of the local fibrations $\pi : \mathcal{U} \to \mathcal{U}/\xi = \bar{\mathcal{U}}$, is given by the following [16]

Theorem 2.3. *A connected Sasakian manifold has constant φ-sectional curvature if and only if the holomorphic sectional curvature of each base manifold $(\bar{\mathcal{U}}, \bar{g}, J)$ is constant.*

Now, let m be a point of an arbitrary Riemannian manifold M and let $G_m(r)$ denote the *geodesic sphere* centered at m and with radius $r < i(m)$, the injectivity radius at m. Every geodesic γ parametrized by arc length and such that $\gamma(0) = m$ leads to a point $p = \gamma(r) = \exp_m(ru) \in G_m(r)$, where we put $u = \gamma'(0)$.

Next, let $\{F_1, \dots, F_n\}$ be an orthonormal frame of parallel vector fields along γ with $F_1(0) = u$ (and hence $F_1 = \gamma'$). For the points $p = \gamma(r) = \exp_m(ru) \in G_m(r)$ we have the following expansions of the curvature tensor $\tilde{R}$, the Ricci tensor $\tilde{\rho}$ and the second fundamental form σ of $G_m(r)$ with respect to $\{F_1, \dots, F_n\}$:

$$\tilde{R}_{abcd}(p) = \frac{1}{r^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + \Big(R_{abcd} - \frac{1}{3}(R_{ubud}\delta_{ac} + R_{uauc}\delta_{bd} - R_{ubuc}\delta_{ad} - R_{uaud}\delta_{bc})\Big)(m) + O(r), \tag{20}$$

$$\begin{aligned}\tilde{\rho}_{ab}(p) = {} & \frac{n-2}{r^2}\delta_{ab} + \Big(\rho_{ab} - \frac{1}{3}\rho_{uu}\delta_{ab} - \frac{n}{3}R_{uaub}\Big)(m) \\ & + r\Big(\nabla_u\rho_{ab} - \frac{1}{4}\nabla_u\rho_{uu}\delta_{ab} - \frac{n+1}{4}\nabla_u R_{uaub}\Big)(m) \\ & + r^2\Big(\frac{1}{2}\nabla^2_{uu}\rho_{ab} - \frac{1}{10}\nabla^2_{uu}\rho_{uu}\delta_{ab} - \frac{n+2}{10}\nabla^2_{uu}R_{uaub} \\ & + \frac{1}{9}R_{uaub}\rho_{uu} - \frac{1}{45}\sum_{\lambda,\mu=2}^{n} R^2_{u\lambda u\mu}\delta_{ab} \\ & - \frac{n+2}{45}\sum_{\lambda=2}^{n} R_{uau\lambda}R_{ubu\lambda}\Big)(m) + O(r^3),\end{aligned} \tag{21}$$

$$\sigma_{ab}(p) = \frac{1}{r}\delta_{ab} - \frac{r}{3}R_{uaub}(m) + O(r^2) \tag{22}$$

for $a, b, c, d = 2, \dots, n$, where $R_{abcd} = g(R_{F_a F_b} F_c, F_d)$ and similarly for the other tensors. We refer to [5], [14], [18] for more details.

Now we turn to the Sasakian case. A point $p \in G_m(r)$ is called a *φ-geodesic point* if it lies on a φ-geodesic ray through m, that is, $u = F_1(0)$ is horizontal. For these points we further specify $\{F_1, \dots, F_n\}$, taking as initial conditions $F_2(0) = \varphi u$ and $F_3(0) = \xi_{|m}$.

When $(M, g, \varphi, \xi, \eta)$ is a Sasakian space form, the second fundamental form σ is known explicitly at these φ-geodesic points (see [2], [4]):

$$\sigma = \alpha\, g + \beta\, \eta \otimes \eta + \delta\, \nu \otimes \nu + \epsilon_1\, \eta \otimes \nu + \epsilon_2\, \nu \otimes \eta, \tag{23}$$

where g denotes the induced metric, ν is the $(0,1)$-tensor on the sphere defined by $\nu(X) = g(X, \varphi\gamma'(r))$ and $\alpha, \beta, \delta, \epsilon_1, \epsilon_2$ depend only on the radius r.

Now, we consider *geodesic tubes*, that is, tubes about a geodesic. We refer to [10], [13], [15], [18], [19] for more details. Let $\vartheta : [a, b] \to M$ be a smooth embedded geodesic and let $P_\vartheta(r)$ denote the tube of radius r about ϑ, where we suppose r to be smaller than the distance from ϑ to its nearest focal point. In that case, $P_\vartheta(r)$ is a hypersurface of M. Let ϑ be parametrized by arc length and denote by $\{e_1, e_2, \dots, e_n\}$ an orthonormal basis of $T_{\vartheta(a)}M$ such that $e_1 = \dot\vartheta(a)$. Further, let $E_1, \dots, E_n$ be the vector fields along ϑ obtained by parallel translation of $e_1, \dots, e_n$. Then $E_1 = \dot\vartheta$ and $\{E_1, \dots, E_n\}$ is a parallel orthonormal frame field along the geodesic ϑ. Next, let $p \in P_\vartheta(r)$ and denote by γ the geodesic through p which cuts ϑ orthogonally at $m = \vartheta(t)$. We parametrize γ by arc length r such that $\gamma(0) = m$ and take $(E_2, \dots, E_n)$ such that $E_2(t) = \gamma'(0) = u$. Finally, let $\{F_1, \dots, F_n\}$ be the orthonormal frame field along γ obtained by parallel translation of $\{E_1(t), \dots, E_n(t)\}$ along γ.

For the hypersurface $P_\vartheta(r)$ one then has the following expansions with respect to this parallel frame field $\{F_1, \dots, F_n\}$ [10], [19]:

$$\tilde R_{1abc}(p) = \Big(R_{1abc} - \frac{1}{2} R_{1ubu}\delta_{ac} + \frac{1}{2} R_{1ucu}\delta_{ab}\Big)(m) + O(r), \tag{24}$$

$$\begin{aligned} \tilde R_{abcd}(p) = {} & \frac{1}{r^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + R_{abcd}(m) \\ & - \frac{1}{3}\Big(R_{budu}\delta_{ac} - R_{bucu}\delta_{ad} + R_{aucu}\delta_{bd} - R_{audu}\delta_{bc}\Big)(m) \\ & + r\Big(\nabla_u R_{abcd} - \frac{1}{4}\nabla_u R_{budu}\delta_{ac} + \frac{1}{4}\nabla_u R_{audu}\delta_{bc} \\ & - \frac{1}{4}\nabla_u R_{aucu}\delta_{bd} + \frac{1}{4}\nabla_u R_{bucu}\delta_{ad}\Big)(m) + O(r^2), \end{aligned} \tag{25}$$

$$\tilde{\rho}_{11}(p) = O(r^0), \tag{26}$$

$$\tilde{\rho}_{1a}(p) = \rho_{1a}(m) - \frac{n-1}{2} R_{1uau}(m) + O(r), \tag{27}$$

$$\begin{aligned}
\tilde{\rho}_{ab}(p) = {} & \frac{n-3}{r^2}\delta_{ab} \\
& + \Big(\rho_{ab} - \frac{n-1}{3} R_{aubu} - \frac{1}{3}\rho_{uu}\delta_{ab} - \frac{2}{3} R_{1u1u}\delta_{ab}\Big)(m) \\
& + r\Big(\nabla_u \rho_{ab} - \frac{n}{4}\nabla_u R_{aubu} - \frac{1}{4}\nabla_u \rho_{uu}\delta_{ab} - \frac{1}{4}\nabla_u R_{1u1u}\delta_{ab}\Big)(m) \\
& + r^2\Big(\frac{1}{2}\nabla^2_{uu}\rho_{ab} - \frac{n+1}{10}\nabla^2_{uu} R_{aubu} + \frac{1}{9}\rho_{uu} R_{aubu} \\
& + \frac{2}{9} R_{1u1u} R_{aubu} - \frac{n+1}{20} R_{1uau} R_{1ubu} - \frac{n+1}{45}\sum_{\lambda=3}^{n} R_{au\lambda u} R_{bu\lambda u} \\
& - \frac{1}{10}\nabla^2_{uu}\rho_{uu}\delta_{ab} - \frac{1}{15}\nabla^2_{uu} R_{1u1u}\delta_{ab} - \frac{1}{3} R^2_{1u1u}\delta_{ab} \\
& - \frac{2}{15}\sum_{\lambda=3}^{n} R^2_{1u\lambda u}\delta_{ab} - \frac{1}{45}\sum_{\lambda,\mu=3}^{n} R^2_{\lambda u\mu u}\delta_{ab}\Big)(m) + O(r^3),
\end{aligned} \tag{28}$$

$$\sigma_{11}(p) = O(r), \tag{29}$$

$$\sigma_{1a}(p) = -\frac{r}{2} R_{1uau}(m) + O(r^2), \tag{30}$$

$$\sigma_{ab}(p) = \frac{1}{r}\delta_{ab} - \frac{r}{3} R_{aubu} + O(r^2) \tag{31}$$

for $a, b, c, d \in \{3, 4, \dots, n\}$.

In the sequel we will use two kinds of geodesic tubes, depending on the direction of the center geodesic ϑ.

First, suppose that the geodesic ϑ is horizontal. Then the tube $P_\vartheta(r)$ is called a *horizontal tube* and a point $p = \exp_m(ru)$ on $P_\vartheta(r)$ is called a *φ-special point* if $u = \varphi\dot{\vartheta}_{|m}$, that is, these points are determined by the initial condition $F_2(0) = \varphi F_1(0)$. Remark that the geodesic ray γ connecting m with p is horizontal. As a supplementary initial condition for the choice of the frame $\{F_1, \dots, F_n\}$ we take $F_3(0) = \xi_{|m}$. For Sasakian space forms, one may use the technique of Jacobi vector fields to compute explicit expressions for the second fundamental form of horizontal geodesic tubes at φ-special points (see [6]). This results in a formula for σ of the same form as (23).

Second, consider geodesic tubes for which the center geodesic ϑ is a ξ-geodesic. They will be called *ξ-geodesic tubes*. Analogously as for spheres, a point $p = \exp_m(ru)$ on $P_\vartheta(r)$ is called a *φ-geodesic point* if the geodesic ray connecting m and p is a φ-geodesic, that is, $u = F_2(0)$ is a horizontal vector. Because ϑ is a ξ-geodesic, $F_1(0) = \xi_{|m}$. Further, it is possible to choose $F_3(0) = \varphi u$. For Sasakian space forms the second fundamental form of ξ-geodesic tubes at φ-geodesic points is again given by a formula of the same form as (23) (see [9]).

By a straightforward computation involving (19), (23) and the Gauss equation, it follows that both for geodesic spheres and tubes the Ricci curvature tensor $\tilde{\rho}$ too is expressed by a formula similar to (23), but with other radial functions $\alpha, \beta, \delta, \epsilon_1, \epsilon_2$.

Finally, a horizontal tangent vector X to a geodesic sphere or tube will be called *strictly horizontal* (with respect to the sphere or tube) if $g(X, \varphi\gamma') = \nu(X) = 0$, with the same meaning of γ as before. Using (2) and (5), it is easy to see that the parallel translates of ξ and $\varphi\gamma'$ along γ are again linear combinations of ξ and $\varphi\gamma'$. Therefore the strictly horizontal tangent vectors at $\gamma(r)$ are spanned by $\{F_4, \dots, F_n\}$.

3. Geodesic spheres

In the next two sections we prove our characterization theorems for Sasakian space forms. First we consider geodesic spheres.

Theorem 3.1. *Let $(M^n, g, \varphi, \xi, \eta)$, $n \geq 5$, be a Sasakian space form. Then for all small geodesic spheres in M we have*

$$\tilde{R}_{XY} \cdot \sigma = 0 = \tilde{R}_{XY} \cdot \tilde{\rho}$$

for all strictly horizontal tangent vectors X, Y at φ-geodesic points of these spheres.

Proof. From (23) it is easy to see that

$$-(\tilde{R}_{XY} \cdot \sigma)(W, W) = 2\eta(\tilde{R}_{XY}W)\Big(\beta\eta(W) + \epsilon_1\nu(W)\Big) + 2\nu(\tilde{R}_{XY}W)\Big(\epsilon_2\eta(W) + \delta\nu(W)\Big).$$

Using (1), (23) and the Gauss equation, we obtain by direct computation that $\eta(\tilde{R}_{XY}W) = -\tilde{R}_{XY\xi W} = 0$ for strictly horizontal tangent vectors to $G_m(r)$. By means of (19), (23) and the Gauss equation it can also be checked easily that $\nu(\tilde{R}_{XY}W) = g(\tilde{R}_{XY}W, \varphi\gamma'(r)) = 0$. This proves the first equality since $\tilde{R}_{XY} \cdot \sigma$ is symmetric.

Because the Ricci curvature tensor $\tilde{\rho}$ has the same form as the second fundamental form σ, it follows in the same way that $\tilde{R}_{XY} \cdot \tilde{\rho} = 0$ for strictly horizontal vectors X, Y at φ-geodesic points. □

Next, we prove the converse.

Theorem 3.2. *Let $(M^n, g, \varphi, \xi, \eta)$, $n \geq 5$, be a Sasakian manifold such that all its small geodesic spheres satisfy one of the conditions*

$$\tilde{R}_{XY} \cdot \sigma = 0 \quad \text{or} \quad \tilde{R}_{XY} \cdot \tilde{\rho} = 0$$

for all strictly horizontal tangent vectors X, Y at φ-geodesic points to these spheres. Then, $(M, g, \varphi, \xi, \eta)$ is a Sasakian space form.

Proof. In terms of the notations introduced in Section 2, the first condition yields that $(\tilde{R}_{ab} \cdot \sigma)_{cd} = 0$ for $a, b \geq 4$ and $c, d \geq 2$. We use (20) and (22) to calculate the power series expansion of the left-hand side of this condition. Then the coefficient of r^{-1} yields

$$-\delta_{ac} R_{udub} + \delta_{bc} R_{udua} - \delta_{ad} R_{ucub} + \delta_{bd} R_{ucua} = 0 \tag{32}$$

at the arbitrarily chosen center point m of the small geodesic sphere. We can take $a = d \neq b$ and $c = 2$ (that is, c represents $F_2(0) = \varphi u$). Then $a \neq c$, $b \neq c$ because $a, b \geq 4$ and we get $R_{u\varphi uub} = 0$ for $b \geq 4$, that is, b represents a vector orthogonal to $\{\xi, u, \varphi u\}$ and since we are working with φ-geodesic points, u is an (arbitrary) horizontal vector at m.

But the horizontality of u implies immediately, using (1), that $R_{u\varphi uu\xi} = 0$, while for $b = u$ the relation holds trivially. So, we obtain that $R_{u\varphi uux} = 0$ for every horizontal vector u and all x orthogonal to φu. Then the result follows in view of Proposition 2.1.

Treating the second condition in a similar way, the coefficients of r^{-2}, r^{-1}, r^0 in the series expansion of $(\tilde{R}_{ab} \cdot \tilde{\rho})_{cd} = 0$ yield:

$$\rho_{x\varphi u} = \frac{n}{3} R_{xu\varphi uu}, \tag{33}$$

$$(\nabla_u \rho)_{x\varphi u} = \frac{n+1}{4} (\nabla_u R)_{xu\varphi uu}, \tag{34}$$

$$\begin{aligned} 0 = {} & \frac{1}{2} (\nabla^2_{uu} \rho)_{x\varphi u} - \frac{n+2}{10} (\nabla^2_{uu} R)_{xu\varphi uu} \\ & + \frac{1}{9} R_{xu\varphi uu} \rho_{uu} - \frac{n+2}{45} \sum_{s=2}^{n} R_{xusu} R_{\varphi uusu}, \end{aligned} \tag{35}$$

where $n = \dim M = 2\ell + 1$, u horizontal and x orthogonal to $\{\xi, u, \varphi u\}$. These conditions are exactly those needed in the proof of [8, Theorem 15] to show that M is a Sasakian space form. □

4. Geodesic tubes

Now, we turn to the two kinds of geodesic tubes. With the terminology introduced in Section 2, we have the following theorem for horizontal geodesic tubes:

Theorem 4.1. *Let $(M^n, g, \varphi, \xi, \eta)$, $n \geq 5$, be a Sasakian manifold. A necessary and sufficient condition for M to be a Sasakian space form is that all its small φ-geodesic tubes satisfy one of the conditions*

$$\tilde{R}_{XY} \cdot \sigma = 0 = \tilde{R}_{XY} \cdot \tilde{\rho}$$

for all strictly horizontal tangent vectors X, Y at φ-special points of these tubes.

Proof. In Section 2 we noticed that for Sasakian space forms the second fundamental form as well as the Ricci tensor of φ-geodesic tubes at φ-special points have the same form as (23). Therefore, the necessity of the conditions follows by completely analogous computations as in the proof of Theorem 3.1.

Conversely, the first condition implies that $(\tilde{R}_{ab} \cdot \sigma)_{1c} = 0$ for $a, b \geq 4$ and $c \geq 3$, in terms of the reference field $\{F_1, \ldots, F_n\}$ introduced in Section 2. We expand this into power series by means of (24), (25), (29)–(31) and then the coefficient of r^{-1} yields $R_{1cab} = 0$. Put $x = F_1(0) = \dot{\sigma}_{|m}$ (which is an arbitrary horizontal vector) and recall that $F_2(0) = \varphi F_1(0) = \varphi x$ and $F_3(0) = \xi_{|m}$. Take $b = c = v$, which must be orthogonal to $\{\xi, x, \varphi x\}$. But then φv is also orthogonal to $\{\xi, x, \varphi x\}$ and we can choose $a = \varphi v$. Hence, $R_{xv\varphi vv} = 0$ for all (unit) horizontal vectors x, v for which v is orthogonal to $\{x, \varphi x\}$. In the same way as in the proof of Theorem 3.2 the result follows by means of Proposition 2.1.

For the second condition, using (24)–(28) to compute the coefficient of r^{-2} in the series expansion of $(\tilde{R}_{ab} \cdot \tilde{\rho})_{1c} = 0$ yields

$$(n-3)R_{1cab} + \delta_{ac}(\rho_{1b} - R_{1ubu}) - \delta_{bc}(\rho_{1a} - R_{1uau}) = 0$$

for $a, b \geq 4$ and $c \geq 3$. With the same choice of a, b, c as above, we obtain $(n-3)R_{xv\varphi vv} - (\rho_{x\varphi v} - R_{xu\varphi vu}) = 0$, where $u = F_2(0) = \varphi x$. But, $R_{x\varphi x\varphi v\varphi x} = R_{x\varphi xvx} = -R_{vx\varphi xx}$, using (7), (3) and the orthogonality properties of the vectors involved. So we obtain

$$\rho_{x\varphi v} = (n-3)R_{xv\varphi vv} - R_{vx\varphi xx} \tag{36}$$

again for all unit horizontal vectors x, v with v orthogonal to $\{x, \varphi x\}$. Using (4), (9) to obtain $\rho_{v\varphi v} = 0$ for horizontal v, we see that (36) holds trivially for $x = v$. Therefore we may suppose that x, v are unit horizontal vectors with v orthogonal to φx. Since this condition is symmetric in

x and v, we can interchange them in the previous equality. But from (4) and (9) we see that $\rho_{x\varphi v} = -\rho_{v\varphi x}$ and so we obtain the relation $(n-4)\{R_{xv\varphi vv} + R_{vx\varphi xx}\} = 0$. Since $n = \dim M \neq 4$ it follows that $R_{xv\varphi vv} = -R_{vx\varphi xx}$. Substituting this into (36) gives $\rho_{x\varphi v} = (n-2)\, R_{xv\varphi vv}$ for any horizontal vector x and any unit vector v orthogonal to $\{\xi, \varphi x\}$.

We will now describe an adaptation to the Sasakian case of the method of polarization used in [7, pp. 198]. Suppose

$$\rho_{x\varphi v} = k\, R_{xv\varphi vv}, \tag{37}$$

for some real constant k and the same choice of vectors x, v as just above.

Replacing v in (37) by $v/\|v\|$ gives a homogenous expression which also holds for non-unit tangent vectors v. Putting $v = \alpha y + \beta z$ into this expression with y, z orthogonal to $\{\xi, \varphi x\}$, yields, by using the coefficient of $\alpha\beta^2$:

$$\rho(x, \varphi y) g(z, z) + 2\rho(x, \varphi z) g(y, z) = k\{R_{y\varphi zzx} + R_{z\varphi yzx} + R_{z\varphi zyx}\}.$$

We want to sum this for z ranging over an orthonormal base to obtain Ricci curvatures in the right-hand side. To do so, we first have to reposition z in the third term, using the first Bianchi identity, and to change φz into z by means of (7) and (9). In this way we obtain

$$\rho(x, \varphi y) g(z, z) - 2\rho(\varphi x, z) g(y, z) = k\{3R_{xz\varphi yz} - R_{\varphi xzyz} + 3g(x, z) g(\varphi y, z)\}.$$

Now we take an orthonormal basis $\{e_1, \dots, e_n\}$ such that $e_{n-1} = \varphi x$ and $e_n = \xi$. Then e_i, $i = 1, \dots, n-2$, are orthogonal to $\{\xi, \varphi x\}$ and we can replace z by e_i . Summing for $i = 1, \dots, n-2$ and simplifying, results in $(n-4k)\rho_{x\varphi y} = 3kR_{yx\varphi xx}$ for (unit) horizontal vectors x, y with y orthogonal to φx. In this expression we interchange x and y. Then we use that $\rho_{y\varphi x} = -\rho_{x\varphi y}$ and combine the obtained equation with the original expression (37) (in which we identify v with y) to get $k(4k-(n+3))R_{xy\varphi yy} = 0$. This holds trivially for $x = \xi$ and because $0 \neq k = n-2 \neq (n+3)/4$, the result follows by Proposition 2.1. □

When we apply the same ideas to ξ-geodesic tubes, we have the following

Theorem 4.2. *Let $(M^n, g, \varphi, \xi, \eta)$, $n \geq 5$, be a Sasakian manifold. A necessary and sufficient condition for M to be a Sasakian space form is that all its small ξ-geodesic tubes satisfy one of the conditions*

$$\tilde{R}_{XY} \cdot \sigma = 0 = \tilde{R}_{XY} \cdot \tilde{\rho}$$

for all strictly horizontal tangent vectors X, Y at φ-geodesic points of these tubes.

Proof. The necessity of these conditions follows by direct computations as explained in the proofs of Theorem 3.1 or 4.1. Conversely, for the first condition, we compute the coefficient of r^{-1} in the series expansion of $(\tilde{R}_{ab} \cdot \sigma)_{cd} = 0$. Because of the similarity of (25), (31) to (20), (22) with respect to the first order terms, this gives the same formula as (32), in which we also take $a = d \neq b$, but $c = 3$ in order to make it represent φu with respect to the reference field $\{F_1, \dots, F_n\}$ introduced for ξ-geodesic tubes in Section 2.

For the second condition we calculate the coefficients of r^{-2}, r^{-1} and r^0 in the series expansion of $(\tilde{R} \cdot \tilde{\rho}_{ab})_{cd} = 0$. Making the same choice as above for a, b, c, d and denoting $x = b$, we get

$$\rho_{x\varphi u} = \frac{n-1}{3} R_{xu\varphi uu}, \tag{38}$$

$$(\nabla_u \rho)_{x\varphi u} = \frac{n}{4} (\nabla_u R)_{xu\varphi uu}, \tag{39}$$

$$\begin{aligned} 0 = \frac{1}{2}(\nabla^2_{uu}\rho)_{x\varphi u} - \frac{n+1}{10}(\nabla^2_{uu}R)_{xu\varphi uu} + \frac{1}{9}\rho_{uu}R_{xu\varphi uu} \\ - \frac{n+1}{20} R_{\xi u\varphi uu} R_{\xi uxu} - \frac{n+1}{45}\sum_{s=3}^{n} R_{su\varphi uu} R_{suxu} \end{aligned} \tag{40}$$

for all horizontal unit vectors u, x with x orthogonal to $\{u, \varphi u\}$. Applying to (38) the method explained in the proof of Theorem 4.1 yields the result if $k = (n-1)/3 \neq (n+3)/4$, that is, if $n \neq 13$. To deal with this dimension we reduce the problem to Kähler geometry. For this purpose we use the local fibration π which projects the ξ-geodesic tube $P_\vartheta(r)$ to a geodesic sphere $\bar{G}_{\bar{m}}(r)$ of $\bar{\mathcal{U}}$ where $\bar{m} = \pi\vartheta$. Projecting the conditions (38)–(40) using (13)–(18) yields

$$\bar{\rho}_{yJv} = \frac{N}{3} \bar{R}_{yvJvv}, \tag{41}$$

$$(\bar{\nabla}_v \bar{\rho})_{yJv} = \frac{N+1}{4} (\bar{\nabla}_v \bar{R})_{yvJvv}, \tag{42}$$

$$\begin{aligned} 0 = \frac{1}{2}(\bar{\nabla}^2_{vv}\bar{\rho})_{yJv} - \frac{N+2}{10}(\bar{\nabla}^2_{vv}\bar{R})_{yvJvv} + \frac{1}{9}\bar{\rho}_{vv}\bar{R}_{yvJvv} \\ - \frac{N+2}{45} \bar{R}_{v\bar{R}_{vJv}v\,vy} + \frac{1}{6}\bar{R}_{yvJvv}, \end{aligned} \tag{43}$$

where $N = n - 1 = 12$ and $v, y \in T_{\bar{m}}\bar{\mathcal{U}}$ are defined by $v = \pi_* u,\ y = \pi_* x$, which can be viewed as arbitrary unit vectors with y orthogonal to $\{v, Jv\}$. But (41), (42) correspond to formulae (34), (37) respectively in the proof of Theorem 12 in [7, pp. 198]. Following the method used in this proof, we get from these two formulae that $(\bar{\mathcal{U}}, \bar{g}, J)$ is locally symmetric. So, when $(\bar{\mathcal{U}}, \bar{g}, J)$ is locally irreducible, then it is an Einstein space and hence the result follows from (41) and Proposition 2.2 and 2.3. Otherwise it is locally a product $\bar{\mathcal{U}}_1 \times \ldots \times \bar{\mathcal{U}}_k$ of Kählerian Einstein spaces. Considering the projections of (41) on the factors, we see by the same argument that they also have constant holomorpic sectional curvatures, say $c_1, \ldots, c_k$ respectively. We will prove that $(\bar{\mathcal{U}}, \bar{g}, J)$ is a complex space form by showing that $c_1 = \ldots = c_k = 0$.

In order to do so, we will first focus on the factors $\bar{\mathcal{U}}_1$, $\bar{\mathcal{U}}_2$. Take unit vectors u_i in $\bar{\mathcal{U}}_i$, $i = 1, 2$, and put $v = (\cos\alpha)u_1 + (\sin\alpha)u_2$. For this unit vector v, the projections of (41) on the first two factors yield the condition (see also [7, p. 200])

$$c_1\{3(N_1+2) - 4N\cos^2\alpha\} = c_2\{3(N_2+2) - 4N\sin^2\alpha\}, \tag{44}$$

where $N_1 = \dim\bar{\mathcal{U}}_1$, $N_2 = \dim\bar{\mathcal{U}}_2$. From this it follows by taking different possible values of α that $c_1 + c_2 = 0$. If $k \geq 3$, we have in the same way that $c_1 + c_3 = 0 = c_2 + c_3$ and then immediately $c_1 = c_2 = c_3 = 0$. Proceeding further for the other factors gives the result.

If $k = 2$, we will need (43) to obtain that $c_1 = c_2 = 0$. Since $(\bar{\mathcal{U}}, \bar{g}, J)$ is locally symmetric, $\bar{\nabla}\bar{R} = 0 = \bar{\nabla}\bar{\rho}$, and from (43) we get

$$10\bar{\rho}_{vv}\bar{R}_{vJv}v + 15\bar{R}_{vJv}v - 24\bar{R}_{v\bar{R}_{vJv}v}v = \beta Jv. \tag{45}$$

Decomposing v as above and projecting on the first factor yields

$$c_1\cos^2\alpha(10\bar{\rho}_{vv} + 15 - 24c_1\cos^2\alpha) = \beta. \tag{46}$$

Doing the same for the second factor and combining this together with $c_2 = -c_1$ gives an equation from which we can conclude that $c_1 = c_2 = 0$ by taking different possible values of α. □

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