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On some arithmetical properties of Stirling numbers

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1. Introduction and the theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of integers, b a non-zero rational integer and $p_1, \dots, p_s$ $(s \geq 0)$ distinct prime numbers. Many numbertheoretical problems can be reduced to equations of the forms

$$(1) \qquad a_n = by^m \qquad \text{in integers} \quad n \geq 0,\ m \geq 2,\ y$$

and

$$(2) \qquad a_n = bp_1^{z_1} \cdot \ldots \cdot p_s^{z_s} \qquad \text{in integers} \quad n,\ z_1, \dots, z_s \geq 0\,.$$

Of particular importance are the cases when, in (1) or (2), a_n is a polynomial in n with rational integer coefficients or a linear recurrence sequence. In these cases, several effective finiteness results have been established for the solutions of (1) and (2); for references see [2], [15] and [13]. These results have been obtained by means of Baker's theory of linear forms in logarithms and its p–adic analogue.

In connection with equation (1), Erdős [5] has shown that the equation

$$(3) \qquad \binom{n+a}{a} = y^m \qquad \text{in integers} \quad a > 1,\ m > 1,\ n \geq 1,\ y > 1$$

has no solutions provided that $a \geq 4$. For a=m=2, there are infinitely many solutions in n, y. The only other known solution is $a = 3,\ m = 2,$ $n = 47,\ y = 140$ and it is likely that there are no more. In this direction, see the results in [9], [6] and [16]. By the result of Tijdeman [16], there are effectively computable upper bounds for the solutions of (3) with $a = 2,\ m \geq 3$ and $a = 3,\ m \geq 2$.

In this paper, we consider equations (1) and (2) in the case when the a_n are Stirling numbers of certain special type. We denote by S_k^n the number of partitions of a set of n elements into k non-empty subsets. These numbers S_k^n are called Stirling–numbers of second kind. For properties of Stirling–numbers, see e.g. [10]. By combining some effective results of Baker [1], Schinzel and Tijdeman [11] and others on superelliptic equations with some well-known arithmetical properties of the numbers S_k^n, we shall prove the theorem below. We denote by S the set of non-zero integers which are not divisible by primes different from $p_1, \dots, p_s$.

Theorem. *Let $a \geq 1$ be an integer. If $S_{n-a}^n \in S$ for some $n > a$ then $n < C_1$. Further, if $S_{n-a}^n \in \mathbf{N}^m$ for some $n > a,\ m \geq 3$ then $n < C_2$. Here C_1 and C_2 are effectively computable positive numbers such that C_1 depends only on a and S, and C_2 only on a.*

In other words, for given $a \geq 1$, there are only finitely many integers $n > a$ with $S_{n-a}^n \in S$ or $S_{n-a}^n \in \mathbf{N}^m,\ m \geq 3$, and all these n can be effectively determined. Since $S_{n-1}^n = \binom{n}{2}$, the second assertion of our Theorem implies Tijdeman's result [16] mentioned above. Finally, we note that the assumption $m \geq 3$ is necessary in the second assertion of the Theorem. Indeed, the equations $x^2 - 2y^2 = 1$ and $x^2 - 2y^2 = -1$ have infinitely many positive integer solutions, and if (x, y) is a solution then $S_{x^2-1}^{x^2} = (xy)^2$ and $S_{2y^2-1}^{2y^2} = (xy)^2$, respectively.

2. Proof of the Theorem

To prove our Theorem, we shall need several lemmas. Denote by $\widetilde{S}_k^n$ the number of partitions of a set of n elements into k subsets having more than 1 element.

Lemma 1. *Let a, n be positive integers such that $n > a \geq 1$. Then we have*

$$S_{n-a}^n = \binom{n}{a+1}\widetilde{S}_1^{a+1} + \binom{n}{a+2}\widetilde{S}_2^{a+2} + \dots + \binom{n}{2a}\widetilde{S}_a^{2a}\,. \tag{4}$$

Proof. See e.g. [10] □

In what follows, let $f(x)$ be a polynomial with rational integer coefficients, and let b be a non-zero rational integer. By the height of a polynomial in $\mathbf{Z}[x]$ we mean the maximum absolute value of its coefficients.

Lemma 2. *Suppose that $f(x)$ has at least two distinct roots. If $f(x) \in bS$ for some $x \in \mathbf{Z}$ then $|x| \leq C_3$, where C_3 is an effectively computable number depending only on b, S and the degree and height of f.*

Proof. This follows from a combination of the results of [8] and [14]. For more explicit and more general versions, see [12], [13] and [7] and the references given there. □

Lemma 3. *Suppose that $f(x)$ has at least two distinct roots and that $m \geq 0$, moreover x and y with $|y| > 1$ are rational integers satisfying*

$$f(x) = by^m \, . \tag{5}$$

Then $m \leq C_4$, where C_4 is an effectively computable number depending only on b and the degree and height of f.

Proof. This is a theorem of Schinzel and Tijdeman [11]. For more explicit and more general versions, see [4], [13] and the references mentioned there. □

Lemma 4. *Let $m \geq 3$ be an integer, and suppose that $f(x)$ has at least two distinct simple roots. If $x, y \in \mathbf{Z}$ satisfy (5) then $\max(|x|, |y|) \leq C_5$ with some effectively computable number C_5 which depends only on b, m and the degree and height of f.*

Proof. This result is due Baker [1] who gave C_5 in an explicit form. For generalizations, see [3] and [13]. We note that Lemmas 2,3 and 4 were proved by means of the theory of linear forms in logarithms and its p–adic analogue. □

Proof of the Theorem. For fixed $a \geq 1$, we consider S^n_{n-a} as a polynomial in n. By Lemma 1, it is a polynomial of degree $2a \geq 2$ with rational coefficients. Hence, putting $f_a(n) = (2a)! S^n_{n-a}$, $f_a(n)$ is a polynomial in n with degree $2a$ and with rational integer coefficients. Further, it follows from (4) that $f_a(n)$ can be written in the form

$$f_a(n) = n(n-1)\dots(n-a)g(n) \tag{6}$$

where $g(n)$ is a polynomial of degree $a-1$ with rational integer coefficients, and by (4), the height of f_a can be bounded above by an explicit expression of a. Then (6) implies that at least two of the roots $0, 1, \dots, a$ of $f_a(n)$ are simple.

First suppose that $S^n_{n-a} \in S$ for some positive integer $n > a$. Then

$$f_a(n) \in bS \qquad \text{for} \qquad b = (2a)!\,.$$

By Lemma 2, we get $n < C_6$ where C_6 is effectively computable and it depends only on a and S.

Next suppose that $S^n_{n-a} \in \mathbf{N}^m$ for some integer $m \geq 3$. Then we get

$$f_a(n) = by^m \qquad \text{for} \qquad b = (2a)! \quad \text{and for some} \quad y \in \mathbf{Z}\,.$$

In what follows, C_7, C_8 and C_9 will denote effectively computable numbers depending only on a. In view of $n > a$, $S^n_{n-a} \neq 0$ and hence $y \neq 0$. If now $|y| = 1$ then, by Lemma 2, $n < C_7$. Further, if $|y| > 1$, then, by Lemma 3, it follows again that $m < C_8$. Finally, by Lemma 4, we get $n < C_9$. $\square$

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