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The operator of composition in Slobodeckij spaces

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Introduction

The so-called Riesz class $A_p = A_p(a,b)$ was introduced by RIESZ in [5] in the following way:

A function u defined in the not necessarily bounded open interval (a,b), belongs to the class A_p with $1 < p < \infty$ if and only if u is absolutely continuous in the interval (a,b) and its derivative u' belongs to the space $L_p(a,b)$. In the same paper, the following characterization of the class A_p was proved: A function u defined in the interval (a,b) belongs to the class A_p if and only if there exists a constant $K > 0$ such that for any system $\{(a_i,b_i) \subset (a,b)\}$ of pairwise disjoint bounded intervals we have

$$\sum_i \frac{|u(b_i) - u(a_i)|^p}{|b_i - a_i|^{p-1}} \le K \tag{1}$$

The sum (1) is called a Riesz sum and the constant can be taken equal to $K = \|u'\|^p_{L_p(a,b)}$. For a bounded interval (a,b) the class A_p coincides with the Sobolev space $W^1_p(a,b)$. In [7] F. SZIGETI, using the above sum, obtained results on the operator of composition in Sobolev spaces of type $W^s_p(a,b)$ where s satisfies an inequality depending on the imbedding theorems involving these spaces. From these results the existence of a solution of an ordinary differential equation in a given space was also obtained. The same author generalized these results to higher dimensional cases (see [8]). First Riesz sums in isotropic spaces $W^s_p(\Omega)$ were introduced where Ω is a domain in $\mathbf{R}^n$ with smooth boundary, $1 < p < \infty$ and s a positive real number satisfying an inequality depending on certain imbedding theorems. From the inequality necessary conditions were proved for the operator of composition to act in the spaces $W^s_p(a,b)$ and, as an application, an existence theorem for differential equations was also obtained. In

[6] J. Rivero and F. Szigeti generalized the above results to the case of the so-called Slobodeckij spaces $W_p^{\vec{s}}(\Omega)$ where Ω is a domain in $\mathbf{R}^n$ with smooth boundary, $1 < p < \infty$ and $\vec{s}$ is a vector in $\mathbf{R}^n$ with components satisfying certain inequalities depending on imbedding theorems for such spaces. More precisely, they proved the following Riesz-inequality in Slobodeckij spaces $W_p^{\vec{s}}(\Omega)$:

Theorem. *Let $\vec{s} = (s_1, \dots, s_n) \in \mathbf{R}_+^n$, $1 < p < \infty$ and let Ω be a domain in $\mathbf{R}^n$ with smooth boundary. Suppose that for all $i = 1, 2, \dots n$ we have*

$$s_i \left(1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{ps_j} \right) \geq 1$$

and let $u \in W_p^{\vec{s}}(\Omega)$. Then there exist constants $K_i > 0$ $(i = 1, \dots, n)$ such that:

a/ for any system $\{(a_{ij}, b_{ij}) \subset \mathbf{R}\}_{j=1}^{I_i}$ of pairwise disjoint bounded intervals, and

b/ for any system $\{\lambda^{ij} \in \mathbf{R}^{n-1}\}_{j=1}^{I_i}$ of vectors with the property

$$\Omega_i^{\lambda^{ij}} = \left\{ (\lambda_1^{ij}, \lambda_2^{ij}, \dots, \lambda_{i-1}^{ij}, t, \lambda_i^{ij}, \dots, \lambda_{n-1}^{ij}) : t = a_{ij} \text{ or } t = b_{ij} \right\} \subset \Omega$$

the estimate

$$\sum_{j=1}^{I_i} = \frac{|u_{i,\lambda^{ij}}(b_{ij}) - u_{i,\lambda^{ij}}(a_{ij})|^p}{|b_{ij} - a_{ij}|^{p-1}} \leq K_i \tag{2}$$

holds where the function $u_{i,\lambda}$ is defined by

$$t \longrightarrow u(\lambda_1, \dots, \lambda_{i-1}, t, \lambda_i, \dots, \lambda_{n-1}) = u_{i,\lambda}(t)$$

for all

$$t \in \Omega_{i,\lambda} = \{\tau : (\lambda_1, \dots, \lambda_{i-1}, \tau, \lambda_i, \dots \lambda_{n-1}) \in \Omega\}.$$

The inequality (2) is the Riesz inequality for the Slobodeckij spaces $W_p^{\vec{s}}(\Omega)$. Using this inequality the mentioned authors obtained sufficient conditions for the operator of composition to act in the spaces $W_p^{\vec{s}}(\Omega)$.

In the present paper we generalize the above results to the case of Slobodeckij type spaces $W_{\vec{p}}^{\vec{s}}(\Omega)$ where the vectors $\vec{s}$ and $\vec{p}$ satisfy a certain vectorial inequality depending on imbedding theorems for the spaces $W_{\vec{p}}^{\vec{s}}(\Omega)$.

In Section 1 some known results /see [1], [2] / on the imbeddings of spaces $W^{\vec{s}}_{\vec{p}}(\Omega)$ are recalled. In Section 2 the Riesz inequality for the spaces $W^{\vec{s}}_{\vec{p}}(\Omega)$ is established and sufficient conditions are obtained under which the operator of composition acts in the spaces $W^{\vec{s}}_{\vec{p}}(\Omega)$. In Section 3 we deduce from these results the existence theorem for a system of second order differential equations.

1. Preliminaries on Slobodeckij spaces

In this section we firstly recall some definitions and results concerning Slobodeckij spaces $W^{\vec{s}}_{\vec{p}}(\Omega)$ where $\vec{p} = (p_1, \dots, p_n)$, $\vec{s} = (s_1, \dots, s_n)$ and Ω denotes a cube $\Omega = \prod_{j=1}^{n}(a_j, b_j)$ in $\mathbf{R}^n$. All results are stated without proofs which can be found in the standard monographs, e.g. in [2].

For $\vec{p} = (p_1, \dots p_n)$, $\vec{q} = (q_1, \dots q_n)$, we shall write $\vec{p} \geq \vec{q}$ and $\vec{p} > \vec{q}$ if $p_i \geq q_i$ and $p_i > q_i$ $(i = 1, 2, \dots n)$ respectively. In particular, the notation $\vec{1} \leq \vec{p} \leq \vec{\infty}$ (where $\vec{1} = (1, \dots, 1)$ and $\vec{\infty} = (\infty, \dots, \infty)$) means that $1 \leq p_i \leq \infty$, for $i = 1, 2, \dots n$.

For given $\vec{p} = (p_1, \dots, p_n)$ with $\vec{1} \leq \vec{p} < \vec{\infty}$, we denote by $L_{\vec{p}}(\Omega)$ the space of all functions u defined and measurable on Ω for which the norm

$$\|u\|_{\vec{p},\Omega} = \left\{ \int_{a_n}^{b_n} \left[\dots \left\{ \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} |u(x)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right\}^{\frac{p_3}{p_2}} \dots \right]^{\frac{p_n}{p_{n-1}}} dx_n \right\}^{\frac{1}{p_n}}$$

is finite. The space $L_{\vec{p}}(\Omega)$ with $\vec{1} \leq \vec{p} < \vec{\infty}$ is a Banach space of functions with the norm defined above.

We shall use the following notation:

A vector $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ with components $\alpha_i \in \mathbf{N}_0$, $i = 1, \dots, n$ (where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$) is said to be a multiindex of dimension n. The number

$$|\vec{\alpha}| = \sum_{i=1}^{n} \alpha_i$$

is called lenght of the multiindex $\vec{\alpha}$. For a vector $\vec{s} = (s_1, \dots, s_n)$ with $\vec{0} < \vec{s} < \vec{\infty}$, we define the number

$$|\vec{\alpha} : \vec{s}\,| = \sum_{i=1}^{n} \frac{\alpha_i}{s_i}.$$

For a function u the generalized derivatives $D^{\vec{\alpha}}u$ are denoted by

$$D^{\vec{\alpha}}u = \frac{\partial^{|\vec{\alpha}|}u}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}.$$

Let $\vec{s} = (s_1, \dots, s_n)$ be a multiindex of dimension n and $\vec{1} \leq \vec{p} < \vec{\infty}$. We shall say that a function u belongs to the Slobodeckij space $W_{\vec{p}}^{\vec{s}}(\Omega)$ if $u \in L_{\vec{p}}(\Omega)$ and it has generalized derivatives $D^{\vec{\alpha}}u$ belonging to $L_{\vec{p}}(\Omega)$ where $|\vec{\alpha} : \vec{s}| \leq 1$.

The norm in the Slobodeckij space $W_{\vec{p}}^{\vec{s}}(\Omega)$ is defined by

$$\|u\|_{\vec{s},\vec{p}} = \|u\|_{\vec{p},\Omega} + \sum_{|\vec{\alpha}:\vec{s}|\leq 1} \|D^{\vec{\alpha}}u\|_{\vec{p},\Omega}.$$

The space $W_{\vec{p}}^{\vec{s}}(\Omega)$ with this norm is a Banach space. For a vector $\vec{s}$ with non-integer components s_i $(i = 1, 2, \dots, n)$, the Slobodeckij space $W_{\vec{p}}^{\vec{s}}(\Omega)$ is defined by the usual interpolation method (see [3]).

Now we recall imbedding theorems for the Slobodeckij space $W_{\vec{p}}^{\vec{s}}(\Omega)$. Let $\vec{p}, \vec{q}, \vec{s} \in \mathbf{R}_+^n$ with $\vec{1} < \vec{p} \leq \vec{q} < \vec{\infty}$. We define the numbers $\rho(\vec{p}, \vec{q}, \vec{s})$ and $\rho(\vec{p}, \vec{s})$ by

$$\rho(\vec{p}, \vec{q}, \vec{s}) = 1 - \sum_{i=1}^{n} (\frac{1}{p_i} - \frac{1}{q_i})\frac{1}{s_i} \quad \text{and} \quad \rho(\vec{p}, \vec{s}) = 1 - \sum_{i=1}^{n} \frac{1}{p_i} s_i$$

For all $j = 1, 2, \dots, n$, we also define the numbers $\rho_j(\vec{p}, \vec{s})$ by

$$\rho_j(\vec{p}, \vec{s}) = 1 - \sum_{\substack{i=1 \\ j\neq i}}^{n} \frac{1}{p_i s_i}.$$

Theorem 1.1. *Let $\vec{p}, \vec{q}, \vec{s} \in \mathbf{R}_+^n$ and $\vec{\lambda} \in \mathbf{R}_+^n$ be such that $\vec{1} < \vec{p} \leq \vec{q} < \vec{\infty}$ and for all $j = 1, 2, \dots, n$, the inequality*

$$\lambda_j \leq s_j \rho(\vec{p}, \vec{q}, \vec{s})$$

holds. Then the imbedding

$$W_{\vec{p}}^{\vec{s}}(\Omega) \hookrightarrow W_{\vec{q}}^{\vec{\lambda}}(\Omega)$$

is a linear, continuous operator and there exists a non-negative constant $C > 0$ such that $\|u\|_{\vec{q},\vec{\lambda}} \leq C\|u\|_{\vec{p},\vec{s}}$ for al $u \in W_{\vec{p}}^{\vec{s}}(\Omega)$ (the constant C depends on $\vec{p}$, $\vec{q}$, $\vec{s}$, $\vec{\lambda}$ and Ω).

Theorem 1.2. *Let $\vec{p}, \vec{s} \in \mathbf{R}^n_+$ such that $\vec{1} < \vec{p} < \vec{\infty}$ and $\rho(\vec{p}, \vec{s}) > 0$. Then the imbedding*

$$W^{\vec{s}}_{\vec{p}}(\Omega) \hookrightarrow b(\Omega)$$

is a linear, continuous operator and there exists a non-negative constant C such that

$$\|u\|_{b(\Omega)} \leq C\|u\|_{\vec{p},\vec{s}} \quad \text{for all} \quad u \in W^{\vec{s}}_{\vec{p}}(\Omega).$$

Here $b(\Omega)$ is the space of all bounded functions defined and continuous on Ω and $\|\cdot\|_{b(\Omega)}$ is given by

$$\|u\|_{b(\Omega)} = \sup_{x\in\Omega} |u(x)|.$$

Theorem 1.3. *(One-dimensional version of the theorem on the trace in $W^{\vec{s}}_{\vec{p}}(\Omega)$). Let $\vec{p}, \vec{s} \in \mathbf{R}^n_+$ be such that $\vec{1} < \vec{p} < \vec{\infty}$ and for all $j = 1, 2, \ldots n$ the inequality*

$$s_j \rho_j(\vec{p}, \vec{s}) > 0$$

holds. Let $u \in W^{\vec{s}}_{\vec{p}}(\Omega)$ and $\vec{\beta} = (\beta_1, \ldots, \beta_{n-1}) \in \mathbf{R}^{n-1}$. Denote

$$\Omega^{j,\vec{\beta}} = \{t \in (a_j, b_j) : (\beta_1, \ldots, \beta_{j-1}, t, \beta_j, \ldots, \beta_{n-1}) \in \Omega\}.$$

Then the one-dimensional trace

$$t \to u(\beta_1, \ldots, \beta_{j-1}, t, \beta_j, \ldots, \beta_{n-1}) = u_{j,\vec{\beta}}(t) \ (t \in \Omega^{j,\vec{\beta}})$$

belongs to the Sobolev space $W^{s_j,\rho_j(\vec{p},\vec{s})}_{p_j}(\Omega^{j,\vec{\beta}})$. In particular, if for all $j = 1, 2, \ldots, n$, the inequality

$$s_j \rho_j(\vec{p}, \vec{s}) \geq 1$$

holds then the functions $u_{j,\vec{\beta}}$ belong to Sobolev space $W^1_{p_j}(\Omega^{j,\vec{\beta}})$.

Theorem 1.4. *Let $\vec{s} \in \mathbf{R}^n_+$ and $\vec{1} < \vec{p} < \vec{\infty}$, such that $s_j \rho(\vec{p}, \vec{s}) > 1$ for all $j = 1, 2, \ldots, n$.*

Then the imbedding

$$W^{\vec{s}}_{\vec{p}}(\Omega) \hookrightarrow b'(\Omega)$$

is a linear, continuous operator and there exists a constant $C > 0$ such that $\|u\|_{b'(\Omega)} \leq C\|u\|_{\vec{p},\vec{s}}$ for all $u \in W^{\vec{s}}_{\vec{p}}(\Omega)$ where $\|u\|_{b'(\Omega)} = \sup_{x\in\Omega} |u(x)| + \sum_{j=1}^{n} \sup_{x\in\Omega} |\frac{\partial u(x)}{\partial x_j}|$.

2. Inequality of Riesz

In the present section we generalize the result for Slobodeckij spaces $W^{\vec{s}}_{\vec{p}}(\Omega)$ where $\vec{s} < \vec{p} < \vec{\infty}$ and $s_i\rho_i(\vec{p},\vec{s}) \geq 1$ for all $i = 1, 2, \dots, n$.

Theorem 2.1. *Let $\vec{s} = (s_1, \dots, s_n) \in \mathbf{R}^n_+$, and $\vec{p} = (p_1, \dots, p_n) \in \mathbf{R}^n_+$ be such that $\vec{1} < \vec{p} < \vec{\infty}$ and $s_i\rho_i(\vec{p},\vec{s}) \geq 1$ for all $i = 1, 2, \dots, n$. If u belongs to the Slobodeckij spaces $W^{\vec{s}}_{\vec{p}}(\Omega)$, then there exist constants $K_i > 0$ with the following properties: For any system $\{(a_{ij}, b_{ij}) \subset (a_i, b_i)\}_{j=1}$ of nonoverlapping bounded intervals, and for any system $\beta^{ij} \in \mathbf{R}^{n-1}$, $j = 1, 2, \dots, n \dots$ such that the points $(\beta^{ij}_1, \dots, \beta^{ij}_{j-1}, t, \beta^{ij}_j, \dots, \beta^{ij}_{n-1})$ with $t = b_{ij}$ or $t = a_{ij}$ belong to Ω, the inequality*

$$\sum_{j=1} \frac{|u_{i,\beta^{ij}}(b_{ij}) - u_{i,\beta^{ij}}(a_{ij})|^{p_i}}{|b_{ij} - a_{ij}|^{p_i - 1}} \leq K_i \tag{2.2}$$

holds. The constants K_i can be chosen as

$$K_i = C_i(\Omega)\|\partial_i u\|^{p_i}_{\vec{p},\vec{s}-\vec{e}_i}$$

where $C_i(\Omega)$ only depends on the domain Ω and $\vec{e}_i = (0, \dots, \overset{i}{1}, 0 \dots, 0)$.

Proof. For all $i = 1, 2, \dots, n$, we define the vectors $\vec{s^i}, \vec{p^i}$ and the cube Ω^i by:

$$\vec{s^i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n), \vec{p^i} = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$$

and $\Omega^i = \Pi^n_{\substack{j=1\\ j\neq i}}(a_j, b_j)$.

Let $\vec{\beta} = (\beta_1, \dots, \beta_{n-1}) \in \mathbf{R}^{n-1}$ be such that $(\beta_1, \dots, \beta_{i-1}, t, \beta_i, \dots, \beta_{n-1}) \in \Omega$ for all $t \in (a_i, b_i)$. Since for all $i = 1, 2, \dots, n$ the inequality

$$s_i\rho_i(\vec{p},\vec{s}) \geq 1$$

holds, from theorem (1.3) we have that the functions $u_{i,\vec{\beta}}$ belong to the isotropic Sobolev spaces $W^1_{p_i}(a_i, b_i)$. Hence

$$u_{i,\vec{\beta}}(b_{ij}) - u_{i,\vec{\beta}}(a_{ij}) = \int_{a_{ij}}^{b_{ij}} (u_{i,\vec{\beta}}(\tau))' d\tau \tag{2.3}$$

Now estimate the norm

$$\|u_{i,\cdot}(b_{ij}) - u_{i,\cdot}(a_{ij})\|_{\vec{p^i},\vec{s^i},\Omega^i}.$$

Since equality (2.3) holds, by Hölder's inequality the following estimate can be obtained:

$$\|u_{i,\cdot}(b_{ij}) - u_{i,\cdot}(a_{ij})\|_{\vec{p}^i,\vec{s}^i,\Omega^i} \le |b_{ij} - a_{ij}|^{(1-\frac{1}{p_i})} \left(\int\limits_{a_{ij}}^{b_{ij}} \|u_{i,\cdot}(\tau)'\|^{p_i}_{\vec{p}^i,\vec{s}^i,\Omega^i} d\tau\right)^{\frac{1}{p_i}}.$$

Hence

$$\sum_{j=1} \frac{\|u_{i,\cdot}(b_{ij}) - u_{i,\cdot}(a_{ij})\|^{p_i}_{\vec{p}^i,\vec{s}^i,\Omega^i}}{|b_{ij} - a_{ij}|^{p_i-1}} \le \int\limits_{a_i}^{b_i} \|(u_{i,0}(\tau))'\|^{p_i}_{\vec{p}^i,\vec{s}^i,\Omega^i} d\tau.$$

For all $i = 1, 2, \dots, n$ the inequality

$$\int\limits_{a_i}^{b_i} \|(u_{i,\cdot}(\tau))'\|^{p_i}_{\vec{p}^i,\vec{s}^i,\Omega^i} d\tau \le \|\partial_i u\|_{\vec{p},\vec{s}-\vec{e}_i,\Omega}$$

holds, therefore for all $i = 1, 2, \dots, n$

$$\sum_{j=1} \frac{\|u_{i,\cdot}(b_{ij}) - u_{i,\cdot}(a_{ij})\|^{p_i}_{\vec{p}^i,\vec{s}^i,\Omega^i}}{|b_{ij} - a_{ij}|^{p_i-1}} \le \|\partial_i u\|^{p_i}_{\vec{p},\vec{s}-\vec{e}_i,\Omega}$$

Now, since for all $i = 1, 2, \dots, n$ the inequality $\rho(\vec{p}^i, \vec{s}^i) > 0$ holds, the imbedding $W^{\vec{s}_i}_{\vec{p}^i}(\Omega^i) \hookrightarrow b(\Omega^i)$ is a linear, continuous operator and there exist non-negative constants C_i such that

$$|u_{i,\beta^{ij}}(b_{ij}) - u_{i,\beta^{ij}}(a_{ij})|^{p_i} \le C_i^{p_i} \|u_{i,\cdot}(b_{ij}) - u_{\cdot}(a_{ij})\|^{p_i}_{\vec{p}^i,\vec{s}^i,\Omega^i}$$

for all systems $\{\beta^{ij} \in \mathbf{R}^{n-1}\}$. Hence for all $i = 1, 2, \dots, n$ the following inequality holds

$$\sum_{j=1} \frac{|u_{i,\beta^{ij}}(b_{ij}) - u_{i,\beta^{ij}}(a_{ij})|^{p_i}}{|b_{ij} - a_{ij}|^{p_i-1}} \le C_i^{p_i} \sum_{j=1} \frac{\|u_{i,\cdot}(b_{ij}) - u_{i,\cdot}(a_{ij})\|^{p_i}_{\vec{p}^i,\vec{s}^i,\Omega^i}}{|b_{ij} - a_{ij}|^{p_i-1}} \le$$
$$\le C_i^{p_i} \|\partial_i u\|^{p_i}_{\vec{p},\vec{s}-\vec{e}_i,\Omega}$$

Taking $K_i = C_i^{p_i} \|\partial_i u\|^{p_i}_{\vec{p},\vec{s}-\vec{e}_i,\Omega}$ the theorem is proved.

Now we shall prove a general theorem on the composition of functions belonging to Slobodeckij spaces $W^{\vec{s}}_{\vec{p}}(\Omega)$. This theorem generalizes an earlier result by J. Rivero and F. Szigeti [6]. The following theorem is a consequence of Riesz' classical result [5] and the above theorem.

Theorem 2.2. *Let $\vec{s} = (s_1, \ldots, s_n)$, $\vec{p} = (p_1, \ldots, p_n)$, $\vec{q} = (q_1, \ldots, q_n)$ and $r \in \mathbf{R}_+$ be such that $\vec{1} < \vec{p}, \vec{q} < \vec{\infty}$ and for all $i = 1, 2, \ldots, n$ the following conditions*

$$s_i \rho_i(\vec{\rho}, \vec{s}) \geq 1 \quad \textit{and} \quad \left(1 - \frac{1}{p_i}\right)\left(1 - \frac{1}{q_i}\right) = 1 - \frac{1}{\tau} \tag{2.4}$$

hold. Let $u \in W_{\vec{p}}^{\vec{s}}(\Omega)$ and g_i $(i = 1, 2, \ldots, n)$ be functions belonging to the isotropic Sobolev spaces $W_{q_i}^1(c,d)$ which are monotonic functions. If the composition $u \circ (g_1, \ldots, g_n)$ can be formed then it belongs to the isotropic Sobolev space $W_r^1(c,d)$. Moreover, there exists a nonnegative constant K such that

$$\|u \circ (g_1, \ldots, g_n)\|_{W_r^1(c,d)} \leq K \left(1 + \sum_{i=1}^{n} \|g_i\|_{W_{q_i}^1(c,d)}^{(1-\frac{1}{p_i})}\right) \|u\|_{\vec{p},\vec{s}}.$$

Proof. Recall that the function $u \circ (g_1 \ldots, g_n)$, belongs to the space $W_r^1(c,d)$ if and only if the function $u \circ (g_1 \ldots, g_n)$ satisfies the inequality of Riesz. To see this, consider a system $\{(c_j, d_j) \subset (c,d)\}$ of nonoverlapping bounded intervals, and for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, let

$$\begin{aligned} &\beta^{ij} = (g_1(d_j), \ldots, g_{i-1}(d_j), g_{i+1,}(c_j), \ldots g_n(c_j)) \in \mathbf{R}^{n-1}, \\ &b_{ij} = g_i(d_j),\ a_{ij} = g_i(c_j) \end{aligned}$$

and $m_i = r(1 - \dfrac{1}{q_i})$. Hence, as the functions g_i $(i = 1, 2, \ldots, n)$ are monotonic, using equality (2.4) and Hölder's inequality we have that

$$\begin{aligned} &\sum_{j=1} \frac{|u \circ (g_1, \ldots, g_n)(d_j) - u \circ (g_1, \ldots, g_n)(c_j)|^r}{|d_j - c_j|^{r-1}} \leq \\ &\leq n^{r-1} \Bigg(\sum_{i=1}^{n} \left(\sum_{g_i(d_j) \neq g_i(c_j)} \frac{|u_{i,\beta^{ij}}(b_{ij}) - u_{i,\beta^{ij}}(a_{ij})|^{p_i}}{|g_i(d_j) - g_i(c_j)|^{p_i - 1}} \right)^{\frac{r}{p_i}} \cdot \\ &\cdot \left(\sum_{j=1} \frac{|g_i(d_j) - g_i(c_j)|^{q_i}}{|d_j - c_j|^{q_i - 1}} \right)^{\frac{m_i}{q_i}} \Bigg). \end{aligned}$$

Since for all $i = 1, 2, \ldots, n$ the inequality

$$s_i \rho(\vec{p}, \vec{s}) \geq 1$$

holds, from theorem (2.1) and the criterion of Riesz, it follows that

$$\sum_{j=1} \frac{|u \circ (g_1, \dots, g_n)(d_j) - u_0(g_1, \dots, g_n)(c_j)|^r}{|d_j - c_j|^{r-1}} \leq$$
$$\leq n^{r-1} \left(\sum_{i=1}^{n} K_i^r \|\partial_i u\|^r_{\vec{p}, \vec{s}-\vec{e}_i, \Omega} \|g_i\|^{r(1-\frac{1}{p_i})}_{L_{q_i}(c,d)} \right).$$

Hence $u \circ (g_1, \dots, g_n)$ belongs to the space $W^1_r(c,d)$. Moreover, the inequality

$$\|u \circ (g_1, \dots, g_n)\|_{L_r(c,d)} \leq n^{\frac{r-1}{r}} K(\Omega) \left(\sum_{i=1}^{n} \|\partial_i u\|^r_{\vec{p}, \vec{s}-\vec{e}_i, \Omega} \|g'_i\|^{r(1-\frac{1}{p_i})}_{L_{q_i}(c,d)} \right)^{\frac{1}{r}}$$

holds.

From the above inequality, using the Sobolev imbedding theorem (1.1), the estimate

$$\|u \circ (g_1, \dots, g_n)\|_{W^1_{r(c,d)}} \leq K \left(1 + \sum_{i=1}^{n} \|g'_i\|^{(1-\frac{1}{p_i})}_{W^1_{q_i}(c,d)} \right) \|u\|_{\vec{p}, \vec{s}}$$

is obtained.

The preceding theorem has a direct generalization:

Theorem 2.3. *Let* $\vec{s} = (s_1, \dots, s_n)$, $\vec{p} = (p_1, \dots, p_n)$, $\vec{q} = (q_1, \dots, q_n)$, $\lambda = (\lambda_1, \dots, \lambda_n)$ *and* $r \in \mathbf{R}_+$ *be such that* $1 < \vec{p} \leq \vec{q} < \vec{\infty}$ *and suppose that, for all* $i = 1, 2, \dots, n$, *the following conditions are satisfied*

$$(2.5) \qquad s_i \rho(\vec{p}, \vec{s}) \left(\lambda_i - \frac{1}{q_i} \right) \geq 1 - \frac{1}{r} \quad \text{and} \quad 1 < \lambda_i < 1 + \frac{1}{q_i}.$$

Let $u \in W^{\vec{s}}_{\vec{p}}(\Omega)$ *and the* g_i $(i = 1, 2, \dots, n)$ *be functions belonging to the isotropic Sobolev spaces* $W^{\lambda_i}_{q_i}(c,d)$ *and being monotonic. If the composition* $u \circ (g_1, \dots, g_n)$ *can be formed, then it belongs to the isotropic Sobolev space* $W^1_r(c,d)$ *and there exists a non-negative constant* K *such that*

$$\|u \circ (g_1, \dots, g_n)\|_{W^1_{r(c,d)}} \leq K \left(1 + \sum_{i=1}^{n} \|g_i\|^{(1-\frac{1}{p_i^0})}_{W^{\lambda_i}_{q_i}(c,d)} \right) \|u\|_{\vec{p}, \vec{s}}$$

where $0 \leq 1 - \frac{1}{p_i^0} = (1 - \frac{1}{r})(\lambda_i - \frac{1}{q_i})^{-1} < 1$ *for all* $i = 1, 2, \dots, n$.

Proof. Let $\vec{s}$, $\vec{p}$, $\vec{q}$, $\vec{\lambda}$ and $r \in \mathbf{R}_+$ be such that conditions (2.5) hold. For all $i = 1, 2, \ldots, n$ we define the numbers q_i^0, p_i^0 and s_i^0 by

$$(1 - \frac{1}{q_i^0}) = \lambda_i - \frac{1}{q_i}, (1 - \frac{1}{p_i^0}) = (1 - \frac{1}{r})(\lambda_i - \frac{1}{q_i})^{-1}$$

and

$$s_i^0 = s_i \rho_1 \vec{p}, \vec{p^0}, \vec{s}) \quad \text{where} \quad \vec{p^0} = (p_1^0, \ldots, p_n^0).$$

Let $\vec{s}$, $\vec{p^0}$, $\vec{q^0}$ be defined by $\vec{s^0} = (s_1^0, \ldots, s_n^0)$, $\vec{p^0} = (p_1^0, \ldots, p_n^0)$ and $\vec{q^0} = (q_1^0, \ldots, q_n^0)$.

Then the following imbeddings hold:

$$W_{\vec{p}}^{\vec{s}}(\Omega) \hookrightarrow W_{\vec{p^0}}^{\vec{s^0}}(\Omega), \; W_{q_i}^{\lambda_i}(c, d) \hookrightarrow W_{q_i^0}^{1}(c, d) \tag{2.6}$$

Moreover, $\vec{s^0}$, $\vec{p^0}$, $\vec{q^0}$ and $r \in \mathbf{R}_+$ satisfy the conditions of theorem (2.2). Indeed it is obvious that the equality

$$\left(1 - \frac{1}{p_i^0}\right)\left(1 - \frac{1}{q_i^0}\right) = 1 - \frac{1}{r}$$

holds for all $i = 1, 2, \ldots, n$. Now we see that for all $i = 1, 2, \ldots, n$ the inequality

$$s_i^0 \rho_i(\vec{p^0}, \vec{s^0}) \geq 1$$

holds, or equivalently

$$1 + \sum_{j=1}^{n} \frac{s_i^0}{p_j^0 s_j^0} \leq s_i^0 + \frac{1}{p_i^0} \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

We clearly have

$$s_i \left(1 - \sum_{j=1}^{n} \frac{1}{p_j s_j}\right)\left(\lambda_i - \frac{1}{q_i}\right) \geq \frac{1}{r} \quad \text{and} \quad \left(1 - \frac{1}{p_i^0}\right)\left(1 - \frac{1}{q_i^0}\right) = 1 - \frac{1}{r}$$

for all $i = 1, 2, \ldots, n$. So

$$s_i(1 - \sum_{j=1}^{n} \frac{1}{p_j s_j}) \geq 1 - \frac{1}{p_i^0} \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

Hence, for all $i = 1, 2, \dots, n$,

$$s_i^0 + \frac{1}{p_i^0} = \left(1 - \sum_{j=1}^{n} \left(\frac{1}{p_j} - \frac{1}{p_j^0}\right) \frac{1}{s_j}\right) s_i + \frac{1}{p_i^0} =$$

$$= s_i \left(1 - \sum_{j=1}^{n} \frac{1}{p_j s_j} + \sum_{j=1}^{n} \frac{1}{p_j^0 s_j}\right) + \frac{1}{p_i^0} \geq$$

$$\geq 1 - \frac{1}{p_i^0} + s_i \sum_{j=1}^{n} \frac{1}{p_j^0 s_j} + \frac{1}{p_i^0} = 1 + \sum_{j=1}^{n} \frac{s_i}{p_j^0 s_j^0} = 1 + \sum_{j=1}^{n} \frac{s_i^0}{p_j^0 s_j^0}.$$

Therefore theorem (2.2) and the imbedding (2.6) imply that there exists $K > 0$ such that

$$\|u \circ (g_1, \dots, g_n)\|_{W^1_{r(c,d)}} \leq K \left(1 + \sum_{i=1}^{n} \|g_i\|_{W^{\lambda_i}_{q_i}(c,d)}^{(1 - \frac{1}{p_i^0})}\right) \|u\|_{\vec{p}, \vec{s}}$$

where $(1 - \frac{1}{p_i^0}) = (1 - \frac{1}{r})(\lambda_i - \frac{1}{q_i})^{-1}$ for all $i = 1, 2, \dots, n$.

Corollary 2.3. *Let* $\vec{s} = (s_1, \dots, s_{2n},\ s_{2n+1})$, $\vec{p} = (p_1, \dots, p_{2n}, p_{2n+1})$, $\vec{q} = (q_1, \dots, q_{2n+1})$ $\vec{\lambda} = (\lambda_1, \dots, \lambda_{2n}, \lambda_{2n+1})$ *and* $r \in \mathbf{R}_+$ *such that*

$$p_{2n+1} = r,\ \lambda_{2n+1} = 1 + \frac{1}{q_{2n+1}} \quad \text{and} \quad s_{2n+1} \leq s_i \left(\lambda_i - \frac{1}{q_i}\right)$$

for all $i = 1, 2, \dots, 2n$.

Suppose that

$$s_i \left(1 - \sum_{j=1}^{2n} \frac{1}{p_j s_j}\right) \left(\lambda_i - \frac{1}{q_i}\right) \geq 1 \quad \text{for all} \quad i = 1, 2, \dots, 2n$$

and

$$s_{2n+1} \left(1 - \sum_{j=1}^{2n} \frac{1}{p_j s_j}\right) \geq 1.$$

If $u \in W^{\vec{s}}_{\vec{p}}(\Omega)$ *and the* g_i $(i = 1, 2, \dots, 2n)$ *are monotonic functions belonging to the isotropic Sobolev spaces* $W^{\lambda_i}_{q_i}(c, d)$*, then the function* $u \circ (g_i, \dots, g_{2n}, I)$ *belongs to* $W^1_r(c, d)$ *and there exists* $K > 0$ *and* $0 <$

$\alpha_i < 1$ *such that*

$$\|u \circ (g_1, \ldots, g_{2n}, I, \|_{W_r^1(c,d)} \leq K \left(1 + \sum_{i=1}^{2n} \|g_i\|^{\alpha_i}_{W^{\alpha_i}_{W^{\alpha_i}_{q_i}(c,d)}}\right) \|u\|_{\vec{p},\vec{s}}.$$

3. Applications to differential equations

In this section, using the above corollary, the Rellich-Kondrashov theorem and Schauder's fixed-point theorem, we deduce an existence theorem for a system of second order differential equations. First we recall some notations and preliminary results.

For each $\rho = 1, 2, \ldots, n$, consider the vectors $\vec{s^\rho} = (s_1^\rho, \ldots, s_{2n}^\rho, s_{2n+1}^\rho)$, $\vec{p^\rho} = (p_1^\rho, \ldots, p_{2n}^\rho, p_{2n+1}^\rho)$, $\vec{q^\rho} = (q_1^\rho, \ldots, q_{2n}^\rho, q_{2n+1}^\rho)$, $\vec{\lambda}^\rho = (\lambda_1^\rho, \ldots, \lambda_{2n}^\rho, \lambda_{2n+1}^\rho)$, and a number $r \in \mathbf{R}_+$ such that

a/ $r = p_{2n+1}^\rho$, $\lambda_{2n+1}^\rho = 1 + \dfrac{1}{q_{2n+1}^\rho}$ $\quad (\rho = 1, 2, \ldots, n)$

b/ $s_{2n+1}^\rho \leq s_i^\rho \left(\lambda_i^\rho - \dfrac{1}{q_i^\rho}\right)$ $\quad (\rho = 1, 2, \ldots, n,\ i = 1, 2, \ldots, 2n)$

c/ $s_j^\rho \left(1 - \sum\limits_{i=1}^{2n} \dfrac{1}{p_i^\rho s_i^\rho}\right)\left(\lambda_j^\rho - \frac{1}{q_j^\rho}\right) \geq 1$ $(\rho = 1, 2, \ldots, n,\ j = 1, 2, \ldots, 2n)$

d/ $s_{2n+1}^\rho \left(1 - \sum\limits_{i=1}^{2n} \dfrac{1}{p_i^\rho s_i^\rho}\right) \geq 1$ $\quad (\rho = 1, 2, \ldots, n)$

e/ $\dfrac{1}{r} \leq \dfrac{1}{q_i^\rho} - \lambda_i^\rho + 2$ $\quad (\rho = 1, 2, \ldots, n,\ i = 1, 2, \ldots, n)$.

For each $\rho = 1, 2, \ldots, n$, consider a function $u_\rho \in W^{\vec{s}^\rho}_{\vec{p}^\rho}(\Omega)$ and let $x_1, \ldots, x_n$, $\dot{x}_1, \ldots, \dot{x}_n$ be monotonic functions such $\vec{p^\rho}$ that for all $i = 1, 2, \ldots, n$ we have

$$x_i \in W^{\lambda_i^\rho}_{q_i^\rho}[0,1], \quad \dot{x}_i \in W^{\lambda_{n+i}^\rho}_{q_{n+i}^\rho}[0,1].$$

Suppose in addition that for all $t \in [0,1]$ we have

$$(x_1(t), \ldots, x_n(t), \dot{x}_1(t), \ldots, \dot{x}_n(t), t) \in \Omega.$$

Then, by the above corollary, we obtain that for each $\rho = 1, 2, \ldots, n$ the composition $u_\rho(x_1(t), \ldots, x_n(t), \dot{x}_1(t), \ldots, \dot{x}_n(t), t)$ belongs to $W_r^1[0,1]$.

Now consider the following initial value problem:
$\ddot{x}_\rho(t) = u_\rho(x_1(t), \ldots, x_n(t), \dot{x}_1(t), \ldots, \dot{x}_n(t), t)$ for all $t \in [0,1]$ and $x_\rho(0) = \nu_\rho$ and $\dot{x}_\rho(0) = \eta_\rho$ for all $\rho = 1, 2, \ldots, n$.

This system is equivalent to the following system of integrodifferential equations

(3.2)

$$x_\rho(t) = \nu_\rho + \eta_\rho t + t\int_t^1 u_\rho(x_1(\tau),\dots,x_n(\tau),\dot{x}_1(\tau),\dots,\dot{x}_n(\tau),\tau)d\tau +$$
$$+\int_0^t \tau u_\rho(x_1(\tau),\dots,x_n(\tau),\dot{x}_1(\tau),\dots,\dot{x}_n(\tau),\tau)d\tau \quad (\rho = 1,2,\dots,n)$$

Since $\ddot{x}_\rho(t) = u_\rho(x_1(t),\dots,x_n(t),\dot{x}(t),\dots,\dot{x}_n(t),t)$ belongs to the space $W_r^1[0,1]$, the function x_ρ belongs to the space $W_r^3[0,1]$.

For each $\rho = 1,2,\dots,n$ define the set D_ρ and the function F_ρ as follows:

$$D_\rho = \Big\{(x,\dots,x_n) \in W_{q_1^\rho}^{\lambda_1^\rho}[0,1]\times\dots\times W_{q_n^\rho}^{\lambda_n^\rho}[0,1] : x_1,\dots,x_n,\dot{x}_1,\dots,\dot{x}_n$$
are monotonic and
$$(x_1(t),\dots,x_n(t),\dot{x},(t),\dots,\dot{x}_n(t),t)\in\Omega\ (t\in[0,1])\Big\}.$$

$$F_\rho(x_1,\dots,x_n)(t) = \nu_\rho + \eta_\rho t +$$
$$+\,t\int_t^1 u_\rho(x_1(\tau),\dots,x_n(\tau),\dot{x}_1(\tau),\dots,\dot{x}_n(\tau),\tau)d\tau +$$
$$+\int_0^t \tau u_\rho(x_1(\tau),\dots,x_n(\tau),\dot{x}_1(\tau),\dots,\dot{x}_n(\tau),\tau)d\tau = x_\rho(t).$$

Then for each $\rho = 1,2,\dots,n$ the function F_ρ maps the set D_ρ into the space $W_r^3[0,1]$, moreover, there exist constants $K_{\rho>0}$ and $0<\alpha_i^\rho<1$ such that

$$\|F_\rho(x_1,\dots,x_n)\|_{W_r^3[0,1]} \le K_\rho\left(1+\sum_{i=1}^n \|x_i\|^{\alpha_i^\rho}_{W_{q_i^\rho}^{\lambda_i^\rho}[0,1]}\right)\|u_\rho\|_{\vec{p}^\rho,\vec{s}^\rho} \tag{3.3}$$

Since $\vec{q^\rho}$, $\vec{\lambda^\rho}$ and r satisfy condition e/, that is

$$\frac{1}{r} \le \frac{1}{q_i^\rho} - \lambda_i^\rho + 2 \quad (\rho = 1,2,\dots,n,\ i = 1,2,\dots,n)$$

the imbedding

$$W_r^2[0,1] \hookrightarrow W_{q_i}^{\lambda_i^\rho}[0,1]$$

is a linear and continuous operator. Therefore there exist constants K_i^ρ such that

$$\|x_i\|_{W^{\lambda_i^\rho}_{q_i^\rho}[0,1]} \leq K_i^\rho \|x_i\|_{W_r^3[0,1]} \tag{3.4}$$

By (3.3) and (3.4) we obtain the existence of constants $K_\rho^* > 0$ and $0 < \alpha_i^\rho < 1$ such that

$$\|F_\rho(x_1, \dots , x_n)\|_{W_r^3[0,1]} \leq K_\rho^* \left(1 + \sum_{i=1}^{n} \|x_i\|^{\alpha_i^\rho}_{W^3_{r[0,1]}}\right) \|u_\rho\|_{\vec{p}^\rho,\vec{s}^\rho} \tag{3.5}$$

For all $\varepsilon > 0$ the following imbeddings are valid

$$W_r^{3-\varepsilon}[0,1] \overset{i_1}{\hookrightarrow} W_r^2[0,1] \quad \text{and} \quad W_r^3[0,1] \overset{i_2}{\hookrightarrow} W_r^{3-\varepsilon}[0,1].$$

Therefore, with the notation used before, for all $\rho = 1, 2, \dots , n$ we can define a function

$$F_\rho : (W_r^{3-\varepsilon}[0,1])^n \longrightarrow W_r^{3-\varepsilon}[0,1] \quad \text{by}$$

$$F_\rho(x_1, \dots , x_n)(t) = i_2(F_\rho(i_1(x_1(t)), \dots , i_1(x_n(t)) \quad \text{such that}$$

$$\|F_\rho(x_1, \dots , x_n)\|_{W_r^{3-\varepsilon}[0,1]} \leq K_\rho^* \left(1 + \sum_{i=1}^{n} \|x_i\|^{\alpha_i^\rho}_{W_r^{3-\varepsilon}[0,1]}\right) \|u_\rho\|_{\vec{p}^\rho,\vec{s}^\rho}. \tag{3.6}$$

Now we define a function

$$F : (W_r^{3-\varepsilon}[0,1])^n \longrightarrow (W_r^{3-\varepsilon}[0,1])^n \quad \text{by}$$

$$F(x_1, \dots , x_n)(t) = (F_1(x_1, \dots , x_n)(t), \dots , F_n(x_1, \dots , x_n)(t)).$$

In the following we shall look for conditions for the function F to satisfy the hypotheses of Schauder's fixed-point theorem.

For each $\rho = 1, 2, \dots , n$, we define

$$R_\rho = K_\rho^* \left(1 + \sum_{i=1}^{n} R_\rho^{\alpha_i^\rho}\right) \|u_\rho\|_{\vec{p}^\rho,\vec{s}^\rho}.$$

Then, for $R > R_\rho$ we have

$$R > K_\rho^* \left(1 + \sum_{i=1}^{n} R^{\alpha_i^\rho}\right) \|u_\rho\|_{\vec{p}^\rho,\vec{s}^\rho}. \tag{3.7}$$

Hence, taking $R > \max_{\rho=1,2,\dots,n}\{R_\rho\}$, we obtain that

$$\|(x_1,\dots,x_n)\|_{(W_r^{3-\varepsilon}[0,1])^n} = \max_{i=1,2,\dots,n} \|x_i\|_{W_r^{3-\varepsilon}[0,1]} \le R$$

implies

$$\|F(x_1,\dots,x_n)\|_{(W_r^{3-\varepsilon}[0,1])^n} = \max_{\rho=1,2,\dots,n} \|F_\rho(x_1,\dots,x_n)\|_{W_r^{3-\varepsilon}[0,1]} \le R.$$

Indeed, from inequalities (3.6) and (3.7) we get

$$\|F_\rho(x_1,\dots,x_n)\|_{W_r^{3-\varepsilon}[0,1]} \le K_\rho^* \left(1 + \sum_{i=1}^{n} \|x_i\|_{W_r^{3-\varepsilon}[0,1]}^{\alpha_i^\rho}\right) \|u_\rho\|_{\vec{p}^\rho,\vec{s}^\rho} \le R.$$

Let us consider now the sets D^1 and D^2 defined as follows:

$$D^1 = \Big\{ (x_1,\dots,x_n) \in (W_r^{3-\varepsilon}[0,1])^n : \max_{i=1,2,\dots,n} \|x_i\|_{W_r^{3-\varepsilon}[0,1]} \le R,$$

$$\dot{x}_1,\dots,\dot{x}_n \ge 0, \quad \ddot{x}_1,\dots,\ddot{x}_n \le 0$$

a.e. in $[0,1]$ and $x_1,\dots,x_n$ satisfy (3.2) $\Big\}$.

$$D^2 = \Big\{ (x_1,\dots,x_n) \in (W_r^{3-\varepsilon}[0,1])^n : \max_{i=1,2,\dots,n} \|x_i\|_{W_r^{3-\varepsilon}[0,1]} \le R,$$

$$\dot{x}_1,\dots,\dot{x}_n, \ddot{x}_1,\dots,\ddot{x}_n \le 0$$

a.e. in $[0,1]$ and $x_1,\dots,x_n$ satisfy (3.2) $\Big\}$.

In terms of these notations, using the above results, we can prove the following

Theorem 3.1. *Suppose that the above conditions $a/, b/, \dots, e/$ are satisfied and for all $\rho = 1,2,\dots,n$ and $\tau = 1,2,\dots,2n+1$ we have*

$$s_\tau^\rho \left(1 - \sum_{i=1}^{2n+1} \frac{1}{p_i^\rho s_i^\rho}\right) \ge 1. \tag{3.8}$$

For each $\rho = 1,2,\dots,n$ let M_ρ denote the norm of the imbedding

$$W_{\vec{p}^\rho}^{\vec{s}^\rho}(\Omega) \hookrightarrow b'(\Omega)$$

Put $r_\rho = 1 + |\nu_\rho| + 2|\eta_\rho| + \frac{7}{4}M_\rho R$, and suppose that $\overline{B_{r_\rho}(0)} \subset \Omega$ and at least one of the following conditions is satisfied:

$a'/ \quad \eta_\rho \ge 0 \quad (\rho = 1,2,\dots,n)$

$b'/ \quad \eta_\rho \le 0 \quad \sup_{\eta \in B_{r_\rho}(0)} u_\rho(\eta) \le -\eta_\rho \quad (\rho = 1,2,\dots,n)$

Then the initial value problem (3.1) has a solution belonging to the space $(W_r^{3-\varepsilon}[0,1])^n$ *for all* $\varepsilon > 0$.

PROOF. Differentiating with respect to t in formula (3.2), by a'/ and b'/ we obtain that

$$F(D^1) \subseteq D^1 \quad \text{and} \quad F(D^2) \subset D^2.$$

Since inequality (3.8) is fulfilled, each component of the function F_ρ is continuous /see Th. 1.4/, so F is continuous. From the Rellich-Kondrasov theorem we know that the inclusions

$$W_r^{3-\varepsilon}[0,1] \stackrel{i_1}{\hookrightarrow} W_r^2[0,1] \quad \text{and} \quad W_r^3[0,1] \stackrel{i_2}{\hookrightarrow} W_r^{3-\varepsilon}[0,1]$$

are compact. Therefore F is a compact function. Thus Schauder's theorem provides a fixed-point for the function F which is a solution to the initial value problem (3.1).

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