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A theorem for the fixed effects one- and two- way analysis of the variance model

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1. Introduction

In his paper [3] Béla Gyires proved the following criterion for the randomized block design ([3], p. 285, Theorem 2). *The expectations of the sample elements can be decomposed into the sum of two quantities corresponding to the block-effect and to the treatment-effect, respectively, if and only if the expectations of the random errors are zero.*

The author dealt with the above-mentioned problem in the case of the Latin square design ([4], [5]) but was unable to obtain the corresponding criterion.

In the fixed effects one-way analysis of the variance model we were able to prove the reverse of the following Theorem. *If the expectations of the sample elements can be decomposed into the sum of two quantities, where the first one is a constant (the so-called overall mean) and the second member corresponds to the effect of the selected level of the single factor having fixed effects, then the expectations of the random errors are zero* ([6], p. 295, Theorem 2). In the proof of the criterion we used the method of Gyires's paper [3]. [6] contains also the minimum dyadical representation of the expectation of the matrix of sample elements ([6], Section 3).

The following problem arises: Is it possible to prove by another method, the reverse of the above-formulated theorem valid for the one-way analysis of the variance model ? There is a positive answer to this question.

We shall give the proof of the reversed theorem for the generalized one-way analysis of the variance model already introduced in [6] applying the method well-known for the homogeneous linear matrix equation ([2], pp. 199–204).

In the present paper we will use the following notations: ξ_{jk}, ε_{jk} random variables; $\boldsymbol{\xi}$, $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$, $\boldsymbol{\zeta}$ matrix-valued random variables with m rows and n columns, that is matrices of dimension $m \times n$ consisting of such random variables that have expectations; $\mathbf{E}$ identity matrix of order m or n; $\mathbf{0}$ zero matrix of dimension $m \times n$; $\mathbf{S}_1$, $\mathbf{S}_2$ are stochastic matrices of order m and n, respectively; $\mathbf{A}$, $\mathbf{B}$ are square matrices of order m and n, respectively; $\mathbf{A}^*$ is the transpose of $\mathbf{A}$; $\mathbf{B}^{-1}$ is the inverse matrix of $\mathbf{B}$; $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, $\mathbf{T}$ orthogonal matrices; $M(\xi_{jk})$ expectation of ξ_{jk}; $M(\xi)$ is the expectation of the matrix ξ, $M(\xi)$ consists of the expectations of the elements of ξ; $\mathbf{A} = \begin{pmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ \cdots \\ a_{m1}, & m_{m2}, & \dots, & a_{mn} \end{pmatrix}$ is a matrix which is given by its elements; $\mathbf{A} = \|a_{jk}\|_{m\times m}$ or $\mathbf{A} = \|a_{jk}\|_{j,k=\overline{1,m}}$ is a square matrix given by its general element; $\mathbf{a}_0$, $\boldsymbol{\lambda}$, $\dots m$-dimensional column vectors; $\mathbf{b}_0$, $\boldsymbol{\mu}$ n-dimensional column-vectors; $\mathbf{b}_0^*$ is the transpose of $\mathbf{b}_0$; $\mathbf{o}$ zero vector; instead of $j = 1, 2, \dots, m$ we use the notation $j = \overline{1,m}$; if necessary, we indicate the dimension of a vector in the form $\lambda_{m\times 1}$.

In the second section we give the generalized form of the one-way analysis of the variance model and the theorems obtained. The third section contains the above-mentioned criterion for the fixed effects one-way analysis of the variance model which is a special case of Theorem 2 in [6] (p. 295).

It is to be noted that this method can be applied also with the randomized blocks to prove the reverse of the corresponding well-known theorem valid for the unreplicated fixed effects two-way layout ([3], p. 285, Theorem 2 and p. 287, a)). The latter proof can be found in the fourth section of this paper.

2. One-way analysis of the variance model and its generalization

Let us assume that equal numbers of observations are made at each level of the single factor having systematic effects on the result. The usual form of such a model is

$$\xi_{jk} = \gamma + \lambda_j + \varepsilon_{jk} \quad (j = \overline{1,m};\ k = \overline{1,n}), \tag{1}$$

where $\sum \lambda_j = 0$, γ is the so-called overall mean, and λ_j is a quantity corresponding to the effect of the j-th level of the factor. The random variables ε_{jk} $(j = \overline{1,m};\ k = \overline{1,n})$ are assumed to be independent and normally distributed with parameters 0 and σ, where σ is positive. The variance σ^2 is unknown.

Usual notations for the various sample means:

$$\bar{\xi}_{j.} = \frac{1}{n}\sum_{k=1}^{n}\xi_{jk}, \quad \bar{\xi} = \frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}\xi_{jk}.$$

The differences $\bar{\xi}_{j.} - \bar{\xi}\,(j = \overline{1,m})$ are discrepancies between the levels of the factor. The differences $\xi_{jk} - \bar{\xi}_{j.}\,(k = \overline{1,n})$ are discrepancies within the j-th level of the systematic factor. The latter differences are said to be random errors. One can prove that they have zero expectations.

By the help of the generalized model we shall prove the following Theorem. *If the expectations of the random errors are zero then the sample elements can be written in the form (1) assuming the existence of the expectations of the sample elements.*

Let $\boldsymbol{\xi}$ be a matrix of dimension $m \times n$. Let its elements be the random variables $\xi_{jk}\,(j = \overline{1,m};\ k = \overline{1,n})$ defined by (1). In this case

$$\boldsymbol{\xi} = \|\gamma\|_{m\times n} + \|\lambda_j\|_{j=\overline{1,m};k=\overline{1,n}} + \|\varepsilon_{jk}\|_{m\times n}. \tag{2}$$

Then $M\{\|\varepsilon_{jk}\|_{m\times n}\} = \mathbf{0}_{m\times n}$ in consequence of (1). (2) can be written in the form

$$M(\boldsymbol{\xi}) = \gamma\mathbf{a}_0\mathbf{b}_0^* + \boldsymbol{\lambda}\mathbf{b}_0^* \tag{3}$$

applying the notations introduced.

Let $\mathbf{S}_1$ be a stochastic matrix of order m having identical elements $\dfrac{1}{m}$. Let $\mathbf{S}_2$ be a stochastic matrix of order n consisting of identical elements $\dfrac{1}{n}$. Then

$$\mathbf{S}_1 = \frac{1}{m}\mathbf{a}_0\mathbf{a}_0^*, \quad \mathbf{S}_2 = \frac{1}{n}\mathbf{b}_0\mathbf{b}_0^*. \tag{4}$$

Let us further define the following matrix-valued random variables of dimension $m \times n$:

$$\boldsymbol{\eta}_2 = \boldsymbol{\xi}\mathbf{S}_2^*, \quad \boldsymbol{\zeta} = \mathbf{S}_1\boldsymbol{\xi}\mathbf{S}_2^*. \tag{5}$$

Then

$$\boldsymbol{\eta}_2 = \begin{pmatrix} \bar{\xi}_{1.}, & \bar{\xi}_{1.}, & \dots, & \bar{\xi}_{1.} \\ \dots & & & \\ \bar{\xi}_{m.}, & \bar{\xi}_{m.}, & \dots, & \bar{\xi}_{m.} \end{pmatrix}_{m\times n}, \quad \boldsymbol{\zeta} = \|\bar{\xi}\|_{m\times n}.$$

In this way

$$\boldsymbol{\xi} - \boldsymbol{\eta}_2 = \|\xi_{jk} - \bar{\xi}_{j.}\|_{j=\overline{1,m};k=\overline{1,n}}$$

is *the matrix of the random errors.*

$$\boldsymbol{\eta}_2 - \boldsymbol{\xi} = \|\bar{\xi}_{j.} - \bar{\xi}\|_{m\times n}$$

is *the matrix of the discrepancies between the effects due to the levels of the systematic factor*. The undermentioned two theorems have been formulated and proved in [6] (p. 295, Theorem 2 and Theorem 3):

Theorem 1. *The decomposition (3) is valid if and only if*

$$M(\boldsymbol{\xi} - \boldsymbol{\eta}_2) = \mathbf{0}_{m\times n}. \tag{6}$$

Theorem 2. *Let (3) be true. Then*

$$M(\boldsymbol{\eta}_2 - \boldsymbol{\zeta}) = \mathbf{0}_{m\times n}$$

if and only if

$$\boldsymbol{\lambda} = c\mathbf{a}_0,$$

where c is a constant.

The following minimum dyadical representation of $M(\xi)$ can be found in [6] (p. 297, (29)):

$$M(\boldsymbol{\xi}) = \begin{pmatrix} \gamma + \lambda_1 \\ \vdots \\ \gamma + \lambda_m \end{pmatrix} (1, \dots, 1)_{1\times n}. \tag{7}$$

Therefore the rank of $M(\boldsymbol{\xi})$ is 1. On the basis of the foregoing we can say for the case $m = n = 1$ that *in the one-way analysis of the variance model the sample elements can be written in the form* (1) *if and only if the expectations of their random errors are zero.*

3. The proof of a criterion with a recent method

We assume the existence of the expectations of the sample elements ξ_{jk} ($j = \overline{1, m}$; $k = \overline{1, n}$).

Theorem 3. *(1) is true for the sample elements if and only if the expectations of the random errors are zero.*

Proof. *If (1) is true for the sample elements then the expectations of the random errors are zero.*

To prove this direction one has to substitute ξ_{jk} from (1) in $M(\xi_{jk} - \bar{\xi}_{j.})$ and to take into consideration the conditions prescribed earlier for the model.

We shall prove Theorem 3 by showing the next theorem. (In the case $m = n = 1$ Theorem 4 is identical with Theorem 3.)

Theorem 4. *(3) is true if and only if*

$$M(\boldsymbol{\xi} - \boldsymbol{\eta}_2) = \mathbf{0}_{m\times n}.$$

PROOF. 1. *If (3) is true then* $M(\boldsymbol{\xi} - \boldsymbol{\eta}_2) = \mathbf{0}$. The truth of this statement follows from the fact that the elements of the expectation of the random error matrix are equal to zero.

2. *If* $M(\boldsymbol{\xi} - \boldsymbol{\eta}_2) = \mathbf{0}$ *then* $M(\boldsymbol{\xi}) = \gamma \mathbf{a}_0 \mathbf{b}_0^* + \boldsymbol{\lambda} \mathbf{b}_0^*$.

In consequence of (5)

$$M(\boldsymbol{\xi} - \boldsymbol{\eta}_2) = M(\boldsymbol{\xi})(\mathbf{E} - \mathbf{S}_2)^*.$$

So (6) can be written in the form:

$$\mathbf{E}M(\boldsymbol{\xi}) - M(\boldsymbol{\xi})\mathbf{S}_2 = \mathbf{0}, \tag{8}$$

where the matrices $\mathbf{E}$ and $\mathbf{S}_2$ have simple structures. Since the equation (8) is of form $\mathbf{AX} - \mathbf{XB} = \mathbf{0}$, that is (8) is a homogeneous linear matrix equation, we can use a well-known theorem to solve the matrix equation (8) ([2], p. 202, Satz 1).

The Jordan normal forms for our symmetric matrices are

$$\mathbf{E}_{m\times m} = \mathbf{V}\mathbf{V}^*, \tag{9}$$

where $\mathbf{V}$ is an orthogonal matrix;

$$\mathbf{S}_2 = \mathbf{U} \begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 0, & \dots, & 0 \\ \vdots & & & \\ 0, & 0, & \dots, & 0 \end{pmatrix}_{n\times n} \mathbf{U}^*, \tag{10}$$

and here

$$\mathbf{U} = \begin{pmatrix} \sqrt{n}^{-1}, & u_{12}, & \dots, & 0 \\ \sqrt{n}^{-1}, & u_{22}, & \dots, & 0 \\ \vdots & & & \\ \sqrt{n}^{-1}, & u_{n2}, & \dots, & u_{nn} \end{pmatrix}_{n\times n}$$

is also an orthogonal matrix. Substituting (9) and (10) in (8) we get

$$\mathbf{V}\mathbf{V}^* M(\boldsymbol{\xi}) - M(\boldsymbol{\xi})\mathbf{U} \begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 0, & \dots, & 0 \\ \vdots & & & \\ 0, & 0, & \dots, & 0 \end{pmatrix} \mathbf{U}^* = \mathbf{0}_{m\times n}. \tag{11}$$

Pre-and post-multiplying (11) by $\mathbf{V}^*$ and $\mathbf{U}$

$$\mathbf{V}^* M(\boldsymbol{\xi})\mathbf{U} - \mathbf{V}^* M(\boldsymbol{\xi})\mathbf{U} \begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 0, & \dots, & 0 \\ \vdots & & & \\ 0, & 0, & \dots, & 0 \end{pmatrix} = \mathbf{0}. \tag{12}$$

With the notation

$$\tilde{M}(\boldsymbol{\xi}) = \mathbf{V}^* M(\boldsymbol{\xi})\mathbf{U},$$

we get from (12)

$$\tilde{M}(\boldsymbol{\xi}) \begin{pmatrix} 0, & 0, & \dots, & 0 \\ 0, & 1, & \dots, & 0 \\ \vdots & & & \\ 0, & 0, & \dots, & 1 \end{pmatrix}_{n\times n} = \mathbf{0}. \tag{13}$$

Suppose that

$$\tilde{M}(\boldsymbol{\xi}) = \|\tilde{m}_{jk}\|_{m\times n}.$$

Therefore from (13)

$$\begin{pmatrix} 0, & \tilde{m}_{12}, & \tilde{m}_{13}, & \dots, & \tilde{m}_{1n} \\ 0, & \tilde{m}_{22}, & \tilde{m}_{23}, & \dots, & \tilde{m}_{2n} \\ \vdots & & & & \\ 0, & \tilde{m}_{m2}, & \tilde{m}_{m3}, & \dots, & \tilde{m}_{mn} \end{pmatrix} = \mathbf{0}.$$

Finally

$$\tilde{M}(\boldsymbol{\xi}) = \begin{pmatrix} \tilde{m}_{11}, & 0, & 0, & \dots, & 0 \\ \tilde{m}_{21}, & 0, & 0, & \dots, & 0 \\ \vdots & & & & \\ \tilde{m}_{m1}, & 0, & 0, & \dots, & 0 \end{pmatrix}.$$

Hence $\tilde{M}(\boldsymbol{\xi})$ involves m free parameters which differ from zero. Since $M(\boldsymbol{\xi}) = \mathbf{V}\tilde{M}(\boldsymbol{\xi})\mathbf{U}^*$, and $\mathbf{U}$ is given at (10), the elements of $M(\boldsymbol{\xi})$ are identical in each row. Let $\tilde{\lambda}_j$ $(j = \overline{1,m})$ be the element of the j-th row of the expectation matrix. For this reason

$$M(\boldsymbol{\xi})_{m\times n} = \begin{pmatrix} \tilde{\lambda}_1, & \tilde{\lambda}_1, & \dots, & \tilde{\lambda}_1 \\ \tilde{\lambda}_2, & \tilde{\lambda}_2, & \dots, & \tilde{\lambda}_2 \\ \vdots & & & \\ \tilde{\lambda}_m, & \tilde{\lambda}_m, & \dots, & \tilde{\lambda}_m \end{pmatrix}. \tag{14}$$

If $\tilde{\lambda}_l \neq 0$ then the minimum dyadical decomposition of $M(\xi)$ on the basis of [1] or [6] is as follows:

$$M(\boldsymbol{\xi}) = \frac{1}{\tilde{\lambda}_l} \begin{pmatrix} \tilde{\lambda}_1 \\ \vdots \\ \tilde{\lambda}_m \end{pmatrix} (\tilde{\lambda}_l, \dots, \tilde{\lambda}_l)_{1\times n} = \begin{pmatrix} \tilde{\lambda}_1 \\ \vdots \\ \tilde{\lambda}_m \end{pmatrix} (1, \dots, 1)_{1\times n}.$$

If

$$\tilde{\lambda}_l = \gamma + \lambda_l \quad (l = \overline{1, m}),$$

then

$$M(\boldsymbol{\xi}) = (\gamma \mathbf{a}_0 + \boldsymbol{\lambda})\mathbf{b}_0^*.$$

This means that Theorem 4 is valid.

Remark 1. It follows from Theorem 4 for the fixed effects one-way analysis of the variance model:

The expectations of the sample elements can be written in the form

$$M(\xi_{jk}) = \gamma + \lambda_j \ (j = \overline{1, m};\ k = \overline{1, n})$$

if and only if

$$M(\xi_{jk} - \bar{\xi}_{j.}) = 0.$$

Remark 2. The method used in this section is applicable with the randomized block design to prove the corresponding theorems. In the proof of the theorems the following theorem must be applied:

The general solution of $\mathbf{AX} - \mathbf{XB} = \mathbf{F}$ $(\mathbf{F} \neq \mathbf{0})$ *is the sum of the general solution of the homogeneous linear matrix equation* $\mathbf{AX} - \mathbf{XB} = \mathbf{0}$ *and of an arbitrary particular solution of the nonhomogeneous linear matrix equation* $\mathbf{AX} - \mathbf{XB} = \mathbf{F}$ ([2], pp. 208–209).

4. A criterion for the fixed effects two-way layout

In the first place we give a matrixical generalization of the unreplicated fixed effects two-way layout on the basis of [3] ([3], p. 284, Corollary 4; [3] pp. 285–287, third section). Then we formulate a criterion for the generalized model. To prove the theorem we shall apply results valid for the general solution of the nonhomogeneous linear matrix equation $\mathbf{AX} - \mathbf{XB} = \mathbf{F}$ and the minimum dyadical representation of a matrix.

Let us consider the matrix

$$\boldsymbol{\xi} = \|\xi_{jk}\|_{m\times n}, \tag{15}$$

where the elements ξ_{jk} ($j = \overline{1,n}$; $k = \overline{1,n}$) have expectations. According to the definition of the randomized block design let

$$M(\xi_{jk}) = \gamma + \lambda_j + \mu_k, \tag{16}$$

where γ is a constant (overall mean), λ_j corresponds to the j-th block-effect and μ_k amounts to the effect of the k-th treatment. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be stochastic matrices of order m and n, respectively. Suppose that $\mathbf{S}_1$ has identical elements $\frac{1}{m}$ and $\mathbf{S}_2$ has identical elements $\frac{1}{n}$, and they have 1 as a simple eigenvalue ([3], p. 284, Corollary 4). Let us further define the matrix-valued random variables

$$\boldsymbol{\eta}_1 = \mathbf{S}_1\boldsymbol{\xi}, \quad \boldsymbol{\eta}_2 = \boldsymbol{\xi}\mathbf{S}_2^*, \quad \boldsymbol{\zeta} = \mathbf{S}_1\boldsymbol{\xi}\mathbf{S}_2^*. \tag{17}$$

On the basis of (16) the expectation of $\boldsymbol{\xi}$ can be given in the form

$$M(\boldsymbol{\xi}) = \gamma\mathbf{a}_0\mathbf{b}_0^* + \boldsymbol{\lambda}\mathbf{b}_0^* + \mathbf{a}_0\boldsymbol{\mu}^*. \tag{18}$$

Here $\mathbf{a}_0$ denotes the m-dimensional column-vector consisting only of components 1, $\mathbf{b}_0^*$ is the row-vector of dimension n composed only of components 1, $\boldsymbol{\lambda}$ is the m-dimensional column-vector with components $\lambda_1, \lambda_2, \dots, \lambda_m$, where λ_j corresponds to the j-th row-effect ($j = \overline{1,m}$) and $\boldsymbol{\mu}^*$ is the n-dimensional row-vector of the k-th column-effects μ_k ($k = \overline{1,n}$).

It seems from (18) that the expectation of $\boldsymbol{\xi}$ is the sum of three dyads. From the definition of $\mathbf{S}_1$ and $\mathbf{S}_2$ one can get

$$\mathbf{S}_1 = \frac{1}{m}\mathbf{a}_0\mathbf{a}_0^*, \quad \mathbf{S}_2 = \frac{1}{n}\mathbf{b}_0\mathbf{b}_0^*. \tag{19}$$

Let us introduce the usual notations for the marginal and total means:

$$\bar{\xi}_{j.} = \frac{1}{n}\sum_{k=1}^{n}\xi_{jk}, \quad (j = \overline{1,m});$$

$$\bar{\xi}_{.k} = \frac{1}{m}\sum_{j=1}^{m}\xi_{jk}, \quad (k = \overline{1,n});$$

$$\bar{\xi} = \frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}\xi_{jk}.$$

Then the elements of the matrix η_1 are equal column by column to the quantities $\bar{\xi}_{.k}$ ($k = \overline{1,n}$), the elements of η_2 are equal row-by-row to the quantities $\bar{\xi}_{j.}$ and each element of $\boldsymbol{\zeta}$ is $\bar{\xi}$.

We call the the differences $\bar{\xi}_{j.} - \bar{\xi}\,(j = \overline{1,m})$, $\bar{\xi}_{.k} - \bar{\xi}\,(k = \overline{1,n})$ *discrepancies between rows* and *discrepancies between columns*, respectively. The quantities $\xi_{jk} - \bar{\xi}_{j.} - \bar{\xi}_{.k} + \bar{\xi}$ are *the random errors* $(j = \overline{1,m};\ k = \overline{1,n})$. It is easy to prove the next theorem for the above - mentioned model applying the definition and the customary assumptions $\sum_{j=1}^{m} \lambda_j = 0$, $\sum_{k=1}^{n} \mu_k = 0$.

If $M(\xi_{jk})$ *can be decomposed into the form of* (16), *then the expectation of the random error equals zero.* Since

$$\begin{aligned} \boldsymbol{\eta}_1 - \boldsymbol{\zeta} &= \begin{pmatrix} \bar{\xi}_{.1} - \bar{\xi}, & \ldots, & \bar{\xi}_{.n} - \bar{\xi} \\ \vdots & & \\ \bar{\xi}_{.1} - \bar{\xi}, & \ldots, & \bar{\xi}_{.n} - \bar{\xi} \end{pmatrix}_{m\times n}, \\ \boldsymbol{\eta}_2 - \boldsymbol{\zeta} &= \begin{pmatrix} \bar{\xi}_{1.} - \bar{\xi}, & \ldots, & \bar{\xi}_{1.} - \bar{\xi} \\ \vdots & & \\ \bar{\xi}_{m.} - \bar{\xi}, & \ldots, & \bar{\xi}_{m.} - \bar{\xi} \end{pmatrix}_{m\times n}, \end{aligned} \tag{20}$$

and $\boldsymbol{\xi} - \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 + \boldsymbol{\zeta}$ is composed of the random errors

$$\xi_{jk} - \bar{\xi}_{j.} - \bar{\xi}_{.k} + \bar{\xi}\ (j = \overline{1,m}; k = \overline{1,n}\,),$$

we call the matrix $\boldsymbol{\eta}_1 - \boldsymbol{\zeta}$ *the matrix of the discrepancies between columns*, the matrix $\boldsymbol{\eta}_2 - \boldsymbol{\zeta}$ is *the matrix of discrepancies between rows* and the matrix $\boldsymbol{\xi} - \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 + \boldsymbol{\zeta}$ is the so-called *random error matrix*.

Theorem 5. *If the expectations of the elements of* ξ *can be decomposed into the sum of three quantities as in (16), then the expectation of the random error matrix is the zero matrix.*

Proof. The usual restrictions for the λ_j and μ_k quantities, as has been mentioned earlier, are

$$\sum_{j=1}^{m} \lambda_j = 0 \quad \text{and} \quad \sum_{k=1}^{n} \mu_k = 0. \tag{21}$$

By the help of (21) we obtain the following formulae:

$$\begin{aligned} M(\boldsymbol{\eta}_1) &= M(\|\bar{\xi}_{.k}\|_{m\times n}) \\ &= \|\gamma + \mu_k\|_{m\times n}, \\ M(\boldsymbol{\eta}_2) &= M(\|\bar{\xi}_{j.}\|_{m\times n}) \\ &= \|\gamma + \lambda_j\|_{m\times n} \quad \text{and} \\ M(\boldsymbol{\zeta}) &= M(\|\bar{\xi}\|_{m\times n}) \\ &= \|\gamma\|_{m\times n}. \end{aligned} \tag{22}$$

So—in consequence of (16) and (22)—$M(\boldsymbol{\xi} - \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 + \boldsymbol{\zeta}) = \mathbf{0}$.

Theorem 5′. *If all the elements of* $\boldsymbol{\xi}$ *have expectations and*

$$M(\boldsymbol{\xi}) = \gamma \mathbf{a}_0 \mathbf{b}_0^* + \mathbf{a}_0 \boldsymbol{\lambda}^* + \boldsymbol{\mu} \mathbf{b}_0^*$$

then

$$M(\boldsymbol{\xi} - \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 + \boldsymbol{\zeta}) = \mathbf{0}.$$

Proof. The truth of the statement of Theorem 5′ comes from Theorem 5.

Now we formulate the converse of Theorem 5′.

Theorem 6. *If* $M(\xi_{jk})$ $(j = \overline{1,m};\ k = \overline{1,n})$ *exists and*

$$M(\boldsymbol{\xi} - \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 + \boldsymbol{\zeta}) = \mathbf{0}, \tag{23}$$

then

$$M(\boldsymbol{\xi}) = \gamma \mathbf{a}_0 \mathbf{b}_0^* + \mathbf{a}_0 \boldsymbol{\mu}^* + \boldsymbol{\lambda} \mathbf{b}_0^*.$$

Remark 3. Theorem 5′ and Theorem 6 are equivalent to Theorem 2 in [3] (p. 285).

Remark 4. Since the rank of $M(\boldsymbol{\xi})$ is 2, which can be seen also with the minimum dyadical decomposition ([1]),

$$M(\boldsymbol{\xi}) = \gamma \mathbf{a}_0 \mathbf{b}_0^* + \mathbf{a}_0 \boldsymbol{\mu}^* + \boldsymbol{\lambda} \mathbf{b}_0^*$$

can be written in the following forms:

$$\begin{aligned} M(\boldsymbol{\xi}) &= \mathbf{a}_0(\gamma \mathbf{b}_0^* + \boldsymbol{\mu}^*) + \boldsymbol{\lambda} \mathbf{b}_0^*, \quad \text{or} \\ M(\boldsymbol{\xi}) &= \mathbf{a}_0 \boldsymbol{\mu}^* + (\gamma \mathbf{a}_0 + \boldsymbol{\lambda}) \mathbf{b}_0^*. \end{aligned}$$

In the proof of Theorem 6 we apply the theorem about the general solution of the nonhomogeneous linear matrix equation $\mathbf{AX} - \mathbf{XB} = \mathbf{F}$ ($\mathbf{F} \neq \mathbf{0}$).

Proof of Theorem 6. In consequence of (17) we get

$$M(\boldsymbol{\xi} - \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 + \boldsymbol{\zeta}) = M[(\mathbf{E} - \mathbf{S}_1)]\boldsymbol{\xi}(\mathbf{E} - \mathbf{S}_2)^*].$$

Therefore the other form of (23) is

$$(\mathbf{E} - \mathbf{S}_1)_{m \times m} M(\boldsymbol{\xi})_{m \times n} (\mathbf{E} - \mathbf{S}_2)^*_{n \times n} = \mathbf{0}. \tag{24}$$

Let

$$\bar{M}(\boldsymbol{\xi}) = (\mathbf{E} - \mathbf{S}_1) M(\boldsymbol{\xi}). \tag{25}$$

Then (24) can be written in the form

$$\mathbf{E}\bar{M}(\boldsymbol{\xi}) - \bar{M}(\boldsymbol{\xi})\mathbf{S}_2 = \mathbf{0}. \tag{26}$$

This is similar to equation (8). Consequently, taking into consideration (14), the solution of (26) is

$$\bar{M}(\boldsymbol{\xi})_{m\times n} = \begin{pmatrix} \hat{\lambda}_1, & \hat{\lambda}_1, & \dots, & \hat{\lambda}_1 \\ \hat{\lambda}_2, & \hat{\lambda}_2, & \dots, & \hat{\lambda}_2 \\ \vdots & & & \\ \hat{\lambda}_m, & \hat{\lambda}_m, & \dots, & \hat{\lambda}_m \end{pmatrix}, \tag{27}$$

On the other hand, from (25)

$$\mathbf{E}M(\boldsymbol{\xi}) - \mathbf{S}_1 M(\boldsymbol{\xi}) = \bar{M}(\boldsymbol{\xi}).$$

This can be written in the form

$$\mathbf{S}_1 M(\boldsymbol{\xi}) - M(\boldsymbol{\xi})\mathbf{E} = -\bar{M}(\boldsymbol{\xi}). \tag{28}$$

The problem is to determine the general solution of the nonhomogeneous linear matrix equation (28). For this reason we give the general solution of the homogeneous linear matrix equation

$$\mathbf{S}_1 M(\boldsymbol{\xi}) - M(\boldsymbol{\xi})\mathbf{E} = \mathbf{0}. \tag{29}$$

Let the Jordan normal form of the symmetric coefficient matrix $\mathbf{S}_1$ be

$$\mathbf{S}_1 = \mathbf{W} \begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 0, & \dots, & 0 \\ \vdots & & & \\ 0, & 0, & \dots, & 0 \end{pmatrix}_{m\times m} \mathbf{W}^*, \tag{30}$$

where $\mathbf{W}$ is an orthogonal matrix which has the form

$$\mathbf{W} = \begin{pmatrix} \sqrt{m}^{-1}, & w_{12}, & \dots, & 0 \\ \sqrt{m}^{-1}, & w_{22}, & \dots, & 0 \\ \vdots & & & \\ \sqrt{m}^{-1}, & w_{m2}, & \dots, & w_{mm} \end{pmatrix}_{m\times m}.$$

Let the Jordan normal form of $\mathbf{E}$ be

$$\mathbf{E}_{n\times n} = \mathbf{T}\mathbf{T}^*, \tag{31}$$

where $\mathbf{T}$ is also an orthogonal matrix. On substituting (30) and (31) in (29)

$$\mathbf{W} = \begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 0, & \dots, & 0 \\ \vdots & & & \\ 0, & 0, & \dots, & 0 \end{pmatrix}_{m\times m} \mathbf{W}^* M(\boldsymbol{\xi}) - M(\boldsymbol{\xi})\mathbf{T}\mathbf{T}^* = \mathbf{0}. \tag{32}$$

Pre- and post-multiplying (32) by $\mathbf{W}^*$ and $\mathbf{T}$ and considering the orthogonality we get

$$\begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0, & 0, & \dots, & 0 \\ \vdots & & & \\ 0, & 0, & \dots, & 0 \end{pmatrix}_{m\times m} \mathbf{W}^* M(\boldsymbol{\xi})\mathbf{T} - \mathbf{W}^* M(\boldsymbol{\xi})\mathbf{T} = \mathbf{0}. \tag{33}$$

Let us introduce the notation

$$\tilde{M}(\boldsymbol{\xi}) = \mathbf{W}^* M(\boldsymbol{\xi})\mathbf{T}. \tag{34}$$

Therefore we obtain from (33)

$$\begin{pmatrix} 0, & 0, & \dots, & 0 \\ 0, & 1, & \dots, & 0 \\ \vdots & & & \\ 0, & 0, & \dots, & 1 \end{pmatrix}_{m\times m} \tilde{M}(\boldsymbol{\xi}) = \mathbf{0}. \tag{35}$$

Let $\tilde{M}(\boldsymbol{\xi}) = \|\tilde{m}_{jk}\|_{m\times n}$. In consequence of (35) there are n free parameters different from zero in $\tilde{M}(\boldsymbol{\xi})$, that is

$$\tilde{M}(\boldsymbol{\xi}) = \begin{pmatrix} \tilde{m}_{11}, & \tilde{m}_{12}, & \dots, & \tilde{m}_{1n} \\ 0, & 0, & \dots, & 0 \\ \vdots & & & \\ 0, & 0, & \dots, & 0 \end{pmatrix}.$$

From (34)

$$M(\boldsymbol{\xi}) = \mathbf{W}\tilde{M}(\boldsymbol{\xi})\mathbf{T}^*.$$

This means that $M(\boldsymbol{\xi})$ consisits of columnwise identical elements. If we introduce the notations $\tilde{\mu}_1, \dots, \tilde{\mu}_n$ for the elements of the columns of $M(\boldsymbol{\xi})$, then

$$M(\boldsymbol{\xi})_{m\times n} = \begin{pmatrix} \tilde{\mu}_1, & \tilde{\mu}_2, & \dots, & \tilde{\mu}_n \\ \tilde{\mu}_1, & \tilde{\mu}_2, & \dots, & \tilde{\mu}_n \\ \vdots & & & \\ \tilde{\mu}_1, & \tilde{\mu}_2, & \dots, & \tilde{\mu}_n \end{pmatrix}. \tag{36}$$

Suppose that $\tilde{\mu}_l \neq 0\,(l = \overline{1, n}\,)$. Then the minimum dyadical representation of (36) is

$$\begin{aligned} M(\boldsymbol{\xi}) &= \frac{1}{\tilde{\mu}_l} \begin{pmatrix} \tilde{\mu}_l \\ \vdots \\ \tilde{\mu}_l \end{pmatrix}_{m\times 1} (\tilde{\mu}_1, \dots, \tilde{\mu}_n) \\ &= \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{m\times 1} (\tilde{\mu}_1, \dots, \tilde{\mu}_n). \end{aligned} \tag{37}$$

With the notation $\tilde{\mu}_l = \gamma + \mu_l\,(l = \overline{1, n}\,)$ one can obtain from (37)

$$\begin{aligned} M(\boldsymbol{\xi}) &= \gamma \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{m\times 1} (1, \dots, 1)_{1\times n} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{m\times 1} (\mu_1, \dots, \mu_n) \\ &= \gamma \mathbf{a}_0 \mathbf{b}_0^* + \mathbf{a}_0 \boldsymbol{\mu}^*. \end{aligned}$$

This expression is the general solution of the homogeneous linear matrix equation (29).

It is easy to prove that $\boldsymbol{\lambda}\mathbf{b}_0^*$ is a particular solution of the nonhomogeneous linear matrix equation (28) by substituting in it $M(\boldsymbol{\xi}) = \boldsymbol{\lambda}\mathbf{b}_0^*$ and $\bar{M}(\boldsymbol{\xi})$ on the basis of (27).

Finally the general solution of the nonhomogeneous linear matrix equation (28) is given by the following expression:

$$M(\xi) = \gamma \mathbf{a}_0 \mathbf{b}_0^* + \mathbf{a}_0 \boldsymbol{\mu}^* + \boldsymbol{\lambda}\mathbf{b}_0^*.$$

This proves the statement of Theorem 6.

Remark 5. Applying a suitable transformation γ may become zero.

In the case of $m = n = 1$ on the base of Theorem $5'$ and of Theorem 6 the following criterion may be formulated for the fixed effects two-way layout.

An arbitrary sample element can be written in the form

$$\xi_{jk} = \gamma + \lambda_j + \mu_k + \varepsilon_{jk}$$

if and only if

$$M(\xi_{jk} - \bar{\xi}_{j.} - \bar{\xi}_{.k} + \bar{\xi}) = 0,$$

where ε_{jk} *is a random variable having normal distribution with the mean zero and unknown variance* σ^2.

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