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Iteration of some integral $T_{p,q}$

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An operational calculus $CO(L^0, L^1, S, T_q, s_q, Q)$ is a set $(L^0, L^1, S, T_q, s_q, Q)$, where L^1, L^0 are linear spaces. The linear operation $S : L^1 \to L^0$ called a derivative is a surjection. The set Q is a set of indices q for the linear operation $T_q : L^0 \to L^1$ and for the linear operation $s_q : L^1 \to \operatorname{Ker} S$ such that

$$ST_q f = f \quad \text{for} \quad f \in L^0, \quad q \in Q\,,$$
$$T_q Sx = x - s_q x \quad \text{for} \quad x \in L^1, \quad q \in Q\,.$$

The operation T_q is called an integral. The operation s_q is called a limit condition. (For the definition and properties of an operational calculus see for example [1–3].)

In this paper we present the formula for the iteration of an operation $T_{p,q}$ given by (6) and the application of this formula to solving the following problem

$$S_p^n x := (S + p\,\mathrm{id})^n x = f,$$
$$s_{p,q}(S + p\,\mathrm{id})^i x = x_i^p \in \operatorname{Ker} S_p \quad \text{for} \quad i = 0, 1, \ldots, n-1\,,$$

where id is an identity operation and the operation $s_{p,q}$ is given by formula (7).

Let an operational calculus $CO(L^0, L^1, S, T_q, s_q, Q)$ be given, where $L^1 \subset L^0$, L^1, L^0 are commutative algebras with unity **1** over the field of real numbers and with the multiplication such that for $x, y \in L^1$

$$S(x \cdot y) = (Sx) \cdot y + x \cdot (Sy)\,, \tag{1}$$
$$s_q(x \cdot y) = (s_q x) \cdot (s_q y)\,. \tag{2}$$

Definition 1. ([5]) We will say that there exists an element $u := E_{\mathbf{1}}^{T_q p}$ if and only if the element $E_{\mathbf{1}}^{T_q p}$ is a solution of the abstract differential equation

$$Su = p \cdot u \tag{3}$$

with condition

$$s_q u = \mathbf{1}\,, \tag{4}$$

where $u \in L^1,\ p \in L^0$ and $E_{\mathbf{1}}^{T_q p} \in \mathrm{Inv}\,.$

Theorem 1. (see [5]) *If there exists an element $E_{\mathbf{1}}^{T_q p}$ then the three operations*

$$S_p u := Su + pu\,, \tag{5}$$

$$T_{p,q} f := \left[T_q \left(f \cdot E_{\mathbf{1}}^{T_q p}\right)\right] \cdot E_1^{-T_q p} \tag{6}$$

$$s_{p,q} := (s_q u) \cdot E_1^{-T_q p} \tag{7}$$

satisfy the axioms of operational calculus, where $u \in L^1,\ f \in L^0$. The operation S_p is a derivative, the operation $T_{p,q}$ is an integral, the operation $s_{p,q}$ is a limit condition.

Definition 2. (see [5]) If there exists an element $E_{\mathbf{1}}^{T_q p}$ then for the elements $x, y \in L^0$ we will define the multiplication $x \circ y$ by the formula

$$x \circ y := x \cdot y \cdot E_{\mathbf{1}}^{T_q p}\,. \tag{8}$$

Corollary 1. (see [5]) *The multiplication $\circ$ satisfies condition* (1) *for the derivative S_p and condition* (2) *for the limit condition $s_{p,q}$. For the multiplication $\circ$ the unity $\mathbf{1}_\circ$ is defined by the formula*

$$\mathbf{1}_\circ = E_{\mathbf{1}}^{-T_q p}\,.$$

Theorem 2. *We have*

$$T_{p,q}^n f = \left[T_q^n \left(f \cdot E_{\mathbf{1}}^{T_q p}\right)\right] \cdot E_{\mathbf{1}}^{-T_q p}, \quad \text{where } f \in L^0\,. \tag{9}$$

PROOF. We proceed by induction. The formula is true for $n = 1$. We must show that

$$T_{p,q}^{n+1} f = \left[T_q^{n+1} \left(f \cdot E_{\mathbf{1}}^{T_q p} \right) \right] \cdot E_{\mathbf{1}}^{-T_q p} . \tag{10}$$

From our inductive assumption and from the definition of the operation $T_{p,q}$ we will get

$$\begin{aligned} & T_{p,q}^{n+1} f = T_{p,q} T_{p,q}^n f = \\ & = \left[T_q \left((T_{p,q}^n f) \cdot E_{\mathbf{1}}^{T_q p} \right) \right] \cdot E_{\mathbf{1}}^{-T_q p} = \\ & = \left[T_q \left\{ \left(\left[T_q^n \left(f \cdot E_{\mathbf{1}}^{T_q p} \right) \right] \cdot E_{\mathbf{1}}^{-T_q p} \right) \cdot E_{\mathbf{1}}^{T_q p} \right\} \right] \cdot E_{\mathbf{1}}^{-T_q p} = \\ & = \left[T_q^{n+1} \left(f \cdot E_{\mathbf{1}}^{T_q p} \right) \right] \cdot E_{\mathbf{1}}^{-T_q p} \end{aligned}$$

□

Corollary 2.

$$T_{p,q}^n \mathbf{1}_\circ = \left[T_q^n 1 \right] \cdot E_{\mathbf{1}}^{-T_q p}$$

Definition 3. (see [2,3]) Let

$$L^n := \left\{ x \in L^{n-1} : \ Sx \in L^{n-1} \right\}, \quad n = 2, 3, \dots .$$

Theorem 3. *The abstract differential equation*

$$(S + p \, \mathrm{id})^n x = f \tag{11}$$

with conditions

$$s_{p,q} S_p^i x = x_i^p \in \operatorname{Ker} S_p \quad \text{for} \ \ i = 0, 1, 2, \dots, n-1 \tag{12}$$

where $x \in L^n, \ p \in L^{n-1}, \ f \in L^0, \ \mathrm{id}\, x = x$ *has only one solution given by the formula*

$$x = x_0^p + T_{p,q} x_1^p + T_{p,q}^2 x_2^p + \cdots + T_{p,q}^{n-1} x_{n-1}^p + T_{p,q}^n f . \tag{13}$$

PROOF. It is known from operational calculus that the abstract differential equation

$$S^n x = f, \quad f \in L^0, \quad x \in L^n$$

with conditions

$$s_q S^i x = x_i \in \operatorname{Ker} S, \quad i = 0, 1, \dots, n-1$$

has only one solution x of the form

$$x = \sum_{i=0}^{n-1} T_q^i s_q S^i x + T_q^n f = \sum_{i=0}^{n-1} T_q^i x_i + T_q^n f\,. \quad \text{(see [2,3])}$$

Making use of the last fact it is easy to notice that formula (13) is true

□

Corollary 3. *If we introduce multiplication* $\circ$ *in* L^0 *then the solution* (13) *of the problem* (11), (12) *can be written in the form*

$$x = x_0^p + x_1^p \circ T_{p,q}\mathbf{1}_\circ + x_2^p \circ T_{p,q}^2 \mathbf{1}_\circ + \cdots + \\ + x_{n-1}^p \circ T_{p,q}^{n-1}\mathbf{1}_\circ + T_{p,q}^n f\,. \tag{14}$$

Theorem 4. *The abstract differential equation* (11) *with conditions*

$$s_q S^i x = x_i \in \operatorname{Ker} S \quad \textit{for} \;\; i = 0, 1, \ldots, n-1 \tag{15}$$

has only one solution.

Proof. The conditions (15) are equivalent to conditions (16), i.e. knowing conditions (15) we can define conditions (12) and conversely. Thus it follows from theorem 3 that the problem (11), (15) has only one solution.

□

Remark. For arbitrary n it is difficult to find the solution of the problem (11), (15). Therefore, as an example we will formulate a theorem which will give the form of the solution for $n = 3$.

Theorem 5. *The abstract differential equation*

$$(S + p\,\mathrm{id})^3 x = f \tag{16}$$

with conditions

$$s_q S^i x = x_i\,, \qquad i = 0, 1, 2 \tag{17}$$

where $x \in L^3$, $p \in L^2$, $f \in L^0$, $x_i \in \operatorname{Ker} S$ *for* $i = 0, 1, 2$ *has only one solution given by the formula*

$$x = x_0 \cdot E_{\mathbf{1}}^{-T_q p} + T_{p,q}\left\{(x_1 + x_0(s_q p)) \cdot E_1^{-T_q p}\right\} + \\ + T_{p,q}^2\left\{\big(x_2 + 2x_1 \cdot (s_q p) + x_0(s_q S p + (s_q p)^2)\big) \cdot E_1^{-T_q p}\right\} + T_{p,q}^3 f\,. \tag{18}$$

PROOF. Applying conditions (17) we can write

$$s_{p,q}x = x_0 \cdot E_{\mathbf{1}}^{-T_q p} = x_0^p\,,$$
$$s_{p,q}S_p x = [x_1 + x_0 \cdot (s_q p)] \cdot E_1^{-T_q p} = x_1^p\,,$$
$$s_{p,q}S_p^2 x = \big[x_2 + 2(s_q p) \cdot x_1 + \big((s_q p)^2 + s_q S_p\big) \cdot x_0\big] \cdot E_{\mathbf{1}}^{-T_q p} = x_2^p\,.$$

Substituting these conditions into formula (13) in theorem 3 for $n = 3$ we will get formula (18) as the solution x of the problem (16), (17). It follows from theorem 4 that it is the only solution of the problem (16), (17). □

Example A. Let us consider an operational calculus with the derivative

$$S\big\{u(x_1, x_2, \ldots, x_m)\big\} := \left\{ \sum_{i=1}^{m} b_i \frac{\partial u(x_1, x_2, \ldots, x_m)}{\partial x_i} \right\}$$

the integral

$$T_{x_m^0}\big\{f(x_1, x_2, \ldots, x_m)\big\} :=$$
$$:= \left\{ \frac{1}{b_m} \int_{x_m^0}^{x_m} f\left(x_1 - \frac{b_1}{b_m}(x_m - \tau), \ldots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau),\, \tau\right) d\tau \right\},$$

and the limit condition

$$s_{x_m^0}\big\{u(x_1, x_2, \ldots, x_m)\big\} :=$$
$$:= \left\{ u\left(x_1 - \frac{b_1}{b_m}(x_m - x_m^0), \ldots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - x_m^0),\, x_m^0\right) \right\},$$

where $u \in L^1 := C^2(R^{m-1} \times \langle x_m^1,\, x_m^2\rangle, R)$,
$f \in L^0 := C^1(R^{m-1} \times \langle x_m^1,\, x_m^2\rangle, R)$,
$x_m^0 \in \langle x_m^1,\, x_m^2\rangle$, $b_i \in R$ for $i = 1, 2, \ldots, m$, $b_m \neq 0$ (see [4]).
For such a model of operational calculus operations S_p, T_{p,x_m^0}, s_{p,x_m^0} are defined in [5]. Following this paper we put

$$S_p\big\{u(x_1, x_2, \ldots, x_m)\big\} := \left\{ \sum_{i=1}^{m} b_i \frac{\partial u(x_1, x_2, \ldots, x_m)}{\partial x_i} + p(x_1, x_2, \ldots, x_m)u(x_1, x_2, \ldots, x_m)\right\}, \tag{19}$$

$$T_{p,x_m^0}\Big\{f(x_1, x_2, \ldots, x_m)\Big\} := \tag{20}$$

$$:= \left\{ e^{-\frac{1}{b_m}\int\limits_{x_m^0}^{x_m} p\left(x_1-\frac{b_1}{b_m}(x_m-\tau),x_2-\frac{b_2}{b_m}(x_m-\tau),\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-\tau),\tau\right)d\tau} \cdot \right.$$

$$\cdot \frac{1}{b_m}\int\limits_{x_m^0}^{x_m} f\left(x_1-\frac{b_1}{b_m}(x_m-\tau),\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-\tau),\tau\right)\cdot$$

$$\left. \cdot\, e^{-\frac{1}{b_m}\int\limits_{x_m^0}^{\tau} p\left(x_1-\frac{b_1}{b_m}(x_m-\xi),\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-\xi)\right)d\xi} d\tau \right\},$$

(21) $$s_{p,x_m^0}\Big\{u(x_1,x_2,\ldots,x_m)\Big\} :=$$

$$:= \Big\{u\left(x_1-\frac{b_1}{b_m}(x_m-x_m^0),\ x_2-\frac{b_2}{b_m}(x_m-x_m^0),\ldots\right.$$

$$\left.\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-x_m^0),\ x_m^0\right)\cdot$$

$$\cdot\, e^{-\frac{1}{b_m}\int\limits_{x_m^0}^{x_m} p\left(x_1-\frac{b_1}{b_m}(x_m-\tau),\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-\tau),\tau\right)d\tau}\Big\},$$

where $u\in L^1,\ f,p\in L^0$.

In [6] there is shown a formula for the iteration of the integral $T_{x_m^0}$. It has the form

(22) $$T^n_{x_m^0}\Big\{f(x_1,x_2,\ldots,x_m)\Big\} =$$

$$= \Big\{\left(\frac{1}{b_m}\right)^n \int\limits_{x_m^0}^{x_m} \frac{(x_m-\tau)^{n-1}}{(n-1)!} f\left(x_1-\frac{b_1}{b_m}(x_m-\tau),\ldots\right.$$

$$\left.\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-\tau),\tau\right)d\tau\Big\}.$$

In [6] it is also shown that for $c\in \operatorname{Ker}\left(\sum\limits_{i=1}^m b_i\frac{\partial}{\partial x_i}\right)$ the formula

(23) $$T^n_{x_m^0}\, c = c\left(\left\{\frac{1}{b_m}\right)^n \frac{(x_m-x_m^0)^n}{n!}\right\}$$

is true. From the formulas (20), (22) and Theorem 2 we have

(24) $$T^n_{p,x_m^0}\Big\{f(x_1,x_2,\ldots,x_m)\Big\} =$$

$$= \left(\frac{1}{b_m}\right)^n \left\{ \int_{x_m^0}^{x_m} \frac{(x_m-\tau)^{n-1}}{(n-1)!} f\left(x_1 - \frac{b_1}{b_m}(x_m-\tau), \ldots \right.\right.$$

$$\left. \ldots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m-\tau), \tau\right) \cdot$$

$$\left. \cdot\, e^{\frac{1}{b_m}\int_{x_m^0}^{\tau} p\left(x_1-\frac{b_1}{b_m}(x_m-\xi),\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-\xi),\xi\right)d\xi} d\tau \right\} \cdot$$

$$\cdot \left\{ e^{-\frac{1}{b_m}\int_{x_m^0}^{x_m} p\left(x_1-\frac{b_1}{b_m}(x_m-\tau),\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-\tau),\tau\right)d\tau} \right\},$$

where $f \in L^0$. It follows from (24) that for

$$c^p = c \left\{ e^{-\frac{1}{b_m}\int_{x_m^0}^{x_m} p\left(x_1-\frac{b_1}{b_m}(x_m-\tau),\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-\tau),\tau\right)d\tau} \right\},$$

where $c \in \operatorname{Ker}\left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i}\right)$, i.e. for $c^p \in \operatorname{Ker} S_p$ the formula

$$T^n_{p,x_m^0}\, c^p = c^p \left\{ \left(\frac{1}{b_m}\right)^n \frac{(x_m - x_m^0)^n}{n!} \right\}$$

is true.

Example B. It follows from theorem 3 and example A that the partial differential equation

$$(25) \qquad \left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i} + p(x_1, x_2, \ldots, x_m)\,\mathrm{id}\right)^n \Big\{x(x_1, x_2, \ldots, x_m)\Big\} = \\ = \Big\{f(x_1, x_2, \ldots, x_m)\Big\}$$

with conditions

$$(26) \quad s_{x_m^0} \left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i} + p(x_i, x_2, \ldots, x_m)\,\mathrm{id}\right)^i \Big\{x(x_1, x_2, \ldots, x_m)\Big\} = \\ = \varphi_i \in \operatorname{Ker}\left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i}\right) \quad \text{for} \quad i = 0, 1, \ldots, n-1,$$

where $x \in L^n$, $p \in L^{n-1}$, $f \in L^0$ (L^0 and L^1 are defined in example A while L^n is defined in definition 3), $x_m \in \langle x_m^1, x_m^2\rangle$, $b_i \in R$ for $i = 1, 2, \ldots, m$, $b_m \neq 0$ has only one solution given by

$$x = \left\{x(x_1, x_2, \ldots, x_m)\right\} =$$

$$= e^{-\frac{1}{b_m}\int\limits_{x_m^0}^{x_m} p\left(x_1 - \frac{b_1}{b_m}(x_m-\tau),\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-\tau),\tau\right)d\tau} \cdot$$

$$\left(\varphi_0 + \frac{1}{b_m}\varphi_1(x_m - x_m^0) + \left(\frac{1}{b_m}\right)^2 \varphi_2 \frac{(x_m - x_m^0)^2}{2!} + \ldots\right.$$

$$\left.\left(\frac{1}{b_m}\right)^{n-1} \varphi_{n-1}\frac{(x_m - x_m^0)^{n-1}}{(n-1)!}\right) + T^n_{p,x_m^0}\left\{f(x_1, x_2, \ldots, x_m)\right\},$$

where $T^n_{p,x_m^0}\{f(x_1, x_2, \ldots x_m)\}$ is defined by the formula (24).

Example C. The partial differential equation (25) for $n = 3$ with conditions

$$\left\{x(x_1, x_2, \ldots, x_m^0)\right\} = \left\{\psi_0(x_1, x_2, \ldots, x_{m-1})\right\},$$

$$\sum_{i=1}^m b_i \frac{\partial}{\partial x_i}\left\{x(x_1, x_2, \ldots, x_{m-1}, x_m^0)\right\} = \left\{\psi_1(x_1, x_2, \ldots x_{m-1})\right\},$$

$$\left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i}\right)^2 \left\{x(x_1, x_2, \ldots, x_{m-1}, x_m^0)\right\} = \left\{\psi_2(x_1, x_2, \ldots, x_{m-1})\right\},$$

where $\psi_i \in C^{4-i}(R^{m-1}, R)$ for $i = 0, 1, 2$ has on the basis of theorem 5 only one solution

$$x = e^{-\frac{1}{b_m}\int\limits_{x_m^0}^{x_m} p(x_1 - \frac{b_1}{b_m}(x_m-\tau),\ldots,x_{m-1}-\frac{b_{m-1}}{b_m}(x_m-\tau),\tau)d\tau}$$

$$\left\{\psi_0(v_1, v_2, \ldots, v_{m-1}) + \frac{1}{b_m}\psi_1(v_1, v_2, \ldots, v_{m-1}) + \right.$$

$$+ \psi_0(v_1, v_2, \ldots, v_{m-1})\ p(v_1, v_2, \ldots, v_{m-1}, x_m^0)(x_m - x_m^0) +$$

$$+ \left[\psi_2(v_1, v_2, \ldots, v_{m-1}) + \right.$$

$$+ 2\psi_1(v_1, v_2, \ldots, v_{m-1})p(v_1, v_2, \ldots, v_{m-1}, x_m^0) +$$

$$+ \psi_0(v_1, v_2, \ldots, v_{m-1})\left(s_{x_m^0} \sum_{i=1}^m b_i \frac{\partial}{\partial x_i}\left\{p(x_1, x_2, \ldots, x_m)\right\} + \right.$$

$$+\left(p(v_1,v_2,\dots,v_{m-1},x_m^0)\right)^2\Big)\Big]\left(\frac{1}{b_m}\right)^2\frac{(x_m-x_m^0)^2}{2!}\Bigg\}+$$
$$+T^3_{p,x_m^0}\Big\{f(x_1,x_2,\dots,x_m)\Big\},$$

where $v_i=x_i-\dfrac{b_i}{b_m}(x_m-x_m^0)$ for $i=1,2,\dots m-1$ and T^3_{p,x_m^0} is defined by the formula (24) for $n=3$.

Similarly, further examples for the application of the theorems formulated can be given, making use of other models of operational calculus (models with different derivatives S).

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