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On the characterization of additive functions on rare sets

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Throughout this paper $f : \mathbf{N} \to \mathbf{R}^k$ denotes an additive function and $\|v\|$ is the (arbitrary, but fixed) norm of the vector $v \in \mathbf{R}^k$.

A set $A = \{a_1 < a_2 < \dots < a_m < \dots\} \subset \mathbf{N}$ is said to be a set of uniqueness if the characterizing condition

$$\text{(C1)} \qquad f(a_m) = 0 \quad m = 1, 2, \dots$$

implies $f \equiv 0$.

Weaker conditions of charecterization are the following:

$$\text{(C2)} \qquad f(a_{m+1}) - f(a_m) \ \text{ is convergent,}$$

$$\text{(C3)} \qquad \|f(a_{m+1}) - f(a_m)\| \ \text{is convergent,}$$

$$\text{(C4)} \qquad \|f(a_{m+1})\| - \|f(a_m)\| \ \text{ is convergent.}$$

R. Freud shoved in [1] that to any function $h : \mathbf{N} \to \mathbf{R}$ there exists a set A satisfying the rarity condition

$$\text{(R1)} \qquad \frac{a_{m+1}}{a_m} > h(m)$$

and (C2) implies $f \equiv 0$. (He dealt with $f : \mathbf{N} \to \mathbf{C}$ functions, but the proof remains valid for $f : \mathbf{N} \to \mathbf{R}^k$ functions as well.)

In this paper we shall prove that any of the weaker characterizing conditions (C3) and (C4) also implies $f \equiv 0$, but instead of (R1) we can guarantee only the rarity condition

$$\text{(R2)} \qquad a_{m+1} - a_m > h(m)\,.$$

Theorem. *a) To any function $h : \mathbf{N} \to \mathbf{R}$ we can construct a set A satisfying* (R2) *such that* (C4) *implies $f \equiv 0$.*

b) The same holds if (C4) *is replaced by* (C3).

Remark 1. It is not difficult to verify that for an arbitrary(not necessarily additive) function f we have (C2) $\Rightarrow$ (C3), (C2) $\Rightarrow$ (C4), (C3) $\not\Rightarrow$ (C2), (C4) $\not\Rightarrow$ (C2), (C3) $\not\Rightarrow$ (C4) and (C4) $\not\Rightarrow$ (C3).

Remark 2. If f is completely additive, then the rarity condition in the Theorem can be replaced even by

$$\text{(R3)} \qquad a_{m+1} > h(a_m)$$

if we consider the set

$$A = \bigcup_{p\,\text{prime}} \left\{ p^{k_{ip}}, p^{s_{ip}}, i = 1, 2, \dots, \lim_{i\to\infty} (s_{ip} - k_{ip}) = \infty \right\},$$

where the values of k_{ip} and s_{ip} can be chosen so that $p^{k_{ip}}$ and $p^{s_{ip}}$ are consecutive elements of A.

Proof of the Theorem. We shall use several times the following proposition.

Lemma. *We can find a constant $\alpha > 1$ and a positive integer B such that given any set of B vectors in $\mathbf{R}^k$ there must be two among them, say u and v, which satisfy*

$$\text{(1)} \qquad \|u + v\| \geq \alpha \min(\|u\|, \|v\|).$$

This is clear for the euclidean norm (e.g. with $\alpha = \sqrt{2}$, and $B = 2^k + 1$) and it can be easily verified for many other norms as well. Intuitively it can be justified for *any* norm by saying that if there are sufficiently many vectors, then there must be a u and v among them which fall nearly in the same direction, hence $\|u + v\|$ is nearly as big as $\|u\| + \|v\|$. This argument implies that (1) holds for all $\alpha < 2$ (of course B depends on α). M. Laczkovich was so kind to provide us with a formal proof (moreover he pointed out that improving the ideas even the intersting fact can be shown that B does *not* depend on the norm).

Proof of the Lemma. We shall give M. Laczkovich's proof for the somewhat stronger statement suggested by the heuristic argument above, i.e. we shall show that to any $\varepsilon > 0$ there is a B such that for arbitrary B vectors, there must be a u and a v among them satisfying

$$\text{(1A)} \qquad \|u + v\| \geq (1 - \varepsilon)(\|u\| + \|v\|).$$

Consider the set $T = \{v; \varepsilon/2 \leq \|v\| \leq 1\}$. We "draw" around each point of T an "open sphere" of "radius" $\varrho = \varepsilon^2/2$. Since T is compact, we can select finitely many, say M, of these spheres which still cover T. We claim that $B = M + 1$ will satisfy the requirements.

Take any B vectors in $\mathbf{R}^k$.After multiplying them uniformly by a suitable constant, we may assume that the maximal norm among them is exactly 1.

If $\|u\| < \varepsilon/2$ holds for some vector u then taking the vector v with $\|v\| = 1$ we have

$$\|u+v\| \geq \|v\| - \|u\| > 1 - \varepsilon/2 > (1-\varepsilon)(1+\varepsilon/2) > (1-\varepsilon)(\|u\| + \|v\|)\,.$$

Hence we have to deal only with the case when all vectors are in T. Since $B > M$, there must be two vectors, u and v, which are in the same small sphere, i.e. $\|u - v\| < 2\varrho$. Since

$$\|u\| + \|v\| = \left\|\frac{u+v}{2} + \frac{u-v}{2}\right\| + \left\|\frac{u+v}{2} - \frac{u-v}{2}\right\| \leq$$
$$\leq 2\left\|\frac{u+v}{2}\right\| + 2\left\|\frac{u-v}{2}\right\| = \|u+v\| + \|u-v\|$$

we have

$$\|u+v\| \geq \|u\| + \|v\| - \|u-v\| > \|u\| + \|v\| - 2\varrho \geq (1-\varepsilon)(\|u\| + \|v\|).$$

(We used $\|u\| \geq \varepsilon/2, \|v\| \geq \varepsilon/2$ in the final inequality.)

We turn now to the proof of part a) of the Theorem. To each fixed $n \geq 2$ let us choose infinitely many finite sequences of primes $> n$ so that the i–th sequence contains exactly i primes. We denote the i–th sequence by $(p_{nij})_{j=1}^{i}$. Let further be

$$A_{n,i} = \bigcup_{1\leq j\leq i} \{p_{nij},\, np_{nij},\, p_{nij}p_{nil}, \quad 0 < l,\ 0 < j - l \leq B\}\,.$$

We can choose the primes p_{nij} so that the elements of the above blocks in $A_{n,i}$ are strictly increasing as j is increasing. Finally let

$$A_m = A_{2,m-1} \cup \cdots \cup A_{j,m+1-j} \cup \cdots \cup A_{m,1} \quad \text{and} \quad A = \bigcup_m A_m\,.$$

By a suitable choice of p_{nij} we can guarantee that the order of $A_{n,t}$ in the definition of A_m denotes the natural increasing order of the elements of $A_{n,t}$ and the ordering of the sets A_m in A conforms to the increasing of m, further the rarity condition (R2) is satisfied.

Let f be a function satisfying (C4). Then we have

$$\lim_{m\to\infty} (\|f(a_{m+1})\| - \|f(a_m)\| = c\,, \tag{2}$$

where $c \geq 0$ obviously.

If for some $n \in \mathbf{N}$ there exists a w such that $\{f(p_{niw}\}_{i=1}^{\infty}$ is not bounded, then by (2) there exists also a sequence (i_t) such that

(3) $$\|f(p_{ni_tj})\| \to \infty \quad \text{as} \quad i_t \to \infty \quad \text{for all} \quad w \le j \le w+B$$

strictly monotonically increasing.

By (1) we can find $r, s \in \{w, w+1, \dots, w+B\}$ $(r \ne s)$ for wich

(4) $$\|f(p_{ni_tr}p_{ni_ts})\| \ge \alpha \min(\,\|f(p_{ni_tr})\|, \|f(p_{ni_ts})\|\,)$$

(r and s may depend on i_t), i.e.

(5) $$\|f(p_{ni_tr}p_{ni_ts})\| - \|f(p_{ni_tz})\| \ge (\alpha-1)\|f(p_{ni_tz})\|$$

holds for $z = r$ or $z = s$.

At the same time p_{ni_tr}, p_{ni_ts} and $p_{ni_tr}p_{ni_ts}$ are in A and there are no more than $L = (B+2)^2$ elements of A between any two of them. Hence if i_t is large enough then (2) implies

$$\|f(p_{ni_tr}p_{ni_ts})\| - \|f(p_{ni_tz})\| \le L \max\{1, 2c\}$$

for $z = r$ or $z = s$, which, by (3), contradicts (5).

So $\{f(p_{nij})\}_{i=1}^{\infty}$ must bounded for any fixed $n, j \in \mathbf{N}$. Hence for a suitable $(v_t)_{t=1}^{\infty}$ we have

(6) $$\lim_{t\to\infty} f(p_{nv_tj}) = c_{n,j}$$

where (v_t) depends on j (and on n). We start with the sequence (v_t) belonging to $j = 1$, retain its first element and thin out the rest to abtain a sequence for $j = 2$, again we retain the first element, thin out, etc. Using this simple selecting process, we arrive at a *universal* sequence (v_t) such that (6) is uniformly satisfied for all j (we keep n fixed).

Let us consider first the case when the limit c in (2) is $\ne 0$. Then we have

(7) $$2c > \|f(a_{m+1})\| - \|f(a_m)\| > c/2$$

if m is large enough. If $j > B$, then

$$p_{nv_tj}, \quad np_{nv_tj}, \quad p_{nv_tj}p_{nv_t,j-B}, \dots, \; p_{nv_tj}p_{nv_t,j-1}, \; p_{nv_t,j+1}$$

are consecutive elements of A, hence (6) and (7) imply

$$\|c_{n,j+1}\| - \|c_{n,j}\| > \frac{Bc}{2}\,,$$

i.e.

(8) $$\|c_{n,j}\| \to \infty \quad \text{as} \quad j \to \infty\,.$$

Let $j > v_t/2$. By similar arguments as in (4) and (5) and using the monotonicity of$\|c_{n,j}\|$, we can find $r, s \in \{j, j+1, \dots, j+B\}$, $r < s$, satisfying

$$\|f(p_{nv_t r}p_{nv_t s})\| - \|f(p_{nv_t r})\| \geq \frac{(\alpha-1)\|c_{n,r}\|}{2} \tag{9}$$

it t is large enough. Here the left hand side is bounded (since $p_{nv_t r}$ and $p_{nv_t s}p_{nv_t r}$ are almost consecutive elements of A and we can use (7)) and the right hand side is unbounded by (8) since $t \to \infty$ and $j > v_t/2$ imply also $r \to \infty$.

Let us turn now to the case $c = 0$ in (2). If for some $n \in \mathbf{N}$ there exists a d for which $f(p_{ni_s d}) \to 0$ for some sequence (i_s), then (2) implies also

$$\|f(n)\| \leq \|f(np_{ni_s d})\| + \|f(p_{ni_s d})\| \to 0 \quad \text{as} \quad s \to \infty\,,$$

i.e. $f(n) = 0$ obviously.

If there is no such a d then let us consider the sequence $(p_{nij})_{i=1}^{\infty}$ for each fixed $j \in \{1, \dots, B\}$. We can choose a $(v_t)_{t=1}^{\infty}$, for wich

$$\lim_{t\to\infty} f(p_{nv_t j}) = c_{n,j} \neq 0\,. \tag{10}$$

By (1) we can find $r, s \in \{1, \dots, B\}$ and v_t large enough so that (9) is satisfied. Using (2) the left hand side of (9) tends to zero as $v_t \to \infty$, which contradicts (10).

We turn now to the proof of part b) of the Theorem. To any $n \in \mathbf{N}$ let us choose a strictly monotonically increasing sequence $(t_{ni})_{i=1}^{\infty}$, for which $(t_{ni}, n) = 1$. Let

$$A_{ni} = \{t_{ni}, nt_{ni}\} \quad \text{and} \quad A = \bigcup\nolimits_{n,i} A_{ni}\,.$$

By a suitable choice of the numbers t_{ni} the two elements of A_{ni} will be consecutive elements in A. So

$$\lim_{i\to\infty} \|f(nt_{ni}) - f(t_{ni})\| = c$$

implies $\|f(n)\| = c$. This, by (1), gives $f \equiv 0$.

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References

[1] R. Freud, On sets characterizing additive arithmetical functions I, *Acta Arith.* **XXXV** (1979), 333–343.

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