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Oscillation of Nonlinear Differential Equations of Second Order

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Abstract. A new oscillation criterion is established for second order differential equations of the form

$$(a(t)x^{\cdot}(t))^{\cdot} + p(t)x^{\cdot}(t) + q(t)|x(t)|^{\lambda} \operatorname{sgn} x(t) = 0, \quad \lambda > 0,$$

where $a, p, q : [t_0, \infty) \to R$ are continuous and $a(t) > 0$ for $t \geq t_0$. The criterion is obtained by using an integral averaging technique and can be applied in some cases in which other known oscillation results are not applicable.

1. Introduction

In this paper we are concerned with the oscillatory behavior of second order ordinary differential equations of the type

$$(1) \qquad \begin{aligned} &(a(t)x^{\cdot}(t))^{\cdot} + p(t)x^{\cdot}(t) + q(t)|x(t)|^{\lambda} \operatorname{sgn} x(t) = 0, \\ &\lambda > 0, \quad \left(\cdot = \frac{d}{dt} \right), \end{aligned}$$

where $a, p, q : [t_0, \infty) \to R$ are continuous and $a(t) > 0$ for $t \geq t_0 \geq 0$.

We restrict our attention to solutions of equation (1) which exist on some ray $[t_0, \infty)$, $t_0 \geq 0$. Such a solution is said to be oscillatory if it has arbitrary large zeros; otherwise it is said to be nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

Recently, GRACE and LALLI [4] considered the second order differential equation

$$(2) \qquad (a(t)x^{\cdot}(t))^{\cdot} + p(t)x^{\cdot}(t) + q(t)f(x(t)) = 0,$$

where the functions a, p and q are defined as in equation (1), $f : R \to R$ is continuous, $xf(x) > 0$ and $f'(x) \geq k > 0$ for $x \neq 0$, $(\, ' = \dfrac{d}{dx})$, and established some oscillation criteria for equation (2) which extend and improve some of the results of BUTLER [1], COLES [2], [3], KAMENEV [5] and WILLETT [6]. It is observed that the results in [4] are not applicable to the special case of differential equation (1) because of the above restriction on the function f.

Therefore, the purpose of this paper is to establish some new theorems for the oscillation of equation (1) by using an averaging condition of the type presented in [4]. The results obtained for the superlinear case, i.e., $\lambda > 1$, extend and improve some of the results of BUTLER [1]. We also mention that the results of this paper for the sublinear case, i.e., $\lambda < 1$ are new and are independent of those in [1], [3], [6] and [7], but that they are similar to those in [1]–[6] when $\lambda = 1$.

2. Main results

Let $\phi(t, t_0)$ denote the class of all positive and locally integrable functions, but not integrable, which contains all bounded functions $t \geq t_0$.

We shall use the following notation. For an arbitrary function $\phi \in \Phi(t, t_0)$ and $\rho \in C' \left[[t_0, \infty), (0, \infty) \right]$ and for any $T \geq t_0$ and all T, we let

$$\alpha[t, T] = \int_T^t \phi(s) ds,$$

$$\eta(t) = \frac{1}{a(t)\rho(t)},$$

$$\gamma(t) = a(t)\rho^{\cdot}(t) - p(t)\rho(t),$$

$$w[t, T] = \eta(t) \left(\int_T^t \eta(s) ds \right)^{-1},$$

$$\nu[t, T^*] = \frac{1}{\phi(t)} \int_{T^*}^{t} \frac{\phi^2(s)}{w[s, T]} ds, \quad \text{for some} \quad T^* > T$$

and

$$A_\phi[t, T] = \frac{1}{\alpha[t, T]} \int_T^t \phi(s) \int_T^s \rho(u) q(u)\, du\, ds.$$

Theorem 1. *Soppuse that there exist functions $\phi \in \Phi(t, t_0)$ and $\rho \in C' \left[[t_0, \infty), (0, \infty) \right]$ such that*

$$\gamma(t) \geq 0 \quad \text{and} \quad \gamma^{\cdot}(t) \leq 0 \quad \text{for} \quad t \geq t_0. \tag{3}$$

$$\lim_{t\to\infty} \inf \int_{t_0}^{t} \rho(s)q(s)ds > -\infty; \tag{4}$$

$$\int^{\infty} \eta(s)ds = \infty, \tag{5}$$

and

$$\int^{\infty} \frac{\alpha^{\mu}[s,T]}{\nu[s,T]} ds = \infty \quad \text{for some} \quad T \geq t_0 \quad \text{and} \quad \mu,\ 0 \leq \mu < 1. \tag{6}$$

If

$$\lim_{t\to\infty} A_{\phi}[t,t_0] = \infty, \tag{7}$$

the equation (1) *is oscillatory for all* $\lambda > 1$.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality, we assume that $x(t) \neq 0$ for $t \geq t_0$. Furthermore, we suppose that $x(t) > 0$ for $t \geq t_0$, since the substitution $w = -x$ transforms equation (1) into an equation of the same form subject to the assumptions of the theorem. Now, we define

$$W(t) = \rho(t)\frac{a(t)x^{\cdot}(t)}{x^{\lambda}(t)} \quad \text{for} \quad t \geq t_0.$$

Then it follows from equation (1) that

$$W^{\cdot}(t) = -\rho(t)q(t) + \gamma(t)\frac{x^{\cdot}(t)}{x^{\lambda}(t)} - \lambda a(t)\rho(t)\frac{\dot{x}^2(t)}{x^{\lambda+1}(t)}, \quad t \geq t_0 \tag{8}$$

and consequently

$$\begin{aligned} a(t)\rho(t)\frac{x^{\cdot}(t)}{x^{\lambda}(t)} &= c_1 - \int_{t_0}^{t} \rho(s)q(s)ds + \int_{t_0}^{t} \gamma(s)\frac{x^{\cdot}(s)}{x^{\lambda}(s)}ds - \\ &- \int_{t_0}^{t} \lambda a(s)\rho(s)\left(\frac{x^{\cdot}(s)}{x^{\beta}(s)}\right)^2 ds, \end{aligned} \tag{9}$$

where $\beta = \dfrac{\lambda+1}{2}$ and $c_1 = a(t_0)\rho(t_0)\dfrac{x^{\cdot}(t_0)}{x^{\lambda}(t_0)}$.
By the Bonnet theorem, for any $t \geq t_0$ there exist a $\xi \in [t_0, t]$ so that

$$\int_{t_0}^{t} \gamma(s)\frac{x^{\cdot}(s)}{x^{\lambda}(s)}ds = \gamma(t_0)\int_{t_0}^{\xi}\frac{x^{\cdot}(s)}{x^{\lambda}(s)}ds = \gamma(t_0)\int_{x(t_0)}^{x(\xi)} u^{-\lambda}du$$
$$= \frac{\gamma(t_0)}{\lambda-1}\left[x^{1-\lambda}(t_0) - x^{1-\lambda}(\xi)\right] \leq \frac{\gamma(t_0)}{\lambda-1}x^{1-\lambda}(t_0) = M_1.$$

Thus, for $t \geq t_0$, we get

$$a(t)\rho(t)\frac{x^{\cdot}(t)}{x^{\lambda}(t)} + \lambda\int_{t_0}^{t} a(s)\rho(s)\left(\frac{x^{\cdot}(s)}{x^{\beta}(s)}\right)^2 ds + \int_{t_0}^{t}\rho(s)q(s)ds \leq L \tag{10}$$

or

$$W(t) + \lambda\int_{t_0}^{t}\frac{1}{a(s)\rho(s)}x^{2(\beta-1)}(s)W^2(s)ds + \int_{t_0}^{t}\rho(s)q(s)ds \leq L, \tag{11}$$

where $L = c_1 + M_1$.

Next, we consider the following three cases for the behavior of $x^{\cdot}$.

Case 1. $x^{\cdot}$ is oscillatory. Thus there exists a sequence $\{t_m\}_{m=1,2,\dots}$ in $[t_0,\infty)$ with $\lim_{m\to\infty} t_m = \infty$ and such that $x^{\cdot}(t_m) = 0$ $(m = 1, 2, \dots)$. Thus (8) gives

$$\int_{t_0}^{t}\lambda a(s)\rho(s)\left(\frac{x^{\cdot}(s)}{x^{\beta}(s)}\right)^2 ds \leq L - \int_{t_0}^{t_m}\rho(s)q(s)ds \ (m = 1, 2, \dots),$$

and hence, by taking into account (3), we conclude that

$$\int_{t_0}^{\infty} a(s)\rho(s)\left(\frac{x^{\cdot}(s)}{x^{\beta}(s)}\right)^2 ds < \infty. \tag{12}$$

So, for some positive constant N, we have

$$\int_{t_0}^{t} a(s)\rho(s)\left(\frac{x^{\cdot}(s)}{x^{\beta}(s)}\right)^2 ds \leq N \quad \text{for} \quad t \geq t_0.$$

By the Schwarz inequality

$$\left|\int_{t_0}^{t}\left(\frac{x^{\cdot}(s)}{x^{\beta}(s)}\right)ds\right|^2 \le \left(\int_{t_0}^{t}\frac{ds}{a(s)\rho(s)}\right)\left(\int_{t_0}^{t}a(s)\rho(s)\left(\frac{x^{\cdot}(s)}{x^{\beta}(s)}\right)^2 ds\right)$$
$$\le N\int_{t_0}^{t}\frac{ds}{a(s)\rho(s)} = N\int_{t_0}^{t}\eta(s)ds,$$

or

$$\left|x^{1-\beta}(t)-x^{1-\beta}(t_0)\right| \le |1-\beta|N^{1/2}\left(\int_{t_0}^{t}\eta(s)ds\right)^{1/2}.$$

There exists a $t_1 > t_0$ and a constant $M > 0$ so that

$$\left|x^{1-\beta}(t)\right| \le M\left(\int_{t_0}^{t}\eta(s)ds\right)^{1/2} \quad \text{for all} \quad t \ge t_1. \tag{13}$$

Using (13) in (8) we get

$$W^{\cdot}(t) \le -\rho(t)q(t)+\gamma(t)\eta(t)W(t)-\frac{\lambda}{M^2}w[t,t_0]W^2(t), \quad t \ge t_1.$$

Now, we proceed in a similar way as in the proof of Theorems 1 and 2 in [4].

As in the above proof, we easily obtain

$$W(t)+\frac{\lambda}{M^2}\int_{t_1}^{t}w[s,t_0]W^2(s)ds \le L_1-\int_{t_1}^{t}\rho(s)q(s)ds, \tag{14}$$

where L_1 is a constant.

We multiply (14) by $\phi(t)$ and integrate from t_1 to t we get

$$\int_{t_1}^{t}\phi(s)W(s)ds+\frac{\lambda}{M^2}\int_{t_1}^{t}\phi(s)\int_{t_1}^{t}w[u,t_0]W^2(u)\,du\,ds \le \alpha[t,t_1]\left[\,L_1-A_{\phi}[t,t_1]\,\right]. \tag{15}$$

Using condition (7), there exists a $t_2 \ge t_0$ such that

$$L_1-A_{\phi}[t,t_1]<0 \quad \text{for} \quad t \ge t_2.$$

Then, for every $t \geq t_2$

$$G(t) = \frac{\lambda}{M^2} \int_{t_1}^{t} \phi(s) \int_{t_1}^{s} w[u, t_0] W^2(u)\, du\, ds < - \int_{t_1}^{t} \phi(s) W(s) ds.$$

Since G is nonnegative we have

$$G^2(t) < \left(\int_{t_1}^{t} \phi(s) W(s) ds \right)^2, \quad t \geq t_2. \tag{16}$$

By the Schwarz inequality, we obtain

$$\begin{aligned} &\left\{ \int_{t_1}^{t} \left(\frac{\phi(s)}{\sqrt{w[s, t_0]}} \right) (\sqrt{w[s, t_0]} W(s)) ds \right\}^2 \leq \\ &\leq \left(\int_{t_1}^{t} \frac{\phi^2(s)}{w[s, t_0]} ds \right) \int_{t_1}^{t} w[s, t_0] W^2(s) ds = \\ &= \frac{M^2}{\lambda} \nu[t, t_1] G^{\cdot}(t), \quad t \geq t_2. \end{aligned} \tag{17}$$

Now,

$$\begin{aligned} G(t) &= \frac{\lambda}{M^2} \int_{t_1}^{t} \phi(s) \int_{t_1}^{t} w[u, t_0] W^2(u)\, du\, ds \geq \\ &\geq \frac{\lambda}{M^2} \int_{t_1}^{t} \phi(s) \left(\int_{t_1}^{t_2} w[u, t_0] W^2(u) du \right) ds = c\alpha[t, t_1], \end{aligned} \tag{18}$$

where

$$c = \frac{\lambda}{M^2} \int_{t_1}^{t_2} w[u, t_0] W^2(u) du.$$

From (16), (17) and (18), we get

$$\frac{\lambda c^{\mu}}{M^2} \frac{\alpha^{\mu}[t, t_1]}{\nu[t, t_1]} \geq G^{\mu-2}(t) G^{\cdot}(t) \quad \text{for all } t \geq t_2 \text{ and some } \mu,\ 0 \leq \mu < 1. \tag{19}$$

Integrating (19) from t_2 to t, we obtain

$$\frac{\lambda c^{\mu}}{M^2}\int_{t_2}^{t}\frac{\alpha^{\mu}[s,t_1]}{\nu[s,t_1]}ds \le \left[\frac{1}{1-\mu}\frac{1}{G^{1-\mu}(t_2)}\right] < \infty,$$

a contradiction.

Case 2. $x^{\cdot} > 0$ on $[T,\infty)$ for some $T \ge t_0$. From (2) and (10) it follows that (12) holds for $t \ge T$, hence we can complete the proof by the procedure of Case 1.

Case 3. $x^{\cdot} < 0$ on $[T,\infty)$ for some $T \ge t_0$. If (12) holds, then we can arrive at a contradiction by the procedure of Case 1. So, we suppose that the integral in (12) diverges. Using (3) in (10) we have

$$-a(t)\rho(t)\frac{x^{\cdot}(t)}{x^{\lambda}(t)} \ge -C + \lambda\int_{T}^{t} a(s)\rho(s)\left(\frac{(x^{\cdot}(s))^2}{x^{\lambda+1}(s)}\right)ds, \tag{20}$$

where C is a constant. By the assumption, we can choose a $T_1 \ge T$ so that

$$\lambda\int_{T}^{T_1} a(s)\rho(s)\frac{(x^{\cdot}(s))^2}{x^{\lambda+1}(s)}ds = 1 + C$$

and then for any $t \ge T_1$ we get

$$\frac{-a(t)\rho(t)\frac{x^{\cdot}(t)}{x^{\lambda}(t)}\left(-\lambda\frac{x^{\cdot}(t)}{x(t)}\right)}{-C+\lambda\int_{T}^{t} a(s)\rho(s)\frac{(x^{\cdot}(s))^2}{x^{\lambda+1}(s)}ds} \ge -\lambda\frac{x^{\cdot}(t)}{x(t)}$$

Integrate the above inequality from T_1 to t we obtain

$$\begin{aligned}\ln\left[-C+\lambda\int_{T}^{t} a(s)\rho(s)\frac{(x^{\cdot}(s))^2}{x^{\lambda+1}(s)}ds\right] &\ge \lambda\int_{t_1}^{t}\left(\frac{-x^{\cdot}(s)}{x(s)}\right)ds \\ &= \ln\left(\frac{x(T_1)}{x(t)}\right)^{\lambda}\end{aligned}$$

which together with (20) yields

$$-a(t)\rho(s)\frac{x^{\cdot}(t)}{x^{\lambda}(t)} \ge \left(\frac{x(T_1)}{x(t)}\right)^{\lambda},$$

from which it follows that

$$x^{\cdot}(t) \le -(x(T_1))^{\lambda} \frac{1}{a(t)\rho(s)} < 0 \quad \text{for} \quad t \ge T_1$$

or

$$x(t) \le x(T_1) - (x(T_1))^{\lambda} \int_{T_1}^{t} \frac{1}{a(s)\rho(s)} ds \to -\infty \quad \text{as} \quad t \to \infty,$$

contradicting the fact that $x(t) > 0$ for $t \ge t_0$. This completes the proof.

The following result is concerned with the oscillatory behavior of equation (1) for all $\lambda > 0$.

Corollary 1. *Let the differentiable function ρ assumed in Theorem 1 be defined by*

$$\rho(t) = \exp\left(\int_{t_0}^{t} \frac{p(s)}{a(s)} ds\right) \quad \text{for} \quad t \ge t_0, \tag{21}$$

and let conditions (3), (5)–(7) *hold. Then equation* (1) *is oscillatory for all $\lambda > 0$.*

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $t \ge t_0$. From (21), we see that $\gamma(t) = 0$ for all $t \ge t_0$. Now, if W is defined as in the proof of Theorem 1, then we obtain (10) or (11).

The rest of the proof is similar to that of Theorem 1 and hence is omitted.

In the following corollary we study the oscillatory behavior of the undamped equation

$$(a(t)x^{\cdot}(t))^{\cdot} + q(t)|x(t)|^{\lambda} \operatorname{sgn} x(t) = 0, \quad \lambda > 0, \tag{22}$$

where the functions a and q are defined as in equation (1).

Corollary 2. *Suppose that there exists a function $\phi \in \Phi(t, t_0)$ such that*

$$\liminf_{t \to \infty} \int_{t_0}^{t} q(s) ds > -\infty, \tag{23}$$

$$\int^{\infty} \frac{1}{a(s)} ds = \infty, \tag{24}$$

$$\int^{\infty} \frac{\alpha^{\mu}[s, T]}{\nu^{*}[s, T]} ds = \infty \text{ for some } T \ge t_0, \text{ and } \mu,\ 0 \le \mu < 1, \tag{25}$$

where

$$\nu^*[t,T] = \frac{1}{\phi(t)} \int_T^t a(s)\phi^2(s) \left(\int_T^s \frac{1}{a(\tau)} d\tau \right) ds.$$

If

$$\lim_{t\to\infty} \frac{1}{\alpha[t,t_0]} \int_{t_0}^t \phi(s) \int_{t_0}^s q(u)\, du\, ds = \infty, \tag{26}$$

then equation (22) *is oscillatory for all* $\lambda > 0$.

PROOF. It follows from Corollary 1 by letting $p(t) = 0$ and $\rho(t) = 1$ for $t \geq t_0$.

The following example is illustrative.

Example 1. Consider the second order differential equation

$$\begin{aligned} &\left(\frac{1}{\sqrt{t}} x^{\cdot}(t)\right)^{\cdot} + \frac{\delta}{t\sqrt{t}} x^{\cdot}(t) \\ &+ \frac{1}{\sqrt{t}} \left[\frac{2+\cos t}{2t} - \sin t\right] |x(t)|^{\lambda} \operatorname{sgn} x(t) = 0 \\ &\qquad \text{for} \quad \lambda > 0 \quad \text{and} \quad t \geq t_0 = \pi \setminus 2. \end{aligned} \tag{27}$$

Here, we take

$$a(t) = \frac{1}{\sqrt{t}}, \qquad p(t) = \frac{\delta}{t\sqrt{t}}, \qquad q(t) = \frac{1}{\sqrt{t}} \left[\frac{2+\cos t}{2t} - \sin t\right],$$
$$\rho(t) = t \quad \text{and} \quad \phi(t) = 1/t, \quad t \geq t_0 = \pi/2.$$

We consider the following two cases:

(i) Let $\delta = 1/2$ and $\lambda > 1$. In this case, we obtain

$$\gamma(t) = \frac{1}{2\sqrt{t}}, \; \alpha[t,\pi/2] = \ln\frac{2t}{\pi}, \quad \eta(t) = \frac{1}{\sqrt{t}},$$
$$w[t,\pi/s] = \frac{1}{2\sqrt{t}[\sqrt{t} - \sqrt{\pi/2}]}, \quad t > \pi/2$$

and

$$\nu[t,\pi] = 2t \left[\ln\frac{t}{\pi} - \sqrt{2}\left(1 - \sqrt{\pi/t}\right)\right] \quad \text{for} \quad t \geq t_1 = \pi > t_0.$$

Now, we can easily calculate that

$$\int^{t} \frac{(\alpha[s,t_1])^{\mu}}{\nu[s,t_1]} ds \geq \frac{1}{2} \int^{t} \frac{1}{s} \left(\ln \frac{s}{\pi}\right)^{\mu-1} ds \to \infty \text{ as } t \to \infty,\ \mu \geq 0$$

and

$$\begin{aligned} &\frac{1}{\int\limits_{t_0}^{t} \phi(s) ds} \int\limits_{t_0}^{t} \rho(u) q(u)\, du\, ds \\ &= \frac{1}{1n\frac{2t}{\pi}} \int\limits_{\pi/2}^{t} \frac{1}{s} \int\limits_{\pi/2}^{s} \sqrt{u} \left[\frac{2+\cos u}{2u} - \sin u\right] du\, ds \\ &\geq \frac{2}{1n\frac{2t}{\pi}} \left[\sqrt{t} - \left(1 + 21n\frac{2t}{\pi}\right)\sqrt{\pi/2}\right] \longrightarrow \infty \quad \text{as} \quad t \to \infty. \end{aligned}$$

Thus, all conditions of Theorem 1 are satisfied and hence equation (27) is oscillatory for all $\lambda > 1$.

(ii) Let $\delta = 1$ and $\lambda > 0$. Then we have

$$\gamma(t) = 0 \quad \text{for} \quad t \geq \pi/2.$$

As in the above case, we see that all conditions of Corollary 1 are satisfied and hence equation (27) is oscillatory for all $\lambda > 0$.

We note that none of the known oscillation criterion can cover this result.

Remarks.

1. In the results obtained here, we note that we do not require that $\int^{\infty} 1/a(s)ds$ to be finite or infinite, also we do not require that the damping coefficient $p(t)$ be a small function.
2. It would be interesting to obtain results similar to those presented here without condition (4), and also, to study equation (2) without the restriction that $f'(x) \geq k > 0$ for $x \neq 0$.

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