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On an Nikolskii-type inequality

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I. Introduction and result

S. M. NIKOLSKII [7] proved in 1951 an inequality of the form

$$(1.1)\qquad \| T_n(x) \|_{L^q[0,2\pi]} \le C n^{\frac{1}{p}-\frac{1}{q}} \| T_n(x) \|_{L^p[0,2\pi]}$$

where C is a constant independent of $T_n(x)$ and n, $1 \le p < q \le \infty$ and $T_n(x)$ is a trigonometric polynomial of order at most n. This inequality has had many applications in approximation theory, for example to embedding problems (see e.g. [10]).

N. X. KY [2] introduced so-called $\{N, \lambda\}$ systems ($\lambda > 0$), that is orthogonal systems $\omega = \{\omega_n(x)\}_{n=0}^{\infty}$, $x \in [a,b]$, $-\infty \le a, b \le \infty$, such that for each $\phi_n \in \pi_n(\omega)$ the following inequality is valid:

$$(1.2)\qquad \| \phi_n(x) \|_{L^q[a,b]} \le C n^{\lambda|\frac{1}{p}-\frac{1}{q}|} \| \phi_n(x) \|_{L^p[a,b]} \quad 1 \le p, q \le \infty,$$

where $\pi_n(\omega)$ is the set of the form $\sum_{k=0}^{n} a_k \omega_k(x)$, $a_k \in \mathbf{R}$, $k=0,1,2,\dots,n$. $n \in \mathbf{P} = \{0,1,2\dots\}$ and C is a constant independent of $p_n(x)$ and n.

From previous literature we can see that the classical orthonormal systems are all of such type (see e.g. [1], [4], [5]). In this paper we give an inequality of type (1.2) for the General Hermite system, that is the system $\{p_n(x) \cdot e^{-x^2/2} |x|^{\alpha/2}\}$ $(\alpha > 0)$, where $p_n(x)$ is the n–th orthonormal polynomial with respect to the weight $w(\alpha, x) = e^{-x^2/2} |x|^{\alpha/2}$ ($\alpha > 0$, $-\infty < x < \infty$). More precisely, we shall prove the following

Theorem. *Let $\alpha > 0$. For each $p_n \in \mathcal{P}$ $(n \in \mathbf{N})$ the following inequality is valid:*

$$(1.3)\qquad \| p_n(x) w(\alpha, x) \|_{L^1(-\infty,\infty)} \le C n^{1/2} \| p_n(x) w(\alpha, x) \|_{L^\infty(-\infty,\infty)}$$

Here (and throughout the paper) C is an absolute constant, which may be different at each occurrence. $\mathcal{P}_n$ is the set of algebraic polynomials of degree at most n. Moreover, the inequality (1.3) is sharp in the sense that there exists a sequence $\{\tilde{p}_n(x)\}_{n=0}^{\infty}$ of polynomials satisfying:

$$\| \tilde{p}_n(x)w(\alpha,x) \|_{L^1(-\infty,\infty)} \geq Cn^{1/2} \| \tilde{p}_n(x)w(\alpha,x) \|_{L^\infty(-\infty,\infty)}$$

II. Notations and lemmas

We shall use the following notations applied in [4].

$$(2.1) \qquad L^p_{U(\alpha)} := \Big\{ f;\ \| f(x)u(\alpha,x) \|_{L^p(0,\infty)} < \infty;\ u(\alpha,x) = e^{-x/2}x^{\alpha/2} \Big\}$$

$$(2.2) \qquad L^p_{H(\alpha)} := \Big\{ f; \| f(x)w(\alpha,x) \|_{L^p(-\infty,\infty)} < \infty;\ w(\alpha,x) = e^{-x^2/2}|x|^{\alpha/2} \Big\} \quad (\alpha > -1,\ 1 \leq p \leq \infty)$$

We denote the Laguerre polynomials and normalized Laguerre polynomials by

$$(2.3) \qquad L_n^\alpha(x) = (n!)^{-1}x^{-\alpha}e^{+x}\left(\frac{d}{dx}\right)^n \left(x^{\alpha+n}e^{-x}\right)$$

and by

$$(2.4) \qquad \hat{L}_n(x) = (-1)^n \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha(x) \quad (x > 0,\ \alpha > -1,\ n \in \mathbf{P})$$

respectively.
The normalized General Hermite polynomials will be denoted by $\hat{H}_n^\alpha(x)$ $(\alpha > 1,\ x \in \mathbf{R})$ i.e.

$$(2.5) \qquad \int_{-\infty}^{\infty} \hat{H}_n^\alpha(x)\hat{H}_m^\alpha(x)e^{-x^2}|x|^\alpha dx = \delta_{nm}$$

where δ_{nm} is the Kronecker symbol.

The Laguerre functions and General Hermite functions are denoted by

$$\mathcal{L}_n^\alpha(x) = e^{-x/2}x^{\alpha/2}\hat{L}_n^\alpha(x) \tag{2.6}$$

and by

$$\mathcal{H}_n^\alpha(x) = e^{-x^2/2}|x|^{\alpha/2}\hat{H}_n^\alpha(x) \tag{2.7}$$

$$(\alpha > -1,\ n \in \mathbf{P})$$

respectively.

Lemma 1. *Let $V_{n,\,2n-1}^{H(\alpha)}(f;x)$ be the de La Vallée–Poussin mean of the General Hermite expansion of a function $f \in L_{H(\alpha)}^p(1 \le p \le \infty)$, then*

$$V_{n,\,2n-1}^{H(\alpha)}(f;x) = \int_{-\infty}^{\infty} f(t) \sum_{k=0}^{2n-1} \Theta_{n,\,2n-1}(k)\hat{H}_k^\alpha(x)\hat{H}_k^\alpha(t)e^{-t^2}|t|^\alpha dt \tag{2.8}$$

where $\Theta_{n,2n-1}(k) = \begin{cases} 1 & ,k \le n \\ \frac{2n-k}{n} & ,n+1 \le k \le 2n-1 \end{cases}$

Proof. For $f \in L_{H(\alpha)}^p$, its Hermite–Fourier series is

$$\sum_{k=0}^{\infty} a_k\hat{H}_k^\alpha(x), \quad \text{where} \quad a_k = \int_{-\infty}^{\infty} f(t)\hat{H}_k^\alpha(t)e^{-t^2}|t|^\alpha dt \tag{2.9}$$

If we put $S_n(f;x) = \sum\limits_{k=0}^{n} a_k\hat{H}_k^\alpha(x)$, then

$$V_{n,\,2n-1}^{H(\alpha)}(f;x) = \frac{1}{n}\left(\sum_{k=0}^{2n-1} S_k(f;x) - \sum_{k=0}^{n-1} S_k(f;x)\right).$$

Using Abel transform, we have

$$\begin{aligned} \sum_{k=0}^{2n-1} S_k(f;x) &= \sum_{k=0}^{2n-1}\sum_{i=0}^{k} \int_{-\infty}^{\infty} f(t)\hat{H}_i^\alpha(t)\hat{H}_i^\alpha(x)e^{-t^2}|t|^\alpha dt \\ &= \sum_{k=0}^{2n-1} \int_{-\infty}^{\infty} f(t)\hat{H}_k^\alpha(t)\hat{H}_k^\alpha(x)e^{-t^2}|t|^\alpha dt \left(\sum_{i=k}^{2n-1} 1\right) \qquad (2.10) \\ &= \sum_{k=0}^{2n-1} (2n-k) \int_{-\infty}^{\infty} f(t)\hat{H}_k^\alpha(t)\hat{H}_k^\alpha(x)e^{-t^2}|t|^\alpha dt \end{aligned}$$

analogously,

$$(2.11)\qquad \sum_{k=0}^{n-1} S_k(f;x) = \sum_{k=0}^{n-1}(n-k)\int_{-\infty}^{\infty} f(t)\hat{H}_k^{\alpha}(t)\hat{H}_k^{\alpha}(x)e^{-t^2}|t|^{\alpha}dt$$

(2.10) and (2.11) lead to (2.8). q.e.d.

Denoting by $(C,1)_n^{H(\alpha)}(f;x)$ the first Cesàro mean of (2.9) we have

$$(2.12)\qquad (C,1)_n^{H(\alpha)}(f;x) = \frac{1}{n}\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k} a_j\hat{H}_j^{\alpha}(x)\right).$$

We can see that

$$(2.13)\quad (C,1)_n^{H(\alpha)}(f;x) = \int_{-\infty}^{\infty} f(t)\left(\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)\hat{H}_k^{\alpha}(x)\hat{H}_k^{\alpha}(t)\right)e^{-t^2}|t|^{\alpha}dt$$

Lemma 2.

$$(2.14)\qquad \hat{H}_{2n}^{\alpha}(x) = \hat{L}_n^{\frac{\alpha-1}{2}}(x^2)$$

$$(2.15)\qquad \hat{H}_{2n+1}^{\alpha}(x) = \hat{L}_n^{\frac{\alpha+1}{2}}(x^2)x \quad (\alpha > -1, x \in \mathbf{R})$$

PROOF. Let $y = x^2$, then

$$\int_{-\infty}^{\infty} \hat{L}_n^{\frac{\alpha-1}{2}}(x^2)\hat{L}_m^{\frac{\alpha-1}{2}}(x^2)e^{-x^2}|x|^{\alpha}dx = \int_0^{\infty}\hat{L}_n^{\frac{\alpha-1}{2}}(y)\hat{L}_m^{\frac{\alpha-1}{2}}(y)e^{-y}y^{\alpha/2}y^{-1/2}dy$$

$$= \int_0^{\infty}\hat{L}_n^{\frac{\alpha-1}{2}}(y)\hat{L}_m^{\frac{\alpha-1}{2}}(y)e^{-y}y^{\frac{\alpha-1}{2}}dy = \delta_{nm}$$

analogously,

$$\int_{-\infty}^{\infty}\hat{L}_n^{\frac{\alpha+1}{2}}(x^2)x\hat{L}_m^{\frac{\alpha+1}{2}}(x^2)xe^{-x^2}|x|^{\alpha}dx$$

$$= \int_0^{\infty}\hat{L}_n^{\frac{\alpha+1}{2}}(y)\hat{L}_m^{\frac{\alpha+1}{2}}(y)e^{-y}y^{\frac{\alpha+1}{2}}dy = \delta_n$$

finally,

$$\int_{-\infty}^{\infty} \hat{L}_n^{\frac{\alpha-1}{2}}(x^2)\hat{L}_m^{\frac{\alpha+1}{2}}(x^2)xe^{-x^2}|x|^\alpha dx = 0 \quad \forall n, m \in \mathbf{P}$$

as the integrand is an odd function. q.e.d.

Lemma 3.

$$(2.16)\qquad |\mathcal{H}_{2n}^\alpha(x)| \le C \begin{cases} \nu^{\frac{\alpha-1}{2}} x^{\frac{2\alpha-1}{2}} & ,0 \le x^2 \le 1/\nu \\ \nu^{-1/4} & ,1/\nu < x^2 \le \nu/2 \\ \left[\nu/x^2\left(\nu^{1/3} + |x^2-\nu|\right)\right]^{-1/4} & ,\nu/2 < x^2 \le 3\nu/2 \\ x^{1/2}e^{-\gamma x^2} & ,3\nu/2 < x^2 \end{cases}$$

$$(2.17)\qquad |\mathcal{H}_{2n+1}^\alpha(x)| \le C \begin{cases} \nu^{\frac{\alpha+1}{2}} x^{\frac{2\alpha-1}{2}} & ,0 \le x^2 \le 1/\nu \\ \nu^{-1/4} & ,1/\nu < x^2 \le \nu/2 \\ \left[\nu/x^2\left(\nu^{1/3} + |x^2-\nu|\right)\right]^{-1/4} & ,\nu/2 < x^2 \le 3\nu/2 \\ x^{1/2}e^{-\gamma x^2} & ,3\nu/2 < x^2 \end{cases}$$

where C, γ are positive constants independent of n, $\nu = 4n + \alpha + 1$.

PROOF. Let $y = x^2$. Using lemma 2. we have

$$\left|\mathcal{H}_{2n}^\alpha(x)\right| = \left|\hat{H}_{2n}^\alpha(x)e^{-x^2/2}|x|^{\alpha/2}\right| = \left|\hat{L}_n^{\frac{\alpha-1}{2}}(x)e^{-y/2}y^{\alpha/4}\right|$$

using [3] , (2.8) we get (2.16). The proof of (2.17) is analogous. q.e.d.

Corollary 1. *If $\alpha > 1/2$ then*

$$(2.18)\qquad \left|\mathcal{H}_{2n}^\alpha(x)\right| \le C \begin{cases} \nu^{-1/12} & ,0 \le x^2 \le 3\nu/2 \\ x^{1/2}e^{-\gamma x^2} & ,3\nu/2 < x^2 \end{cases}$$

$$(2.19)\qquad \left|\mathcal{H}_{2n+1}^\alpha(x)\right| \le C \begin{cases} \nu^{3/4} & ,0 \le x^2 \le 3\nu/2 \\ x^{1/2}e^{-\gamma x^2} & ,3\nu/2 < x^2 \end{cases}$$

where C, γ are positive constants, $\nu = 4n + \alpha + 1$.

Lemma 4.

$$(2.20)\qquad \| \hat{H}_n^\alpha(x) \|_{L^p_{H(\alpha)}} \sim \begin{cases} n^{1/2p-1/4} & ,1 \le p \le 4 \\ n^{-(1/12+1/6p)} & ,4 < p \le \infty \end{cases}$$

$$(2.21)\qquad \| \hat{H}^{\alpha}_{n+1}(x) - \hat{H}^{\alpha}_{n-1}(x) \|_{L^p_{H(\alpha)}} \sim n^{1/2p-1/4} \quad (n \to \infty)$$

PROOF. Assume that n is even, i.e. $n = 2m$ ($m \in \mathbf{N}$) (the case when n is odd, leads to a slight modification only).
Let $y = x^2$. We have

$$\|\hat{H}^{\alpha}_{2m}(x)\|_{L^p_{H(\alpha)}} = \|\hat{L}^{\frac{\alpha-1}{2}}_m(x^2)\|_{L^p_{H(\alpha)}} = \left(\int_{-\infty}^{\infty} \left|\hat{L}^{\frac{\alpha-1}{2}}_m(x^2)e^{-x^2/2}|x|^{\alpha/2}\right|^p dx \right)^{1/p}$$

$$= \left(\int_0^{\infty} \left|\hat{L}^{\frac{\alpha-1}{2}}_m(y)e^{-y/2}y^{\alpha/4-1/2p}\right|^p dy \right)^{1/p} = \| \hat{L}^{\frac{\alpha-1}{2}}_m(y) \|_{L^p_{U(\alpha/2-1/p)}}$$

Put $a = \alpha/2 - 1/p$, $a + b = (\alpha-1)/2$, then $a + b > -1$ and $a > -2/p$ as $\alpha > -1$. From [4], (2.9) we get

$$\| \hat{H}^{\alpha}_{2m}(x) \|_{L^p_{H(\alpha)}} = \| \hat{L}^{\frac{\alpha-1}{2}}_m(y) \|_{L^p_{U(\alpha/2-1/p)}} \sim$$

$$\sim \begin{cases} m^{1/2p-1/4} & ,1 \le p \le 4 \\ m^{-(1/12-1/6p)} & ,4 < p \le \infty. \end{cases} \sim \begin{cases} n^{1/2p-1/4} & ,1 \le p \le 4 \\ n^{-(1/12+1/6p)} & ,4 < p \le \infty \end{cases}$$

(2.21) is proved from (2.20) and the fact that $n^{-(1/12+1/6p)} < n^{1/2p-1/4}$ for $n \in \mathbf{N}$, $4 < p \le \infty$. q.e.d.

Lemma 5.

$$(2.22)\qquad \text{ess sup}\Lambda_n(t;\ \alpha,\ \alpha-1/2,\ 1) \le C \qquad ,\ \alpha > -1/2$$

$$(2.23)\qquad \Lambda_n(t;\ \alpha,\ \alpha-1/2,\ 0) \le C \qquad ,\ \alpha \ge -1/2$$

Here $\Lambda_n(t;\alpha,\gamma,\delta)$ is the Lebesgue function, defined by

$$(2.24)\qquad \Lambda_n(t;\alpha,\gamma,\delta) = (A^{\delta}_n)^{-1} \int_0^{\infty} \Big|\sum_{k=0}^{n} A^{\delta}_{n-k} L^{\alpha}_k(x) L^{\alpha}_k(t)\Big| (t/x)^{\frac{\alpha-\gamma}{2}} dx$$

where

$$(2.25)\qquad A^{\delta}_n = \binom{n+\delta}{n}$$

PROOF. (2.23) was proved in ([3] p. 30–31). Putting $r = -1/4$ into [8], Corollary of theorem 1, we obtain

$$\| (C,1)^{\alpha}_n \|_{[L^1_{U(\alpha-1/2)}]} \le C\ .$$

However

$$\| (C,1)_n^\alpha \|_{[L^1_{U(\alpha-1/2)}]} = \text{ess sup}\Lambda_n(t;\alpha,\alpha-1/2,1)$$

(c.f. [3], (4.19)) so we get (2.22). q.e.d.

III. The proof of the Theorem

Using the notation (2.2) we shall show the existence of a constant C such that

$$\| p_n(x) \|_{L^1_{H(\alpha)}} \leq Cn^{1/2} \| p_n(x) \|_{L^\infty_{H(\alpha)}} \tag{3.1}$$

for every $p_n \in \mathcal{P}_n$, and this inequality is sharp. Let $p_n \in \mathcal{P}_n$ be arbitrary, then

$$\begin{aligned} \| p_n(x) \|_{L^1_{H(\alpha)}} &= \| V^{H(\alpha)}_{n,2n-1}(p_n;x) \|_{L^1_{H(\alpha)}} \\ &= \| 2(C,1)^{H(\alpha)}_{2n-1}(p_n;x) - (C,1)^{H(\alpha)}_{n-1}(p_n;x) \|_{L^1_{H(\alpha)}} \\ &\leq 2 \| (C,1)^{H(\alpha)}_{2n-1}(p_n;x) \|_{L^1_{H(\alpha)}} + \| (C,1)^{H(\alpha)}_{n-1}(p_n;x) \|_{L^1_{H(\alpha)}} \\ &\leq \{2I(2n-1) + I(n-1)\} \| P_n(x) \|_{L^\infty_{H(\alpha)}} \end{aligned}$$

where

$$I(n) = \int_{-\infty}^{\infty} \left\| \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \hat{H}^\alpha_k(t)\hat{H}^\alpha_k() \right\|_{L^1_{H(\alpha)}} e^{-t^2/2}|t|^{\alpha/2}dt$$

$$= \int_{-\infty}^{\sqrt{3\nu/2}} + \int_{-\sqrt{3\nu/2}}^{+\sqrt{3\nu/2}} + \int_{\sqrt{3\nu/2}}^{\infty} =: I_1(n) + I_2(n) + I_3(n).$$

$$I_3(n) \leq \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \int_{\sqrt{3\nu/2}}^{\infty} \| \hat{H}^\alpha_k(x) \|_{L^1_{H(\alpha)}} \left|\hat{H}^\alpha_k(t)\right| e^{-t^2/2}t^{\alpha/2}dt$$

$$\leq \sum_{k=0}^{n} \| H^\alpha_k(x) \|_{L^1_{H(\alpha)}} \int_{\sqrt{3\nu/2}}^{\infty} \left|\hat{H}^\alpha_k(t)\right| e^{-t^2/2}t^{\alpha/2}dt$$

Hence, using lemma 3. and lemma 4. we obtain

$$I_3(n) \le C\sum_{k=0}^{n} k^{1/4} \int\limits_{\sqrt{3\nu/2}}^{\infty} x^{1/2}e^{-\gamma x^2}dx$$

$$\le Cn^{5/4} \int\limits_{\sqrt{3\nu/2}}^{\infty} y^2 e^{-\gamma\nu^{3/4}y}dy \tag{3.2}$$

$$\le Cn^{5/4}e^{-\gamma\nu}n^{-1/4} \le Cne^{\gamma n} \le C$$

Analogously, we can see that

$$I_1(n) \le C \tag{3.3}$$

Now, let us prove that $I_2(n) \le Cn^{1/2}$ Let $n = 2m+1$ (the case $n = 2m$ can be dealt with in a similar way with slight modification)

$$\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)\hat{H}_k^{\alpha}(x)\hat{H}_k^{\alpha}(t) = \sum_{k=0}^{2m+1}\left(1-\frac{k}{2m+2}\right)\hat{H}_k^{\alpha}(x)\hat{H}_k^{\alpha}(t)$$

$$=\sum_{j=0}^{m}\left(1-\frac{2j}{2m+2}\right)\hat{H}_{2j}^{\alpha}(x)\hat{H}_{2j}^{\alpha}(t)+\sum_{j=0}^{m}\left(1-\frac{2j+1}{2m+2}\right)\hat{H}_{2j+1}^{\alpha}(x)\hat{H}_{2j+1}^{\alpha}(t)$$

$$=\sum_{j=0}^{m}\left(1-\frac{j}{m+1}\right)\hat{L}_j^{\frac{\alpha-1}{2}}(x^2)\hat{L}_j^{\frac{\alpha-1}{2}}(t^2)$$

$$+\sum_{j=0}^{m}\left(1-\frac{j}{m+1}\right)\hat{L}_j^{\frac{\alpha+1}{2}}(x^2)\hat{L}_j^{\frac{\alpha+1}{2}}(t^2)xt$$

$$-\frac{1}{(2m+2)}\sum_{j=0}^{m}\hat{L}_j^{\frac{\alpha+1}{2}}(x^2)\hat{L}_j^{\frac{\alpha+1}{2}}(t^2)xt =: A_1+A_2+A_3$$

$$I_2(n) = \int\limits_{-\sqrt{3\nu/2}}^{\sqrt{3\nu/2}}\int\limits_{-\infty}^{\infty}\left|\sum_{k=0}^{2m+1}\left(1-\frac{k}{2m+2}\right)\hat{H}_k^{\alpha}(x)\hat{H}_k^{\alpha}(t)\right| e^{-\frac{x^2+t^2}{2}}|xt|^{\alpha/2}dxdt$$

$$\le 4\sum_{j=1}^{3}\int\limits_{0}^{\sqrt{3\nu/2}}\int\limits_{0}^{\infty}|A_j|e^{-\frac{x^2+t^2}{2}}(xt)^{\alpha/2}dxdt$$

Putting $x = \xi^2,\ t = \tau^2$, we get

$$I_2(n) = C\int_0^{3\nu/2}\int_0^{\infty}\left|\sum_{j=0}^{m}\left(1-\frac{j}{m+1}\right)\hat{L}_j^{\frac{\alpha-1}{2}}(\xi)\hat{L}_j^{\frac{\alpha-1}{2}}(\tau)\right|e^{-\frac{\xi+\tau}{2}}(\xi\tau)^{\alpha/4-1/2}d\xi d\tau$$

$$+\,C\int_0^{3\nu/2}\int_0^{\infty}\left|\sum_{j=0}^{m}\left(1-\frac{j}{m+1}\right)\hat{L}_j^{\frac{\alpha+1}{2}}(\xi)\hat{L}_j^{\frac{\alpha+1}{2}}(\tau)\right|e^{-\frac{\xi+\tau}{2}}(\xi\tau)^{\alpha/4}d\xi d\tau$$

$$+\,C\int_0^{3\nu/2}\int_0^{\infty}\left|\frac{1}{(2m+2)}\sum_{j=0}^{m}\hat{L}_j^{\frac{\alpha+1}{2}}(\xi)\hat{L}_j^{\frac{\alpha+1}{2}}(\tau)\right|e^{-\frac{\xi+\tau}{2}}(\xi\tau)^{\alpha/4}d\xi d\tau$$

$$= C\int_0^{3\nu/2}\int_0^{\infty}\left|\sum_{j=0}^{m}\left(1-\frac{j}{m+1}\right)\mathcal{L}_j^{\frac{\alpha-1}{2}}(\xi)\mathcal{L}_j^{\frac{\alpha-1}{2}}(\tau)\right|(\xi\tau)^{-1/4}d\xi d\tau$$

$$+\,C\int_0^{3\nu/2}\int_0^{\infty}\left|\sum_{j=0}^{m}\left(1-\frac{j}{m+1}\right)\mathcal{L}_j^{\frac{\alpha+1}{2}}(\xi)\mathcal{L}_j^{\frac{\alpha+1}{2}}(\tau)\right|(\xi\tau)^{-1/4}d\xi d\tau$$

$$+\,\frac{C}{(2m+2)}\int_0^{3\nu/2}\int_0^{\infty}\left|\sum_{j=0}^{m}\mathcal{L}_j^{\frac{\alpha+1}{2}}(\xi)\mathcal{L}_j^{\frac{\alpha+1}{2}}(\tau)\right|(\xi\tau)^{-1/4}d\xi d\tau$$

$$= C\int_0^{3\nu/2}\Lambda_m(\tau\,;\frac{\alpha-1}{2},\ \frac{\alpha-2}{2},\ 1)\tau^{-1/2}d\tau$$

$$+\,C\int_0^{3\nu/2}\Lambda_m(\tau\,;\frac{\alpha+1}{2},\ \frac{\alpha}{2},\ 1)\tau^{-1/2}d\tau$$

$$+\,\frac{C}{(2m+2)}\int_0^{3\nu/2}\Lambda_m(\tau\,;\frac{\alpha+1}{2},\ \frac{\alpha}{2},\ 0)\tau^{-1/2}d\tau$$

Hence (2.22) and (2.23) of lemma 5. yield

$$I_2(n) \le C\int_0^{3\nu/2}\tau^{-1/2}d\tau + C\frac{m^{1/6}}{2m+2}\int_0^{3\nu/2}\underline{1}d\tau \tag{3.4}$$

$$\le C\nu^{1/2} \le Cn^{1/2}$$

(3.2), (3.3) and (3.4) lead to (3.1).

Finally, we show that (3.1) is sharp. We can choose the test polynomials as follows:

$$\tilde{p}_n(x) := \left\{ \hat{H}_n^\alpha(x) - \hat{H}_{n-2}^\alpha(x) \right\} \qquad (x \in \mathbf{R},\ n \geq 2) \tag{3.5}$$

Then, for $1 \leq p \leq \infty$, by (2.21)

$$\| \tilde{p}_n(x) \|_{L^p_{H(\alpha)}} = \| \hat{H}_n^\alpha(x) - \hat{H}_{n-2}^\alpha(x) \|_{L^p_{H(\alpha)}} \sim n^{1/2p-1/4}$$

Consequently,

$$\frac{\|\tilde{p}_n(x)\|_{L^1_{H(\alpha)}}}{\|\tilde{p}_n(x)\|_{L^\infty_{H(\alpha)}}} \quad \sim \quad n^{1/2}$$

Thus there exists a constant $C > 0$, such that for each $n \in \mathbf{N}$

$$\| \tilde{P}_n(x) \|_{L^1_{H(\alpha)}} \geq C n^{1/2} \| \tilde{P}_n(x) \|_{L^\infty_{H(\alpha)}}$$

This completes the proof of the Theorem. q.e.d.

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