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Convergence rates in the law of large numbers for arrays[1]

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1. Introduction

The classical MARCINKIEWICZ and ZYGMUND [16] strong law of large numbers states the following. Let $X_1, X_2, \dots$ be independent identically distributed (i.i.d.) random variables (r.v.'s) with $EX_i = 0$ if $E|X_i| < \infty$, $S_n = X_1 + \dots + X_n$, $0 < q < 2$. Then $\lim_{n\to\infty} S_n/n^{1/q} = 0$ almost surely if and only if $E|X_i|^q < \infty$.

The rate of convergence to zero of quantities $P(|S_n|/n^{1/q} > \varepsilon)$ is described by the well-known theorem of BAUM and KATZ [2]:

Theorem BK. *Let $X_1, X_2, \dots$ be i.i.d. r.v.'s (with $EX_i = 0$ if $E|X_i| < \infty$), $S_n = X_1 + \dots + X_n$. Let $t > 0$, $r \geq 1$, $r/t > 1/2$. Then $E|X_i|^t < \infty$ is the necessary and sufficient condition that*

$$\sum_{n=1}^{\infty} n^{r-2} P(|S_n| > n^{r/t}\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{1.1}$$

For $r = t = 2$ this theorem is due to HSU and ROBBINS [11] and ERDŐS [3] and [4]. If $r = t = 1$ the theorem has been proved by SPITZER [17]. The other cases have been obtained by KATZ [14] and BAUM and KATZ [2]. In this paper we shall refer to (the sufficiency part of) Theorem BK as Spitzer's theorem if $r = 1$ and as the Katz theorem if $r > 1$.

Theorem BK has been extended in several directions such as to Banach space valued random variables (see e.g. JAIN [13], WOYCZYŃSKI [18]), to subsequences (see e.g. GUT [8]), to random variables with multidimensional indices (see e.g. GUT [7] and FAZEKAS [5]), etc.

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If inequality (1.1) is satisfied with $r = 2$ and $t = 2q$ then it is said that the sequence $S_n/n^{1/q}$ converges to 0 completely. Complete convergence for arrays of r.v.'s has been studied recently by Hu, Móricz and Taylor [12] and Gut [9]. Hu, Móricz and Taylor [12] established the following result:

Let $\{X_{nk},\ k = 1, ..., n,\ n = 1, 2, ...\}$ be an array of rowwise independent r.v.'s, $EX_{nk} = 0$ for all n and k, $S_n = \sum_{k=1}^n X_{nk}$. Suppose that $\{X_{nk}\}$ is weakly dominated by a random variable X such that $E|X|^{2q} < \infty$, $1 \le q < 2$. Then $S_n/n^{1/q}$ converges to 0 completely.

Gut [9] completed the above result and simplified its proof.

The present paper is devoted to extensions of some results of Gut [90]. We use the concept of weak mean domination (Definition 2.1) introduced by Gut [90]. This condition is less restrictive than the previously used weak domination condition (in other words uniformly bounded tail probabilities, see e.g. Hu, Móricz and Taylor [12], Woyczyński [18]). We formulate our results in terms of Banach space valued r.v.'s. However, in view of Gut [9], they seem to be new in the case of real r.v.'s, too.

In Section 2 basic definitions and lemmas are given. Section 3 is devoted to extensions of the Katz theorem to weakly mean dominated arrays of Banach space valued r.v.'s. Motivated by Jain [13], Theorem 3.5 deals with the case of general Banach spaces. In this case boundedness in probability of the normalized partial sums is required. Thereom 3.1 is a consequence of Theorem 3.5 for Banach spaces of type p. However, a straightforward proof of Theorem 3.1 is included because it is short and easy especially for real r.v.'s.

In Section 4 Spitzer's theorem is generalized to arrays. Section 5 is devoted to the necessity part of Theorem BK. Proposition 5.1 offers the converse statement to theorems 3.1, 3.5 and 4.1 in the case of identically distributed r.v.'s. Proposition 5.2 deals with non-identically distributed r.v.'s. In this case the necessary condition is expressed with tail probabilities of X_{nk}.

In Section 6 arrays of the form $\{X_{nk},\ k = 1, ..., k_n,\ n = 1, 2, ...\}$ are studied. Theorems 6.2 and 6.4 offer Katz and Spitzer type results for $S_{k_n} = \sum_{l=1}^{k_n} X_{nl}$. Here we mention only that in the case of "rapidly" increasing sequences $\{k_n\}$ the sufficient moment condition can be significantly weaker than in the case of $k_n = n$ (Remark 6.5b). The results of this section seem to be new even for i.i.d. real r.v.'s (compare with Gut [8]).

2. Notation and preliminary results

Let B be a real separable Banach space with norm $|\,.\,|$. We suppose that B is equipped with its Borel σ-field $\mathcal{B}$. Let $\{X_{nk},\ k = 1, \dots, n,\ n = 1, 2, \dots\}$ be a triangular array of rowwise independent B-valued random variables (r.v.'s). We do not assume independence between rows. Let $S_n = \sum_{k=1}^{n} X_{nk},\ n = 1, 2, ...,$ denote the row sums.

Throughout the paper we assume that

$$(2.1)\quad EX_{nk} = 0 \quad \text{whenever} \quad E|X_{nk}| < \infty\,, \qquad k = 1, ..., n,\ n = 1, 2, \dots .$$

(If $E|X| < \infty$, then EX is to be understood in the Bochner sense.)

Definition 2.1 (Hu, Móricz, Taylor [12], Gut [9]). We say that the array $\{X_{nk},\ k = 1, \dots, n,\ n = 1, 2, \dots\}$ is

(a) *weakly dominated* by the r.v. X if

$$\text{(WD)}\quad P(|X_{nk}| > x) \le P(|X| > x) \qquad \text{for all } x > 0 \text{ and for all } k \text{ and } n;$$

(b) *weakly mean dominated* by the r.v. X if, for some $\gamma > 0$,

$$\text{(WMD)}\quad \frac{1}{n}\sum_{k=1}^{n} P(|X_{nk}| > x) \le \gamma P(|X| > x) \qquad \text{for all } x > 0 \text{ and for all } n.$$

The (WMD) assumption is strictly weaker than the (WD) assumption (Gut [9], Example 2.1). The following example shows that this is true for sequences, as well.

Example. Let $|X_k| = k^{\frac{1}{4}}$ if $k = 2^l$ $(l = 1, 2, \dots)$ and $|X_k| = 1$ otherwise. Then $\{X_k\}$ satisfies (WMD) but does not satisfy (WD).

We shall use the well-known inequality of P. Lévy (see Hoffmann-Jørgensen [10]).

Lemma 2.2. *If $X_1, \dots, X_n$ are independent, symmetric, B-valued r.v.'s, $S_n = X_1 + \dots + X_n$, then for $\lambda > 0$*

$$(2.2)\qquad P\left(\max_{1\le j\le n} |S_j| > \lambda\right) \le 2\, P(|S_n| > \lambda).$$

The most powerful tool used in this paper is the following inequality (see Hoffmann-Jørgensen [10], Jain [13]).

Lemma 2.3. *Let $X_1, \dots, X_n$ be independent, symmetric, B-valued r.v.'s, $S_n = X_1 + \dots + X_n$. If s, t are nonnegative real numbers, then*

$$P(|S_n| > 2t + s) \le P\left(\max_{1\le k\le n} |X_k| > s\right) + 4(P\,(|S_n| > t))^2. \tag{2.3}$$

Moreover, if j is a positive integer, then

$$P(|S_n| > 3^j t) \le A_j P\left(\max_{1\le k\le n} |X_k| > t\right) + B_j(P(|S_n| > t))^{2^j}, \tag{2.4}$$

where A_j and B_j are nonnegative constants which depend only on j.

In the symmetrization/desymmetrization procedure we use the following inequalities. Let $X^* = X - X'$ be a symmetrization of the r.v. X. Then

$$P(|X^*| \ge t) \le 2\, P(|X - a| \ge t/2) \quad \text{for every } a \in B \text{ and } t \ge 0. \tag{2.5}$$

On the other hand we have

$$P(|X^*| \ge t/2) \ge P(|X| > t)\; P(|X'| < t/2) \quad \text{for every } t \ge 0. \tag{2.6}$$

Let Φ denote the set of functions $\varphi : [0, \infty) \to [0, \infty)$ that are strictly increasing, unbounded, and $\varphi(0) = 0$.

The following lemma is a version of Lemma 2 of Bakštys and Norvaiša [1].

Lemma 2.4. *Let $X_1, X_2, \dots$ be a sequence of B-valued r.v.'s and let $\{X_i^* = X_i - X_i',\ i = 1, 2, \dots\}$ be its symmetrization. Let $\varphi \in \Phi$ and let t and δ be positive numbers. Suppose that*

$$P(|X_i| < \varphi(i)\, t/2) > \delta \quad \textit{for} \quad i > n_{t,\delta}. \tag{2.7}$$

Then there exists an $n_0 = n_0(t, \delta)$ such that

$$P\left(\max_{1\le i\le n} |X_i^*| > \varphi(n)\, t/2\right) > \delta P\left(\max_{1\le i\le n} |X_i| > \varphi(n)\, t\right) \quad \textit{for} \quad n > n_o. \tag{2.8}$$

Moreover

$$P\left(\sup_{i\ge n} \frac{|X_i^*|}{\varphi(i)} > \frac{t}{2}\right) \ge \delta P\left(\sup_{i\ge n} \frac{|X_i|}{\varphi(i)} > t\right) \quad \textit{for} \quad n > n_0. \tag{2.9}$$

Proof. Let n_0 be so large that

$$\inf_{j<n} P(|X_j| < \varphi(n)\, t/2) > \delta \quad \text{for all} \quad n > n_0. \tag{2.10}$$

Let $n > n_0$ and set

$$B_i = \{|X_i'| < \varphi(n)\, t/2\}, \quad A_i = \{|X_i| > \varphi(n)\, t\}, \quad i = 1, ..., n. \tag{2.11}$$

Then $\{|X_i^*| > \varphi(n)\, t/2\} \supset A_i B_i$ for $i = 1, ..., n$. An application of the "Lemma for events" (Loève [15] p. 258) yields (2.8). The proof of (2.9) is similar.

We adopt the following notation used by Jain [13]. If φ and ψ are in Φ (the class of functions introduced just before stating Lemma 2.4), then θ denotes the composite function

$$\theta = \varphi \circ \psi. \tag{2.12}$$

Define a sequence $\{\beta(n),\ n = 1, 2, \dots\}$ by

$$\beta(n) = \theta(n+1) - \theta(n), \qquad n = 1, 2, \dots. \tag{2.13}$$

We assume that the sequence $\{\beta(n)\}$ satisfies the following condition

$$\text{for some } C_1, C_2 > 0, \quad C_1 \le C_2\, \beta(n+1) \le \beta(n) \text{ for all } n. \tag{2.14}$$

We suppose also that φ satisfies the so called Δ_2-condition, which is equivalent to the following

$$\text{for some } C < \infty,\ \varphi(3x) \le C\, \varphi(x) \quad \text{for all } x > 0. \tag{2.15}$$

We need the following lemmas of Jain [13].

Lemma 2.5. *Let X be a B-valued r.v. and $\varphi \in \Phi$. Then*

$$E\, \varphi(|X|) < \infty \quad \textit{iff} \quad \sum_{n=1}^{\infty} \beta(n)\, P\,(|X| > \psi(n)) < \infty. \tag{2.16}$$

In particular, for $r > 0,\ t > 0$

$$E|X|^t < \infty \quad \textit{iff} \quad \sum_{n=1}^{\infty} n^{r-1} P\left(|X| > n^{r/t}\right) < \infty. \tag{2.17}$$

Lemma 2.6. *Let $\{X_{nk},\ k = 1, \dots, n,\ n = 1, 2, \dots\}$ be an array of rowwise independent, symmetric B-valued r.v.'s, $S_n = \sum_{k=1}^n X_{nk}$, $n = 1, 2, \dots.$ Let $\varphi \in \Phi$ satisfy condition (2.15). Assume that there exists a nonnegative sequence $\{\gamma_n,\ n = 1, 2, \dots\}$ such that $\{|S_n|/\gamma_n\}$ is bounded in probability. Then there exists a constant $A < \infty$ such that*

$$E\, \varphi(|S_n|) \le 2C \sum_{k=1}^{n} E\, \varphi\,(|X_{nk}|) + 8\, A\, C\varphi(\gamma_n) \quad \textit{for all } n. \tag{2.18}$$

The proof of this lemma is the same as Theorem 3.1 of Jain [13].

The following lemma is a variant of Lemma 2.1 of Gut [9].

Lemma 2.7. *Let the array $\{X_{nk}\}$ be weakly mean dominated by X.*

(a) $\sum_{k=1}^n E|X_{nk}| \le n\gamma E|X|$.

(b) *If $\varphi \in \Phi$ then $\{\varphi(|X_{nk}|)\}$ is weakly mean dominated by $\varphi(|X|)$.*

(c) *If $A > 0$ then*

$$\sum_{k=1}^n E|X_{nk}|I\{|X_{nk}| > A\} \le \gamma n E|X|I\{|X| > A\}. \tag{2.19}$$

Here, and in the sequel, $I\{\ .\ \}$ denotes the indicator of a set.

We also need the following statement of Woyczyński [18] (Prop. 1.3):

Lemma 2.8. *Let $\{X_i\}$ be independent, symmetric B-valued r.v.'s, $S_n = \sum_{k=1}^n X_k$. Let $\{a_i\}$, $\{b_i\}, \{c_i\}$ be sequences of positive numbers such that $a_i \uparrow \infty$, $c_i \downarrow 0$ and $\sum_{i=1}^j 2^i b_{2^i} = O(2^j c_{2^j})$. If*

$$\sum_{n=1}^\infty c_n P(|S_n|/a_n > \varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0, \tag{2.20}$$

then

$$\sum_{n=1}^\infty b_n P\left(\sup_{k\ge n}\{|S_k|/a_k\} > \varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{2.21}$$

We recall that the Banach space B is of (Rademacher) type p $(0 < p \le 2)$ iff there exists a $C > 0$ such that

$$E\,|\sum_{i=1}^n X_i|^p \le C \sum_{i=1}^n E|X_i|^p \tag{2.22}$$

for every independent B-valued r.v.'s $X_1, \dots, X_n$ with $E|X_i|^p < \infty$ (and $EX_i = 0$, if $p \ge 1$), $i = 1, \dots, n$. We remark that there is no Banach space of type p for $p > 2$ and every Banach space is of type p for $0 < p \le 1$. Moreover, if B is of type p and $p' < p$, then B is of type p'.

The following version of the Marcinkiewicz-Zygmund inequality is due to Woyczyński [18]:

Lemma 2.9. *Let B be of type p $(1 \le p \le 2)$, $q > p$. Let $X_1, \dots, X_n$ be independent B-valued r.v.'s with $E|X_i|^q < \infty$ and $EX_i = 0$, $i = 1, \dots, n$. Then there exists an $A > 0$ such that*

$$E\,|\sum_{i=1}^n X_i|^q \le A\, n^{(q-p)/p} \sum_{i=1}^n E|X_i|^q. \tag{2.23}$$

For $q = p$ inequality (2.23) reduces to inequality (2.22).

3. Generalizations of the Katz theorem

In this section we generalize the Katz theorem for weakly mean dominated Banach space valued arrays. In Theorem 3.5 the case of general Banach spaces and general normalizing functions are considered. This theorem has been obtained by Jain [13] for i.i.d. B-valued sequences. Our proof relies heavily on the method of Jain [13]. Theorem 3.1 is a consequence of Theorem 3.5 for Banach spaces of type p and for power functions as normalizing functions. However, we give an independent proof of Theorem 3.1 because it is short due to the direct applicaton of the Hoffmann-Jørgensen and the Marcinkiewicz-Zygmund inequalities (see also Theorem 2.1 of Gut [9]).

Theorem 3.1. *Let* $\{X_{nk},\ k = 1, \dots, n,\ n = 1, 2, \dots\}$ *be an array of rowwise independent* B*-valued r.v.'s. Suppose that this array satisfies condition* (WMD) *with* X *such that* $E|X|^t < \infty$ *for some* $t > 0$. *Set* $S_n = \sum_{k=1}^{n} X_{nk},\ n = 1, 2, \dots$. *Suppose that* B *is of type* p *for some* $0 < p \le t \wedge 2$. *If* $r > t/p$, *then*

$$\sum_{n=1}^{\infty} n^{r-2} P(|S_n| > \varepsilon n^{r/t}) < \infty \quad \text{for all } \varepsilon > 0. \tag{3.1}$$

Proof. First we assume symmetry. By inequality (2.4)

$$\begin{aligned} P(|S_n| > 3^j \varepsilon n^{r/t}) &\le \\ &\le A_j \sum_{k=1}^{n} P(|X_{nk}| > \varepsilon n^{r/t}) + B_j (P(|S_n| > \varepsilon n^{r/t}))^{2^j}. \end{aligned} \tag{3.2}$$

Let us consider the second term in the right-hand side of (3.2). If $t \le 1$, by Markov's inequality, the c_r-inequality and Lemma 2.7

$$P(|S_n| > \varepsilon n^{r/t}) \le \frac{E|S_n|^t}{\varepsilon^t n^r} \le \frac{1}{\varepsilon^t n^r} \sum_{k=1}^{n} E|X_{nk}|^t \le \frac{\gamma E|X|^t}{\varepsilon^t n^{r-1}}. \tag{3.3}$$

If $t \ge 1$, then Markov's inequality, the Marcinkiewicz-Zygmund inequality and Lemma 2.7 give

$$P(|S_n| > \varepsilon n^{r/t}) \le \frac{E|S_n|^t}{\varepsilon^t n^r} \le \frac{C n^{t/p-1}}{\varepsilon^t n^r} \sum_{k=1}^{n} E|X_{nk}|^t \le \frac{C E|X|^t}{\varepsilon^t n^{r-t/p}}. \tag{3.4}$$

By (3.2)–(3.4)

$$\sum_{n=1}^{\infty} n^{r-2} P(|S_n| > 3^j \varepsilon n^{r/t}) \leq \tag{3.5}$$

$$\leq A_j \gamma \sum_{n=1}^{\infty} n^{r-1} P(|X| > \varepsilon n^{r/t}) + B_j \sum_{n=1}^{\infty} n^{r-2} \left(\frac{\gamma E|X|^t}{\varepsilon^t n^{r-\lambda}} \right)^{2^j},$$

where $\lambda = 1$ if $t \leq 1$ and $\lambda = t/p$ if $t > 1$. The first term on the right-hand side of (3.5) is finite (by $E|X|^t < \infty$ and Lemma 2.5) while the second term can be made finite by choosing an appropriate j if $r > \lambda$, i.e. if $r > 1$ whenever $t \leq 1$ and if $r > t/p$ whenever $t > 1$. The proof is complete in the case of symmetric r.v.'s.

In the general case let $\{X_{nk}^*\}$ be a symmetrization of $\{X_{nk}\}$. By (2.5), the array $\{X_{nk}^*\}$ satisfies condition (WMD) with $2X$ and 2γ. Therefore the sums S_n^* corresponding to $\{X_{nk}^*\}$ satisfy (3.1). On the other hand, (3.3) and (3.4) imply that

$$P(|S_n| < \varepsilon n^{r/t}) > \delta \quad \text{if} \quad n > n_\delta. \tag{3.6}$$

Thus an application of inequality (2.6) completes the proof.

Remark 3.2. If we specialize Theorem 3.1 we obtain some previous results. In the case of $B = \mathbf{R}$ and $r = 2$ we get Theorem 2.1 of Gut [9]. In the case of $B = \mathbf{R}$, $r = 2$ and condition (WD) we get Theorem 2 of Hu, Móricz and Taylor [2]. In the case of a sequence $\{X_i\}$ of weakly dominated, independent B-valued r.v.'s and $1 < t < 2$ we obtain (a part of) Theorem 4.3 of Woyczyński [18].

Remark 3.3. Using the notation of Theorem 3.1, let $S_{nl} = \sum_{k=1}^{l} X_{nk}$, $l = 1, \dots, n$, $n = 1, 2, \dots$. Under conditions given in Theorem 3.1 we have

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1 \leq l \leq n} |S_{nl}| > \varepsilon n^{r/t} \right) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{3.7}$$

The proof of this fact is the same as that of Theorem 3.1 if one uses Lévy's inequality in the symmetric case and inequality (2.8) instead of (2.6) in the general case.

Corollary 3.4. *Let us suppose that the array in Theorem 3.1 is given by a sequence: $X_{nk} = X_k$ for all n and k. Then, under conditions of Theorem 3.1,*

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\sup_{k \geq n} \{|S_k| / k^{r/t}\} > \varepsilon \right) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{3.8}$$

PROOF. As $r > 1$ we can apply Lemma 2.8 in the symmetric case. The general case follows from the symmetric case if we use (2.9).

Assume that functions φ and ψ belong to Φ, functions θ and β are defined by (2.12) and (2.13), respectively, and conditions (2.14) and (2.15) hold.

Theorem 3.5. *Let* $\{X_{nk},\ k = 1,\dots,n,\ n = 1,2,\dots\}$ *be an array of rowwise independent B-valued r.v.'s. Suppose that this array satisfies condition* (WMD) *with* X *such that* $E\ \varphi(|X|) < \infty$ *for a* $\varphi \in \Phi$*. Set* $S_n = \sum_{k=1}^n X_{nk},\ n = 1,2,\dots$ *. Suppose that there exists a sequence* γ_n *such that* $\{|S_n|/\gamma_n\}$ *is bounded in probability and*

$$(n \vee \varphi(\gamma_n))/\theta(n) = O((\log n)^{-\delta} \wedge (\beta(n))^{-\delta}) \quad \text{for a} \quad \delta > 0. \tag{3.9}$$

Then

$$\sum_{n=1}^{\infty} (\beta(n)/n)\, P\left(|S_n| > \varepsilon\psi(n)\right) < \infty \quad \text{for every} \quad \varepsilon > 0. \tag{3.10}$$

PROOF. We first assume symmetry. By inequality (2.4)

(3.11)

$$P\left(|S_n| > 3^j\varepsilon\psi(n)\right) \le A_j \sum_{k=1}^{n} P\left(|X_{nk}| > \varepsilon\psi(n)\right) + B_j \left(P(|S_n| > \varepsilon\psi(n))\right)^{2^j}.$$

By Markov's inequality, lemmas 2.6 and 2.7, and condition (3.9) we have

$$\begin{aligned} P\left(|S_n| > \varepsilon\psi(n)\right) &\le P\left(\varphi(|S_n|) > \varepsilon'\theta(n)\right) \le \frac{E\varphi(|S_n|)}{\varepsilon'\theta(n)} \le \\ &\le \left[2C\sum_{k=1}^{n} E\varphi(|X_{nk}|) + 8AC\varphi(\gamma_n)\right]/[\varepsilon'\theta(n)] \le \\ &\le \left[2CnE\varphi(|X|) + 8AC\varphi(\gamma_n)\right]/[\varepsilon'\theta(n)] = \\ &= O\left(\frac{n \vee \varphi(\gamma_n)}{\theta(n)}\right) = O\left((\log n)^{-\delta} \wedge (\beta(n))^{-\delta}\right). \end{aligned} \tag{3.12}$$

Multiplying (3.11) by $\beta(n)/n$, summing on n, using condition (WMD) and inequality (3.12) we obtain

$$\sum_{n=1}^{\infty}(\beta(n)/n)P\left(|S_n| > 3^j\varepsilon\psi(n)\right) \le \tag{3.13}$$

$$\le\ A_j\gamma\sum_{n=1}^{\infty}\beta(n)P(|X|>\varepsilon\psi(n)) + B_j\sum_{n=1}^{\infty}\frac{\beta(n)}{n}[O\left((\log n)^{-\delta}\wedge(\beta(n))^{-\delta}\right)]^{2^j}.$$

The first term on the right-hand side of (3.13) is finite by Lemma 2.5. The second term is

$$(3.14)\qquad B_j \sum_{n=1}^{\infty} \frac{\beta(n)}{n} O(\log n \vee \beta(n))^{-\delta 2^j} = \sum_{n=1}^{\infty} O\left(\frac{1}{n}\frac{1}{(\log n)^{1+\alpha}}\right) < \infty$$

(where $\alpha > 0$ can be attained by choosing an appropriate j). Therefore, (3.10) is proved in the symmetric case.

In the general case we symmetrize . Let $\{X^*_{nk} = X_{nk} - X'_{nk}\}$ be a symmetrized version of $\{X_{nk}\}$ and $\{S^*_n\}, \{S'_n\}$ the partial sums corresponding to $\{X^*_{nk}\}$ and $\{X'_{nk}\}$, respectively. $\{X^*_{nk}\}$ satisfies condition (WMD) with $2X$ and 2γ, $E\varphi(|2X|) < \infty$. Inequality (2.5) shows that $\{|S^*_n| \,/\gamma_n\}$ is bounded in probability. Therefore

$$(3.15)\qquad \sum_{n=1}^{\infty} (\beta(n)/n)\ P(|S^*_n| > \varepsilon\psi(n)) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

By (3.9), $\lim_{n\to\infty} \gamma_n/\psi(n) = 0$. Since $\{|S_n|/\gamma_n\}$ is bounded in probability, we get

$$(3.16)\qquad \lim_{n\to\infty} P(|S_n| > \varepsilon\psi(n)) = 0 \quad \text{for all} \quad \varepsilon > 0.$$

Now, an application of inequality (2.6) completes the proof.

Remark 3.6. Let B be of type p $(1 \le p < 2)$, $E|X|^p < \infty$. Then Markov's inequality, the condition of type p (or the Marczinkiewicz-Zygmund inequality), and (WMD) give

$$(3.17)\qquad P(|S_n| \,/\gamma_n > \varepsilon) \le \varepsilon^{-1} A E|X|^p\, n^{1/p}/\gamma_n \quad \text{for all} \quad \varepsilon > 0.$$

Therefore, if $\{n^{1/p}/\gamma_n\}$ is bounded, then $\{|S_n| \,/\gamma_n\}$ is bounded in probability in the case considered.

We mention the following : if B is of type φ, where φ is a submultiplicative Orlicz function, $E\varphi(|X|) < \infty$, then the boundedness of $\{n\varphi(1/\gamma_n)\}$ is sufficient that $\{|S_n|/\gamma_n\}$ be bounded in probability. The proof is a modification of the method used in Theorem 4.1 of Fazekas [6].

Remark 3.7. Using the above remark one can specialize Theorem 3.5 to get Theorem 3.1.

4. A generalization of Spitzer's theorem

In this section we generalize Spitzer's theorem to weakly mean dominated arrays of B-valued r.v.'s, where B is of type $p+\delta$. For real r.v.'s this theorem has been obtained by GUT [9], Theorem 6.1. For weakly dominated sequences this theorem has been proved by WOYCZYŃSKI [18]. To prove our theorem we follow the methods of Theorem 4.3 of WOYCZYŃSKI [18].

Theorem 4.1. *Let* $\{X_{nk},\ k=1,\dots,n,\ n=1,2,\dots\}$ *be an array of rowwise independent* B*-valued r.v.'s,* $S_n=\sum_{k=1}^n X_{nk}$*. Suppose that* $\{X_{nk}\}$ *satisfies condition* (WMD) *with* X *such that*

$$E|X|^p(\log^+|X|)^u<\infty \tag{4.1}$$

for some $0<p<2,\ u\geq 0$*. Assume that* B *is of type* $p+\delta$ *for some* $\delta>0\quad(p+\delta\leq 2)$*. Then*

$$\sum_{n=1}^{\infty} n^{-1}(\log n)^u P\left(|S_n|>n^{1/p}\varepsilon\right)<\infty \quad \text{for all} \quad \varepsilon>0. \tag{4.2}$$

PROOF. Introduce the truncated variables $Y_{nk}=X_{nk}\,I\{\,|X_{nk}|\leq n^{1/p}\}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1}(\log n)^u P\left(|S_n|>n^{1/p}\varepsilon\right)\leq&\\ \leq\sum_{n=1}^{\infty} n^{-1}(\log n)^u P\left(\max_{1\leq k\leq n}|X_{nk}|>n^{1/p}\right)&+\\ +\sum_{n=1}^{\infty} n^{-1}(\log n)^u P\left(|\sum_{k=1}^n Y_{nk}|>n^{1/p}\,\varepsilon\right)&.\end{aligned} \tag{4.3}$$

Using condition (WMD), Lemma 2.5, and (4.1) one can see that the first term on the right-hand side of (4.3) is finite. To estimate the second term define

$$X'=X'(n,p)=X\,I\{|X|\leq n^{1/p}\}+n^{1/p}I\{|X|>n^{1/p}\}. \tag{4.4}$$

If $p<1$, then Markov's inequality, the c_p-inequality, and condition (WMD) imply that

$$\begin{aligned}P\left(|\sum_{k=1}^n Y_{nk}|>n^{1/p}\varepsilon\right)\leq&\\ \leq n^{-\frac{p+\delta}{p}}\varepsilon^{-(p+\delta)}\sum_{k=1}^n E|Y_{nk}|^{p+\delta}\leq Cn^{-\frac{\delta}{p}}E|X'|^{p+\delta}&,\end{aligned} \tag{4.5}$$

where $p + \delta \le 1$. Now, consider the case $p \ge 1$. We have (recall that $EX_{nk} = 0$)

$$(4.6)\quad |\sum_{k=1}^{n} EY_{nk}| = |\sum_{k=1}^{n} EX_{nk} I\{|X_{nk}| > n^{1/p}\}| \le$$
$$\le nE|X| I\{|X| > n^{1/p}\} \le$$
$$\le \frac{n}{n^{(p-1)/p}} E|X|^p I\{|X| > n^{1/p}\} = o(n^{1/p}) \quad \text{if} \quad n \to \infty.$$

Therefore, if n is large enough, say $n > n_\varepsilon$, then

$$(4.7)\quad P\left(|\sum_{k=1}^{n} Y_{nk}| > n^{1/p}\, \varepsilon\right) \le P\left(|\sum_{k=1}^{n} (Y_{nk} - EY_{nk})| > n^{1/p}\, \varepsilon/2\right).$$

Using Markov's inequality, the assumption that B is of type $p + \delta$, the c_p-inequality, and condition (WMD) we get ($p \ge 1$)

$$P\left(|\sum_{k=1}^{n} (Y_{nk} - EY_{nk})| > n^{1/p}\, \varepsilon\right) \le$$
$$(4.8)\qquad C_1 n^{-(p+\delta)/p} \sum_{k=1}^{n} E|Y_{nk} - EY_{nk}|^{p+\delta} \le$$
$$\le C_2\, n^{-(p+\delta)/p} \sum_{k=1}^{n} E|Y_{nk}|^{p+\delta} \le C_3\, n^{-\delta/p}\, E|X'|^{p+\delta}.$$

Now, (4.7) and (4.8) imply that (4.5) is true for $n > n_\varepsilon$ in the case of $p \ge 1$, as well. We estimate $E|X'|^{p+\delta}$:

$$(4.9)\qquad E|X'|^{p+\delta} = \int_0^{n^{(p+\delta)/p}} P(|X|^{p+\delta} > x)dx =$$
$$= C \int_0^1 n^{1+\delta/p} s^{\delta/p} P\left(|X| > n^{1/p} s^{1/p}\right) ds.$$

By (4.5) and (4.9)

$$(4.10)\qquad \sum_{n=1}^{\infty} n^{-1} (\log n)^u P\left(|\sum_{k=1}^{n} Y_{nk}| > n^{1/p} \varepsilon\right) \le$$
$$\le C_3 \int_0^1 s^{\delta/p} \sum_{n=1}^{\infty} (\log n)^u P\left(|X s^{-1/p}| > n^{1/p}\right) ds \le$$

$$\leq C_4 \int_0^1 s^{\delta/p} \, E|Xs^{-1/p}|^p (\log^+ |Xs^{-1/p}|)^u ds \leq$$

$$\leq C_5 E|X|^p (\log^+ |X|)^u \int_0^1 s^{\delta/p-1-\rho} ds < \infty$$

because $\rho > 0$ can be chosen to be arbitrarily small. This completes the proof.

Remark 4.2. The original Spitzer theorem is obtained by putting $u = 0$.

Corollary 4.3. *Let $\{X_k\}$ be a sequence of independent B-valued r.v.'s, $S_n = \sum_{k=1}^n X_k$. Suppose that $\{X_k\}$ satisfies condition* (WMD) *with X such that $E|X|^p \log^+ |X| < \infty$, $0 < p < 2$. Assume that B is of type $p + \delta$, where $\delta > 0$. Then*

$$\sum_{k=1}^{n} n^{-1} P\left(\sup_{n\leq k} \{|S_k| / k^{1/p}\} > \varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{4.11}$$

To prove this corollary one can apply Lemma 2.8.

5. Some converse statements

In the case of identically distributed r.v.'s, Theorem 3.5 has the following converse.

Proposition 5.1. *Let $\{X_{nk},\ k = 1, \dots, n,\ n = 1, 2, \dots\}$ be an array of rowwise independent B-valued r.v.'s, $S_n = \sum_{k=1}^n X_{nk}$. Suppose that X_{nk} are identically distributed and φ satisfies* (2.15). *If inequality* (3.10) *is satisfied for some $\varepsilon > 0$, then $E\,\varphi\,(|X_{11}|) < \infty$.*

Proof. In the symmetric case Lévy's inequality implies

$$\begin{aligned} &P\left(\max_{1\leq k\leq n} |X_{nk}| > 2x\right) \leq \\ &\qquad \leq P\left(\max_{1\leq k\leq n} |\sum_{j=1}^{k} X_{nj}| > x\right) \leq 2P(|S_n| > x). \end{aligned} \tag{5.1}$$

Hence

$$\sum_{n=1}^{\infty} (\beta(n)/n)\, P\left(\max_{1\leq k\leq n} |X_{nk}| > 2\varepsilon\psi(n)\right) < \infty \quad \text{for some} \quad \varepsilon > 0. \tag{5.2}$$

By Lemma 2.3 of Jain [13] $E\varphi(|X_{nk}|) < \infty$. The symmetrization procedure is the same as in Theorem 3.3 of Jain [13].

We remark that the above proposition contains the converse of theorems 3.1 and 4.1, too.

In the non-identically distributed case our aim is to prove that

$$\sum_{n=1}^{\infty} n^{\beta} P(|S_n| > n^{\alpha}\varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0 \tag{5.3}$$

implies

$$\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{n} P(|X_{nk}| > n^{\alpha}\varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{5.4}$$

Gut [9] has proved that for $\alpha > 0$ and $\beta = 0$ (5.3) implies (5.4) if X_{nk} are symmetrically distributed or weakly dominated r.v.'s. Using the method of Erdős [3] we extend this result.

Proposition 5.2. *Let $\{X_{nk},\ k = 1, \dots, n,\ n = 1, 2, \dots\}$ be an array of rowwise independent B-valued r.v.'s, $S_n = \sum_{k=1}^{n} X_{nk}$. Suppose that the r.v.'s X_{nk} are symmetric or satisfy condition* (WD). *Suppose that* (5.3) *is satisfied for an $\alpha > 0$ and $S_n/n^{\alpha} \to 0$ in probability (when $\beta < 0$). Then* (5.4) *holds true.*

Proof. First we suppose symmetry. Inequality (5.1) implies that

$$\lim_{n\to\infty} P\left(\max_{1\le k\le n} |X_{nk}| > 2n^{\alpha}\varepsilon\right) = 0. \tag{5.5}$$

Therefore

$$\lim_{n\to\infty} P\left(|\sum_{\substack{l=1\\ l\neq k}}^{n} X_{nl}| > n^{\alpha}\varepsilon\right) = 0 \quad \text{uniformly in} \quad k, \tag{5.6}$$

since $S_n/n^{\alpha} \to 0$ in probability.

We have

$$\bigcup_{k=1}^{n} \{|X_{nk}| > 2n^{\alpha}\varepsilon\} \left\{|\sum_{\substack{l=1\\ l\neq k}}^{n} X_{nl}| < n^{\alpha}\varepsilon\right\} \subset \{|S_n| > n^{\alpha}\varepsilon\}. \tag{5.7}$$

The same calculation as on p.290 of ERDŐS [3] gives

$$
\begin{aligned}
&P(|S_n| > n^\alpha \varepsilon) \geq \\
&\geq \sum_{k=1}^{n} P(|X_{nk}| > 2n^\alpha \varepsilon) \left[P\left(|\sum_{\substack{l=1 \\ l \neq k}}^{n} X_{nl}| < n^\alpha \varepsilon \right) - \right. \\
&\quad \left. - P\left(\max_{1 \leq l \leq k} |X_{nl}| > 2n^\alpha \varepsilon \right) \right].
\end{aligned} \tag{5.8}
$$

By (5.5), (5.6) and (5.8) we obtain that for a $\delta > 0$

$$
P(|S_n| > n^\alpha \varepsilon) \geq \delta \sum_{k=1}^{n} P(|X_{nk}| > 2n^\alpha \varepsilon) \quad \text{if} \quad n > n_\delta. \tag{5.9}
$$

This establishes the symmetric case.

The symmetrization procedure is the following. Denote by $Y^* = Y - Y'$ the symmetrization of a r.v. Y. Then inequality (2.5) shows that $\{X^*_{nk}\}$ satisfies the conditions of our proposition. Therefore (5.3) is true for $\{S^*_n\}$. By inequality (2.6) and condition (WD), (5.3) is true for $\{S_n\}$ as well.

The following simple example shows that without conditions on X_{nk} or S_n (5.3) does not imply (5.4).

Example 5.3. Let $X_{nk} = 0$ for all k if n is odd and let $X_{nk} = (-1)^k\, n^{\alpha+1}$ if n is even. Then $S_n = 0$ for all n, but $P(|X_{nk}| > n^\alpha \varepsilon) = 1$ for every k if $n > \varepsilon$ and n is even.

6. General arrays

Let $\{k_n,\ n = 1, 2, \dots\}$ be a strictly increasing sequence of positive integers. Let $\{X_{nk},\ k = 1, \dots, k_n,\ n = 1, 2, \dots\}$ be a rowwise independent array of B-valued r.v.'s, $S_{k_n} = \sum_{k=1}^{k_n} X_{nk},\ n = 1, 2, \dots$. Following GUT [8], introduce the functions Ψ and M:

$$
\Psi(x) = \text{ Card}\{n : k_n \leq x\} \quad \text{for} \quad x > 0 \quad \text{and} \quad \Psi(0) = 0, \tag{6.1}
$$

$$
M(x) = \sum_{n=1}^{[x]} k_n. \tag{6.2}
$$

The following lemma is a generalization of Lemma 2.1 of GUT [8].

Lemma 6.1. *Let $\rho > 0$ and suppose that*

$$\limsup_{n\to\infty} k_n/M(n-1) < \infty \tag{6.3}$$

if $\rho > 1$. Then

$$E\,(M(\Psi(|X|)))^{\rho} < \infty \tag{6.4}$$

implies

$$\sum_{n=1}^{\infty} M^{\rho-1}(n)\, k_n\, P(|X| > k_n) < \infty. \tag{6.5}$$

The proof can be accomplished by methods of Jain [13], Lemma 2.2 and Gut [8], Lemma 2.1. We remark that a similar converse statement also holds.

The following theorem is a generalization of Theorem 4.1 of Gut [9].

Theorem 6.2. *Let $\{X_{nk}\}$ be the array above. Let $r \geq 1$, $t > 0$ and suppose that* (6.3) *is satisfied if $r > 2$. Let B be of type p for some $p \in (0,2]$. Let $\{X_{nk}\}$ satisfy* (WMD) *with an X for which*

$$E\,(M(\Psi(|X|^{t/r})))^{r/2} < \infty \tag{6.6}$$

and

$$E\,|X|^s < \infty \tag{6.7}$$

for an $s \geq p$. Suppose that $r > t/p$ if $s > 1$ while $r > t/s$ if $s \leq 1$. Then

$$\sum_{n=1}^{\infty} (M(n))^{r/2-1}\, P\left(|S_{k_n}| > k_n^{r/t}\varepsilon\right) < \infty \quad \text{for every } \varepsilon > 0. \tag{6.8}$$

Proof. In the symmetric case we use inequalities (2.4) and (2.23) (as in Theorem 3.1) to get

$$\begin{aligned} &\sum_{n=1}^{\infty} (M(n))^{r/2-1}\, P\left(|S_{k_n}| > k^{r/t}\,\varepsilon\, 3^j\right) \leq \\ &\leq C_1 \sum_{n=1}^{\infty} (M(n))^{r/2-1}\, k_n\, P(|X| > k_n^{r/t}\,\varepsilon) + \\ &\quad + C_2 \sum_{n=1}^{\infty} (M(n))^{r/2-1} \left(k_n^{s/p - sr/t} E|X|^s\right)^{2^j}. \end{aligned} \tag{6.9}$$

By Lemma 6.1 this expression is finite. The symmetrization procedure is the same as in Theorem 3.1.

Let us mention some special cases.

Remark 6.3. (a) Let $k_n = n,\ n = 1, 2, \ldots.$ Then $M(n) \asymp n^2$ and $M(\Psi(x)) \asymp x^2$. Condition (6.6) is reduced to $E|X|^t < \infty$. If we take $s = t$, then we obtain Theorem 3.1.

(b) Let $k_n = l^n,\ n = 1, 2, \ldots.$ Then $M(n) \asymp l^n$ and $M(\Psi(x)) \asymp x$. Condition (6.6) is reduced to $E|X|^{t/2} < \infty$. Our theorem is meaningless for $r < 2$. If $r = 2$, then $s > t/2$, therefore (6.7) is strictly stronger than (6.6). On the other hand, if t and r are given and $r > 2$ (and B is of type p for some $p \in (t/r, 2]$), then one can choose s so that $t/r < s < t/2$. Therefore, in this special case, (6.7) can be omitted.

(c) Let $k_n = n^d$, where d is a positive integer. Then $M(n) \asymp n^{d+1}$, $M(\Psi(x)) \asymp x^{(d+1)/d}$ and (6.6) is equivalent to $E\,|X|^{\frac{t(d+1)}{2d}} < \infty$. If, moreover, $r > 2d/(d+1)$, then (6.6) implies (6.7).

(d) If $r = 2$ and $B = \mathbf{R}$, then our Theorem 6.2 is the same as Theorem 4.1 of Gut [9].

Theorem 6.4. *Let* $\{X_{nk}\}$ *be an array as above. Let* $r \geq 1$ *and* $t > 0$ *and suppose that* (6.3) *is satisfied if* $r > 2$. *Assume that* B *is of type* p *for some* $p \in (0, 2]$. *Suppose that* $\{X_{nk}\}$ *satisfies* (WMD) *with an* X *such that*

$$E\,|X|^{t/r} < \infty, \tag{6.10}$$

$$E\,(M(\Psi(|X|^{t/r})))^{r/2} < \infty, \tag{6.11}$$

$$\int_0^1 s^{p-1} E(M(\Psi((|X|s^{-1})^{t/r})))^{r/2} ds < \infty. \tag{6.12}$$

Then

$$\sum_{n=1}^{\infty} (M(n))^{r/2-1} P(|S_{k_n}| > k_n^{r/t}\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{6.13}$$

Proof. The proof is similar to that of Theorem 4.1. Let $Y_{nk} = X_{nk}\, I\{|X_{nk}| < k_n^{r/t}\}$ and

$$X' = X'(n, r, t) = X\, I\{|X| < k_n^{r/t}\} + k_n^{r/t}\ I\{|X| \geq k_n^{r/t}\}. \tag{6.14}$$

Then

$$\sum_{n=1}^{\infty} (M(n))^{r/2-1}\ P(|S_{k_n}| > k_n^{r/t}\varepsilon) \leq \tag{6.15}$$

$$\leq \sum_{n=1}^{\infty}(M(n))^{r/2-1}\, k_n\, P(|X| > k_n^{r/t})+$$

$$+\sum_{n=1}^{\infty} M(n)^{r/2-1} P\left(|\sum_{k=1}^{k_n} Y_{nk}| > k_n^{r/t}\varepsilon\right).$$

The first term on the right hand side is finite by Lemma 6.1. The second term is

$$\sum_{n=1}^{\infty}(M(n))^{r/2-1}\, P\left(|\sum_{n=1}^{k_n} Y_{nk}| > k_n^{r/t}\varepsilon\right) \leq \tag{6.16}$$
$$\leq C\sum_{n=1}^{\infty} M(n)^{r/2-1}\, k_n^{1-pr/t}\, E|X'|^p.$$

By (6.12), the last expression is finite.

Remark 6.5. (a) Let $k_n = n$. Then (6.11) is $E|X|^t < \infty$ and this implies (6.10). Furthermore, (6.12) is satisfied iff $p > t$. Therefore our result is in accordance with the classical ones. Theorem 6.4 covers the classical results if $r = 1$, while Theorem 6.2 does not (because, if $r = 1$, then $s > t$; therefore (6.7) is more restrictive than the classical assumptions).

(b) Let $k_n = l^n$. Then condition (6.11) is the same as $E|X|^{t/2} < \infty$ and this implies condition (6.10) when $r \geq 2$. Condition (6.12) is satisfied iff $p > t/2$. For $r = 2$ we get the following: $E|X|^q < \infty$ implies that

$$\sum_{n=1}^{\infty} P\left(|S_{l^n}| > (l^n)^{\frac{1}{q}}\varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > 0$$

if $0 < q < p$. For $B = \mathbf{R}$ it is contained in Theorem 4.2 of Gut [9].

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References

[1] G. Bakštys and R. Norvaiša, On the rate of convergence in the law of large numbers in Banach spaces, *Litovsk. Mat. Sb.* **22** Vol. 2 (1980), 10–19, (in Russian).

[2] L.E. Baum and M. Katz, Convergence rates in the law of large numbers., *Trans. Amer. Math. Soc.* **120** (1965), 108–123.

[3] P. Erdős, On a theorem of Hsu and Robbins, *Ann. Math. Statist.* **20** (1949), 286–291.

[4] P. Erdős, Remark on my paper "On a theorem of Hsu and Robbins", *Ann. Math. Statist.* **21** (1950), 138.
[5] I. Fazekas, Convengence rates in the Marcinkiewicz strong law of large numbers for Banach space valued random variables with multidimensional indices, *Publ. Math. Debrecen* **32** (1985), 203–209.
[6] I. Fazekas, The law of the iterated logarithm in Banach spaces of type φ, *Publ. Math. Debrecen* **36** (1989), 65–74.
[7] A. Gut, Marcinkiewicz laws and convergence rates in the law of large numbers for random variables with multidimensional indices, *Ann. Probab.* **6** (1978), 469–482.
[8] A. Gut, On complete convergence in the law of large numbers for subsequences, *Ann. Probab.* **13** (1985), 1286–1291.
[9] A. Gut, Complete convergence for arrays, *Preprint, Uppsala University* (1990).
[10] J. Hoffmann-Jørgensen, Sums of independent Banach space valued random variables, *Studia Math.* **LII** (1974), 159–186.
[11] P.L. Hsu and H. Robbins, Complete convergence and the law of large numbers, *Proc. Nat. Acad. Sci. USA* **33** (1947), 25–31.
[12] T.-C. Hu, F. Móricz and R.L. Taylor, Strong laws of large numbers for arrays of rowwise independent random variables, *Acta Math. Acad. Sci. Hungar.* **54** (1989), 153–162.
[13] N. C. Jain, Tail probabilities for sums of independent Banach space valued random variables, *Wahrscheinlichkeitstheorie verw. Gebiete* **33** (1975), 155–166.
[14] M. Katz, The probability in the tail of a distribution, *Ann. Math. Statist.* **34** (1963), 312–318.
[15] M. Loève, Probability Theory, 4th ed. *Springer*, *New York*, 1977.
[16] J. Marcinkiewicz and A. Zygmund, Sur les fonctions indépendantes, *Fund. Math.* **29** (1937), 60–90.
[17] F.L. Spitzer, A combinatorial lemma and its application, *Trans. Amer. Math. Soc.* **82** (1956), 323–339.
[18] W. A. Woyczyński, On Marcinkiewicz- Zygmund laws of large numbers in Banach spaces and related rates of convergence, *Probab. Math. Statist.* **1** (1980), 117–131.

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